Math 6441 - Spring 2018
Homework 2

Read Chapter 0 in Hatcher.

Work all these problems and talk to me if you have any questions on them, but carefully write up and turn in **only** *problems 4, 5, 6, 8, and 10.* **Due: In class on January 29.**

1. Let γ and η be two paths in X with the same image and traverse the image in the same direction. Show there is some function $f : [0,1] \to [0,1]$ such that $\gamma = \eta \circ f$ and that γ and η are homotopic.
2. Show that any homomorphism $\mathbb{Z} = \pi_1(S^1, (1,0))$ to itself can be realized as f_* for some map $f : S^1 \to S^1$.
3. Let A a path connected subspace of a space X and $x_0 \in A$. Show that in inclusion map $i : A \to X$ induces a surjective map $\pi_1(A, x_0) \to \pi_1(X, x_0)$ if and only if any path with endpoints in A is homotopic, rel end points, to a path in A.
4. Show a path connected space X is simply connected if and only if every map $S^1 \to X$ extends to a map $D^2 \to X$.
5. Recall from class if $p \in S^1$ is a fixed point then $\pi_1(X, x_0) = [S^1, X]_0$ (that is homotopy classes of base point preserving maps $(S^1, p) \to (X, x_0)$). There is a natural map

$$\Psi : \pi_1(X, x_0) \to [S^1, X],$$

   where recall $[S^1, X]$ is the set of homotopy classes of map (with no condition on the base point). Show Ψ is onto if X is path connected. Also show that $\Psi([\gamma]) = \Psi([\eta])$ if and only if $[\gamma]$ and $[\eta]$ are conjugate in $\pi_1(X, x_0)$. Thus when X is path connected Φ induces a one-to-one correspondence between $[S^1, X]$ and the conjugacy classes of elements in $\pi_1(X, x_0)$.
6. Use the fundamental group to show there is no retraction of the Möbius band to its boundary.
7. If G is a topological group with unit element e, then for two loops γ and η based at $e$we can define the loop $(\gamma \cdot \eta)(t) = \gamma(t)\eta(t)$. Show that $\gamma * \eta$ is homotopic to $\gamma \cdot \eta$ rel end points. HINT: Think about reparameterizing loops.
8. Using the proof in the pervious problem show that $\pi_1(G, e)$ is abelian for a topological group G.
9. Let $f, g : X \to Y$ be homotopic maps. Show there is a path $\gamma : [0,1] \to Y$ from $f(x_0)$ to $g(x_0)$ so that the maps $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ and $g_* : \pi_1(X, x_0) \to \pi_1(Y, g(x_0))$ are related by $\phi_\gamma \circ g_* = f_*$ where $\phi_\gamma : \pi_1(Y, g(x_0)) \to \pi_1(Y, f(x_0))$ is the isomorphism from class.
10. Given any point $p \in S^1$ let γ_p be the loop that starts at p and goes counterclockwise around S^1. (Here we think of S^1 as the unit circle in $\mathbb{C}$ and the patch can be given by $\gamma_p(t) = p \cdot e^{2\pi i t}$.) In class we showed that we can take $[\gamma_p]$ to be a generator of $\pi_1(S^1, p) \cong \mathbb{Z}$. With this generator for any p, show that given any path $\lambda : [0,1] \to S^1$ the isomorphism $\phi_\lambda : \pi_1(S^1, \lambda(0)) \to \pi_1(S^1, \lambda(1))$ is the identity map. Hint: Path lifting might be helpful.
    If $f : S^1 \to S^1$, then show using the fundamental group that f induces a homomorphism from $\mathbb{Z}$ to $\mathbb{Z}$ that is independent of the homotopy class of f. (Note this is not completely obvious if f or the homotopy does not preserve base points, but follows from the above observation.) The integer corresponding to the homomorphism induced by f is called the

degree of f and denoted $\deg(f)$. Show a map $f : S^1 \to S^1$ extends to a map $D^2 \to S^1$ if and only if the degree of f is zero.