Math 6441 - Spring 2018
Homework 3

Read Chapter 1.2 and 1.3 in Hatcher.

Work all these problems and talk to me if you have any questions on them, but carefully write up and turn in **only** *problems 1, 5, 9, 10, and 11.* **Due: In class on February 7.**

1. Let G_1 and G_2 be non-trivial groups. Show that $G_1 * G_2$ is non-abelian, has elements of infinite order, and has trivial center.
2. Prove that a finite order element of $G_1 * G_2$ is either an element in G_1, in G_2 or a conjugate of such an element.
3. Show that the maximal order of an element of $\mathbb{Z}_n * \mathbb{Z}_m$ is the maximum of n and m.
4. Suppose a group G has a presentation $\langle x_1, \ldots, x_n | r_1, \ldots r_m \rangle$ where the relators are
$$r_i = x_{i_1}^{s_{i_1}} \cdots x_{i_{k_i}}^{s_{i_{k_i}}}$$
for $i = 1, \ldots m$, and the s_i are ± 1. Then show that if H is any other group and $h_1, \ldots, h_n$ are any elements of H that satisfy
$$h_{i_1}^{s_{i_1}} \cdots h_{i_{k_i}}^{s_{i_{k_i}}} = e_h,$$
where e_H is the identity element in H, then there is a unique homomorphism $\phi : G \to H$ such that $\phi(x_i) = h_i$.
5. Prove that the groups $\langle x, y | x^2 y^2 \rangle$ and $\langle a, b | baba^{-1} \rangle$ are isomorphism.
6. Show that $\mathbb{Z} \oplus \mathbb{Z}$ has presentation $\langle x, y | xyx^{-1}y^{-1} \rangle$.
7. Consider the rational numbers $\mathbb{Q}$ as a group under addition. Show that $\mathbb{Q}$ has presentation
$$\langle x_i, i = 1, 2, 3, \ldots | x_n^n = x_{n-1}, i > 1 \rangle.$$
Hint: try to construct a map by sending x_i to $\frac{1}{n!}$.
8. Let X be the union of the unit sphere S^2 in $\mathbb{R}^2$ and the unit disk D^2 in the xy-plane. Compute $\pi_1(X)$.
9. Take two copies of the torus $S^1 \times S^1$ and let X be the space obtained by identifying $S^1 \times \{pt\}$ in one torus with $S^1 \times \{pt\}$ in the other torus using the identity map on S^1. Compute $\pi_1(X, x_0)$.
10. Given a map $f : X \to X$ the **mapping torus T_f of f** is the space obtained from $X \times [0, 1]$ by identifying $(x, 0)$ with $(f(x), 1)$ for all $x \in X$. If $X = S^1 \vee S^1$ and f is a base point preserving map, write a presentation for $pi_1(T_f)$ in terms of the map $f_* : \pi_1(X, x_0) \to \pi_1(X, x_0)$. Do the same for $X = S^1 \times S^1$.
11. Let X and Y be two non-empty spaces. If X is path connected then show that the join $X * Y$ simply-connected.
12. Let X_1 and X_2 be the torus $S^1 \times S^1$ with an open disk removed and $C'' = S^1 \times \{pt\}$ in X_1 (we assume the disk that is removed is disjoint from C''). Let X be $X_1 \cup X_2$ with the boundaries glued together. So X is a genus 2 surface. Let C be the boundary of X_1 sitting in X and C' be C'' thought of as a subset of X. Show there is no retraction from X to C but there is a retraction from X to C'. (Hint: what is C in the abelianization of $\pi_1(X)$?)