Math 6441 - Spring 2021
Homework 2

Read Chapter 1.1 in Hatcher.

Work all these problems and talk to me if you have any questions on them, but carefully write up and turn in **only** *problems 1, 4, 5, 6, 8, and 15.* **Due: In class on February 12.**

1. Let $\gamma : [0,1] \to X$ and $\eta : [0,1] \to X$ be two paths in X. If there is a homeomorphism $f : [0,1] \to [0,1]$ (that takes 0 to 0) such that $\gamma = \eta \circ f$ then show that γ and η are homotopic. (That is re-parameterizations of paths are homotopic.)
2. Show that any homomorphism $\mathbb{Z} = \pi_1(S^1,(1,0))$ to itself can be realized as f_* for some map $f : S^1 \to S^1$.
3. Let A a path connected subspace of a space X and $x_0 \in A$. Show that in inclusion map $i : A \to X$ induces a surjective map $\pi_1(A, x_0) \to \pi_1(X, x_0)$ if and only if any path with endpoints in A is homotopic, rel end points, to a path in A.
4. Show a path connected space X is simply connected if and only if every map $S^1 \to X$ extends to a map $D^2 \to X$.
5. Recall from class if $p \in S^1$ is a fixed point then $\pi_1(X, x_0) = [S^1, X]_0$ (that is homotopy classes of base point preserving maps $(S^1, p) \to (X, x_0)$). There is a natural map

$$\Psi : \pi_1(X, x_0) \to [S^1, X],$$

where recall $[S^1, X]$ is the set of homotopy classes of map (with no condition on the base point). Show Ψ is onto if X is path connected. Also show that $\Psi([\gamma]) = \Psi([\eta])$ if and only if $[\gamma]$ and $[\eta]$ are conjugate in $\pi_1(X, x_0)$. Thus when X is path connected Φ induces a one-to-one correspondence between $[S^1, X]$ and the conjugacy classes of elements in $\pi_1(X, x_0)$.
6. Use the fundamental group to show there is no retraction of the Möbius band to its boundary.
7. If G is a topological group with unit element e, then for two loops γ and η based at e we can define the loop $(\gamma \cdot \eta)(t) = \gamma(t)\eta(t)$. Show that $\gamma * \eta$ is homotopic to $\gamma \cdot \eta$ rel end points. HINT: Think about reparameterizing loops.
8. Using the proof in the pervious problem show that $\pi_1(G, e)$ is abelian for a topological group G.
9. Given any point $p \in S^1$ let γ_p be the loop that starts at p and goes counterclockwise around S^1. (Here we think of S^1 as the unit circle in $\mathbb{C}$ and the patch can be given by $\gamma_p(t) = p \cdot e^{2\pi i t}$.) In class we showed that we can take $[\gamma_p]$ to be a generator of $\pi_1(S^1, p) \cong \mathbb{Z}$. Show that given any path $\lambda : [0,1] \to S^1$ the isomorphism $\phi_\lambda : \pi_1(S^1, \lambda(1)) \to \pi_1(S^1, \lambda(0))$ send $\gamma_{\lambda(1)}$ to $\gamma_{\lambda(0)}$.
   Hint: Path lifting might be helpful.
   Notice that this means if we always identify $\pi_1(S^1, p)$ with $\mathbb{Z}$ by the isomorphism $[\gamma_p]^n \mapsto n$, then we can think of ϕ_λ as a map from $\mathbb{Z}$ to $\mathbb{Z}$, and as such a map it is the identity map.
   If $f : S^1 \to S^1$, then show using the fundamental group that f induces a homomorphism from $\mathbb{Z}$ to $\mathbb{Z}$ that is depends only the the homotopy class of f. (Note this is not completely obvious if f or the homotopy does not preserve base points, but follows from the above observation.) The integer corresponding to the homomorphism induced by f is called the degree of f and denoted $\deg(f)$. Show a map $f : S^1 \to S^1$ extends to a map $D^2 \to S^1$ if and only if the degree of f is zero.

10. Let G_1 and G_2 be non-trivial groups. Show that $G_1 * G_2$ is non-abelian, has elements of infinite order, and has trivial center.
11. Prove that a finite order element of $G_1 * G_2$ is either an element in G_1, in G_2 or a conjugate of such an element.
12. Show that the maximal finite order of an element of $\mathbb{Z}_n * \mathbb{Z}_m$ is the maximum of n and m.
13. Suppose a group G has a presentation $\langle x_1, \ldots, x_n | r_1, \ldots r_m \rangle$ where the relators are

$$r_i = x_{i_1}^{s_{i_1}} \cdots x_{i_{k_i}}^{s_{i_{k_i}}}$$

for $i = 1, \ldots m$, and the s_i are ± 1. Then show that if H is any other group and $h_1, \ldots, h_n$ are any elements of H that satisfy

$$h_{i_1}^{s_{i_1}} \cdots h_{i_{k_i}}^{s_{i_{k_i}}} = e_h,$$

where e_H is the identity element in H, then there is a unique homomorphism $\phi : G \to H$ such that $\phi(x_i) = h_i$.
14. Prove that the groups $\langle x, y | x^2 y^2 \rangle$ and $\langle a, b | baba^{-1} \rangle$ are isomorphism.
15. Show that $\mathbb{Z} \oplus \mathbb{Z}$ has presentation $\langle x, y | xyx^{-1}y^{-1} \rangle$.
16. Consider the rational numbers $\mathbb{Q}$ as a group under addition. Show that $\mathbb{Q}$ has presentation

$$\langle x_i, i = 1, 2, 3, \ldots | x_n^n = x_{n-1}, i > 1 \rangle.$$

Hint: try to construct a map by sending x_i to $\frac{1}{n!}$.