Math 6441 - Spring 2020
Supplement 1: H and H'-spaces

A pointed space (Y, y_0) is called an *H-space* if there are maps

$$\mu : Y \times Y \to Y \text{ and } \nu : Y \to Y,$$

such that

- $\mu \circ i_1 \sim id_Y$ and $\mu \circ i_2 \sim id_Y$ where $i_1 : Y \to Y \times Y : y \to (y, y_0)$ and similarly for i_2,
- $\mu \circ (id_Y \times \mu) \sim \mu \circ (\mu \times id_Y)$ as maps from $Y \times Y \times Y$ to Y, and
- $\mu \circ (id_Y \times \nu)$ is homotopic to a constant map.

1. Show that $[X, Y]_0$ has a natural group structure for every pointed space X if and only if Y is an H-space. Here natural means that if $f : X \to X'$ is a continuous map then the induced map $[X', Y]_0 \to [X, Y]_0$ is a homomorphism.
   Hint: Given an H-space note that $[f], [g] \in [X, Y]_0$ then $f \times g : X \to Y \times Y$. Define multiplication by $[f] * [g] = [\mu \circ (f \times g)]$.
   For the other implication assume take $X = Y \times Y$ and let $p_i : Y \times Y \to Y$ be projection to the i^{th} factor. Now let μ be a representative of $[p_1] * [p_2]$ and ν a representative of $[id_Y]^{-1}$.
2. Let (Y, y_0) be a pointed space. The *loop space* $\Omega(T)$ of Y is the space of based continuous maps from (S^1, x_0) to (Y, y_0). Show $\Omega(Y)$ is an H-space.

Recall that if (X, x_0) and (Y, y_0) are pointed space then the wedge product $X \vee Y$ is the subset $(X \times \{y_0\}) \cup (\{x_0\} \times Y)$ in $X \times Y$, with base point (x_0, y_0).

A pointed space (Y, Y_0) is called an H'-space if there are mappings

$$\mu : Y \to Y \vee Y \text{ and } \nu : Y \to Y$$

such that

- $p_1 \circ \mu \sim id_Y$ and $p_2 \circ \mu \sim id_y$ where $p_i : Y \vee Y \to Y$ is projection to the i^{th} factor,
- $(id_Y \vee \mu) \circ \mu \sim (\mu \vee id_Y) \circ \mu$ as maps from Y to $Y \vee Y \vee Y$, and
- $(id_Y \vee \nu) \circ \mu$ is homotopic to the identity on Y

3. Show that $[Y, X]_0$ has a natural group structure for every pointed space X if and only if Y is an H'-space. Here natural means that if $f : X \to X'$ is a continuous map then the induced map $[Y, X]_0 \to [Y, X']_0$ is a homomorphism.

Let X be a topological space, the *suspension* of X is

$$\Sigma X = X \times [0, 1]/ \sim$$

where $\sim$ indicates that $X \times \{0\}$ is collapsed to a point and so $X \times \{1\}$ to another point. If (X, x_0) is a pointed space then the *reduced suspension* of X is $\Sigma X = X \times [0, 1]/ \sim$ where you collapse as before, but also collapse $\{x_0\} \times [0, 1]$. Notice that this means that $\sim$ in this case just collapses $(X \times \{0, 1\}) \cup (\{x_0\} \times [0, 1])$ to a point. Call this point the new base point of the suspension. For pointed spaces ΣX will always mean reduced suspension.

4. Show that the suspension of S^n is S^{n+1}. If we choose a base point of S^n show that it's reduced suspension is also S^{n+1}.
5. If (Y, y_0) is a pointed space then show that ΣY is an H'-space.

Notice that we now know that

$$\pi_n(X, x_0) = [S^n, X]_0$$

is a group for all $n \geq 1$!

6. If (X, x_0) is an H'-space and (Y, y_0) is an H-space, then show that the product structures on $[X, Y]_0$ coming form X as an H'-space and from Y as an H-space agree. Also show that the product is commutative.
   Hint: Let $+$ be the multiplication from the H'-space structure and $\cdot$ be the one from the H-space structure. Denote μ for the H'-space structure by $\mu_\vee$ and the μ for the H-space structure by μ_x. Note that $[f_1] \cdot [f_2] = [\mu_x \circ f_1 \times f_2]$ and $[f_1] + [f_2] = [\nabla \circ (f_1 \vee f_2) \circ \mu_\vee]$ where $\nabla : Y \vee Y \to Y$ send (y, y_0) and (y_0, y) to y. Let $\Delta : X \to X \times X : x \mapsto (x, x)$. Now given $f_1, f_2 \in [X, Y]_0$, show that $\nabla \circ (f_1 \wedge f_2) \circ \mu_\vee$ and $\mu_x \circ (f_1 \times f_2) \circ \Delta$ are homotopic. For commutativity note that if G is a group and $\rho : G \times G \to G : (g, h) \mapsto gh$ is a homomorphism then multiplication is commutative.

The *smash product* of pointed spaces X adn Y is

$$X \wedge Y = (X \times Y)/(X \vee Y).$$

7. For a pointed space X show that $\Sigma X = X \wedge S^1$.
8. Show that

$$[\Sigma X, Y]_0 = [X, \Omega Y]_0.$$

   Hint: Note that there are obvious map $\Psi : C^0(X|timesY, Z) \to C^0(X, X^0(Y, Z))$ defined by letting $\Psi(g)(x)$ be the function $y \mapsto g(x, y)$ is a homeomorphism if Y is locally compact (that is has an open neighborhood with compact closure) and Hausdorff and X is Hausdorff. And the inverse that sends $f : X \to C^0(Y, Z)$ to $\Psi^{-1}(f)(x, y) = f(x)(y)$.
   Not this shows that $[X \times Y, Z] = [X, C^0(Y, Z)]$.
   Now show that for based spaces $[X \wedge Y, Z]_0 = [X, C^0_{based}(Y, Z)]_0$.

Notice that this shows that

$$\pi_n(X, x_0) = [S^n, X]_0 = [\Sigma S^{n-1}, X]_0 = [S^{n-1}, \Omega X]_0.$$

So if $n \geq 2$ then S^{n-1} is an H'-space and ΩX is an H-space, so $\pi_n(X, x_0)$ is abelian.