Math 6441 - Spring 2021
Homework 4

Read Chapter 2.1 in Hatcher's book.

Work all these problems and talk to me if you have any questions on them, but carefully write up and turn in **only** *problems 1, 2, 4, 8, 9, and 10.* **Due: In class on March 17.**

1. Let (A_*, ∂_A) and (B_*, ∂_B) be two chain complexes and $\{f_n\} : (A_*, \partial_A) \to (B_*, \partial_B)$ be a chain map. Define a new complex $M_*(f)$ called the mapping cone of $\{f_n\}$ where $M_n(f) = A_{n-1} \oplus B_n$ and
$$\partial_f(a, b) = (-\partial_A a, \partial_B b + f_{n-1}(a)).$$
Show that this defines a complex and that there is a long exact sequence
$$\ldots \to H_n(A_*, \partial_A) \to H_n(B_*, \partial_B)) \to H_n(M_*(f), \partial_f) \to H_{n-1}(A_*, \partial_A) \to \ldots$$
Moreover show that in the above long exact sequence the first map is induced by $\{f_n\}$ and thus $\{f_n\}$ induces an isomorphism on homology if and only if $H_n(M_*(f), \partial_f) = 0$ for all n. Hint: Note there is a natural inclusion $B_n \to M_n(f)$ that is a chain map and if we let A^+_* be the complex with $A^+_n = A_{n-1}$ and $\partial_{A^+} a = -\partial_A a$ then there is a natural projection $M_n(f) \to A^+_n$ that give chain maps. Note that $H_n(A^*_*) = H_{n-1}(A_*)$.
2. Recall if $p : Y \to X$ is a covering space the induced map p_* on π_1 is injective. Show that the map induced on H_1 need not be injective. Hint: Consider a wedge of circles.
3. Let $r : X \to A$ be a retract for X onto a subspace $A \subset X$. Let $i : A \to X$ be the inclusion map. Show that $r_* : H_n(X) \to H_n(A)$ is surjective and $i_* : H_n(A) \to H_n(X)$ is injective for all n and that $H_n(X)$ is the direct sum of $\ker r_*$ and $\operatorname{im} i_*$.
4. Let $X = X_1 \cup X_2 \cup X_3$, where each X_i is open. If all X_i, $X_i \cap X_j$ and $X_1 \cap X_2 \cap X_3$ are either contractible or empty. Show that $H_n(X) = 0$ for all $n \geq 2$.
5. Show that chain homotopy of chain maps is an equivalence relation.
6. Given an exact sequence $A \to B \to C \to D \to E$, show that $C = 0$ if and only if the map $A \to B$ is surjective and the map $D \to E$ is injective.
7. If $X = T^2$ and A is a finite set of points, then compute $H_n(X, A)$.
8. The suspension $S(X)$ of a space X is $X \times [0, 1]$ with $X \times \{0\}$ collapsed to a point and $X \times \{1\}$ collapsed to a separate point (that is $S(X)$ is two copies of the cone on X glued together along X). Show $\widetilde{H}_{n+1}(S(X)) = \widetilde{H}_n(X)$.
9. Show that $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$ have the same homology groups but their universal covers do not have the same homology groups.
10. Use the Mayer-Vietoris sequence to compute the homology of real projective n-space $\mathbb{R}P^2$ and $\mathbb{R}P^3$.
11. Use the Mayer-Vietoris sequence to compute the homology of X where X is obtained by gluing the boundary of the Möbius band to $S^1 \times \{pt\}$ in $S^1 \times S^2$.