Math 6441 - Spring 2018
Homework 4

Read Chapter 1.3 and Appendix 1.A and 1.B in Hatcher.
Work all these problems and talk to me if you have any questions on them, but carefully write up and turn in **only** *problems 1, 3, 6, 8, and 10.* **Due: In class on February 16.**

1. Use covering spaces to show that the free group of rank 2 has a normal subgroup of index 3 and a non-normal subgroup of index 3. If the free group is generated by elements a and b the give explicit generators for the subgroups.
2. If X is a space with contractible universal cover then show that any map $S^n \to X$, $n \geq 2$, can be extended to a map $D^{n+1} \to X$.
3. If X is space with contractible universal cover and Y is any path connected CW-complex with a vertex labeled y_0, then show that any homomorphism $\phi : \pi_1(Y, y_0) \to \pi_1(X, x_0)$ is induced by a continuous map $f : Y \to X$. That is f takes y_0 to x_0 and $f_* = \phi$.
4. If F_n is the free group on n generators and G is a subgroup in index k. Then you know from Hatcher Appendix 1.A that G is a free group. How many generators is it a free group on?
5. Let X be a connected, locally pathwise connected, and semi-locally simply connected topological space (so a space with a universal cover). Prove that a connected n-fold covering spaces of X correspond to representations of $\pi_1(X)$ to the symmeric group S_n that acts transitively on $\{1, \ldots, n\}$.
   More specifically, show that given a connected covering space $\widetilde{X} \to X$ there is an associated homomorphism $h : \pi_1(X) \to S_n$ (such that for any i and j there is some $g \in \pi_1(X)$ such that $h(g)(i) = j$) and conversely give such a homomorphism there is a connected covering space realizing this homomorphism.
6. Using the previous problem you can describe an n fold covering space of $S^1 \vee S^1$ by labeling the loops with elements of S_n (so that the two elements generate a transitive subgroup of S_n). Draw the covering spaces corresponding to the following labelings
   (a) Label with elements (1 2) and id in S_2.
   (b) Label with elements (1 2 3) and id in S_3.
   (c) Label with elements (1 2) and (2 3) in S_3.
7. Using the above idea list all 3 fold covers of $S^1 \vee S^1$.
8. Let X be a path connected, locally path connected space with $\pi_1(X, x_0)$ finite, show that any map $X \to S^1$ is nullhomotopic.
9. If F is a free group and N is a non-trivial infinite index normal subgroup of F, then show using covering spaces that N is not finitely generated. In particular show that if F is a free group on more than one generator then the commutator subgroup of F is not finitely generated.
10. Show that a finitely generated group has only a finite number of index n subgroups for a fixed n. Hint: Consider the case of free groups using graphs and covering space theory. Then prove the general result by the fact that any group is a quotient group of a free group.
11. Let $p : X \to Y$ and $q : Y \to Z$ be covering spaces. If each are of finite degree show that $q \circ p : X \to Z$ is a covering space.

12. Show that if in the previous problem p is of infinite degree then the composition does not have to be a covering space. For this consider $Z = \cup_{i=1}^{\infty} C_i$, where C_i is the circle of radius $1/i$ centered at $(0, 1/i)$ in $\mathbb{R}^2$. Let Y be the union of the x-axis and $C = \cup_{i=2}^{\infty} C_i$ and all the translates of C by vectors $(n, 0)$. Show there is an obvious infinite cover of Y over Z. Now find a 2 fold cover of Y such that the composition with the infinite cover is not a covering map.