Math 6441 - Spring 2021
Homework 5

Read Chapter 3.1 and 3.2 in Hatcher's book.
Work all these problems and talk to me if you have any questions on them, but carefully write up and turn in **only** *problems 5, 6, 7, 8, 9, and 10.* **Due: April 2.**

1. Given a homomorphism $f : G \to H$ of abelian group show there is an induced homomorphism $f_* : H_n(X; G) \to H_n(X; H)$.
2. If $0 \to G_1 \xrightarrow{a} G_2 \xrightarrow{b} G_3 \to 0$ is a short exact sequence of abelian groups then for any topological space there is a long exact sequence
$$\ldots \to H_n(X; G_1) \xrightarrow{a_*} H_n(X; G_2) \xrightarrow{b_*} H_n(X; G_3) \xrightarrow{\beta} H_{n-1}(X; G_1) \ldots$$
The map β is called the Bockstein map.
3. Let X be a CW complex of dimension n with a finite number of cells. Let l_k be the number of k cells in X. Show
$$\sum_{i=0}^{n}(-1)^i l_k = \sum_{i=0}^{n}(-1)^i \operatorname{rank} H_i(X).$$
This number is called the Euler characteristic of X and denoted $\chi(X)$. Note the formula above implies that you can (1) compute $\chi(X)$ easily from the cell structure on X, but it is an invariant of the homotopy type of X.
4. If X is a CW complex that is the union of Y and Z where Y, Z, and $Y \cap Z$ are subcomplexes, then
$$\chi(X) = \chi(Y) + \chi(Z) - \chi(Y \cap Z).$$
5. Show that if $\widetilde{X}$ is a n fold covering space of the CW complex X then $\chi(\widetilde{X}) = n\chi(X)$. Hint: you need to say something about the CW structure on $\widetilde{X}$. For this problem you just need to describe the structure, but don't need to prove the structure is correct (though you should think through this).
6. Show that a surface of genus g can be a covering space of a surface of genus h if and only if $g = n(h-1)+1$ for some integer n. (Here all surfaces are assumed to be oriented.)
7. Compute the homology of $\mathbb{R}P^n$ with coefficients in $\mathbb{Z}$ and $\mathbb{Z}/2$.
8. Given a sequence of finitely generated (and hence presented) abelian groups
$$G_0 = \mathbb{Z}, G_1, \ldots, G_n$$
show there is a CW complex X such that $H_i(X) \cong G_i$. (If you are comfortable with infinite constructions, think about the situation when the G_i are not finitely generated, but you do not have to write this up for this problem.)
9. Let X be the space obtained from S^1 by attaching a 2-cell by a map of degree 2 and attaching another 2-cell by a map of degree 3. Use cellular homology to compute the homology of X.
10. Let X be the union of the unit sphere in $\mathbb{R}^3$ and the portion of the z-axis running from the south to the north pole of the sphere. Put a CW structure on X and use cellular homology to compute the homology of X.