Math 6441 - Spring 2021
Homework 1

Read Chapter 0 in Hatcher.

Work all these problems and talk to me if you have any questions on them, but carefully write up and turn in **only** *problems 1, 2, 4, 5, 7, and 8.* **Due: by 5:00 pm on February 1.**

1. Given a topological space X and points $x, y \in X$, let $e_x : \{*\} \to X$ be the function from the 1 point set $\{*\}$ that sends $*$ to x, and similarly for $e_y : \{*\} \to X$. Show that x and y are connected by a path if and only if e_x is homotopic to e_y.
2. Show that a space X is contractible if and only if every map $f : X \to Y$, for arbitrary Y, is null-homotopic if and only if every map $f : Y \to X$, for arbitrary Y, is null-homotopic.
3. Show that spaces X and Y are homotopy equivalent if and only if for any space Z there is a one-to-one correspondence

$$\phi_Z : [X, Z] \to [Y, Z]$$

such that for all continuous maps $h : Z \to Z'$ we have $h_* \circ \phi_Z = \phi_{Z'} \circ h_*$ where h_* is the map induced on homotopy classes of maps, that is the following diagram commutes

$$\begin{array}{ccc} [X, Z] & \xrightarrow{\phi_Z} & [Y, Z] \\ \downarrow h_* & & \downarrow h_* \\ [X, Z'] & \xrightarrow{\phi_{Z'}} & [Y, Z']. \end{array}$$

4. Show that $S^n * S^m = S^{n+m+1}$ where $X * Y$ stands for the join of X and Y (see Hatcher Chapter 0 for the definition). Hint: work this out for n, m equal to 0 or 1 first.
5. In Chapter 0 of Hatcher's book (page 7) he describes a CW structure on S^∞ with two cells of each dimension, enumerate all the subcomplexes of S^∞ with that CW structure.
6. Show that $f : X \to Y$ is a homotopy equivalence if there exist maps $g, h : Y \to X$ such that $f \circ g \sim id_Y$ and $h \circ f \sim id_X$. More generally, show that f is a homotopy equivalence if $f \circ g$ and $h \circ f$ are homotopy equivalences.
7. Given CW complexes X and Y, explicitly describe the CW structure on $X \times Y$ in terms of the CW structures on X and Y. (That is describe the cells of $X \times Y$ and describe the attaching maps.) Choose a cell structure on S^1 and explicitly write out the CW structure on $S^1 \times S^1$ coming from the product structure.
8. If G is a group and a topological space, then it is called a *topological group* if the maps

$$\mu : G \times G \to G : (g, h) \mapsto g \cdot h$$

and

$$i : G \to G : g \mapsto g^{-1}$$

are continuous, where $g \cdot h$ is the multiplication operation in G. Examples of topological groups include $\mathbb{R}$, S^1 (the unit circle in $\mathbb{C}$), and the group of $n \times n$ matrices with real (or complex) entries. If G is a topological group then show that for any space X the set $[X, G]$ is a group and for any continuous map $f : X \to Y$ the map $f^* : [Y, G] \to [X, G]$ defined in class is a homomorphism.

Fact: $[X, S^1]$ is the first cohomology group $H^1(X)$, though our definition later will be quite different.