II Homology Theory

There are many types of (ordinary) homology theories

singular, simplicial, cubical, cellular, ...

these all give the same results on CW-complexes

we will discuss the "most general" one singular homology

and then derive an easily computable one cellular homology

There are also generalized homologies, like bordism theory, K-theory ...

these are different from ordinary homology

we might discuss them briefly

Underlying these theories is homological algebra which is a purely

algebraic theory of "chain complexes"

such objects show up in many contexets and give even

more "homology theories", like Floer-homology

such theories are not really about algebraic topology

A Singular Homology

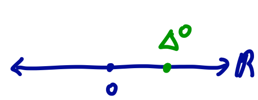
the standard p-simplex is

$$\Delta^p = \left\{ \sum_{i=0}^{p} t_i e_i \in \mathbb{R}^{p+1} : \sum_{i=0}^{p} t_i = 1,\ t_i \geq 0 \right\}$$

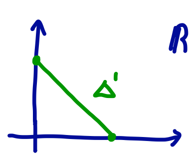
where $e_0 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, e_1 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots e_p = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$ are the standard basis vectors for $\mathbb{R}^{p+1}$

examples:

1)

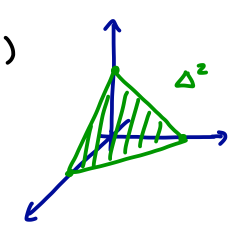


2)



$\mathbb{R}^2$

3)



given $v_0, \dots v_n \in \mathbb{R}^n$ denote by $[v_0, \dots, v_p]$ the map

$$\Delta^p \longrightarrow \mathbb{R}^n$$
$$(t_0, \dots t_p) \longmapsto \sum t_i v_i$$

example:

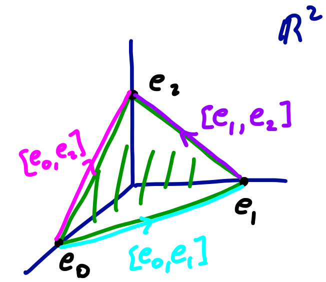
1) $[e_0, \dots, e_p] : \Delta^p \to \mathbb{R}^{p+1}$ parameteizes a copy of Δ^p

2) $[e_0, \dots, \hat{e}_i \dots, e_p] : \Delta^{p-1} \to \Delta^p$ parameterizes i^{th} face of Δ^p

means leave out i^{th} element

i.e. $(t_0, \dots t_{p-1}) \in \Delta^{p-1} \longmapsto \sum_{j<i} t_j e_j + \sum_{j \geq i} t_j e_{j+1}$

e.g. $p = 2$

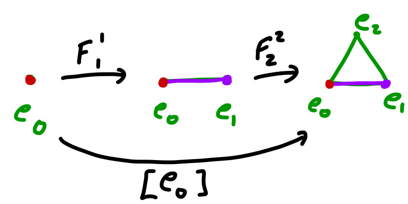


indicates dim. of simplex

we call this the i^{th} face map and also denote it $F_i^p = [e_0, \dots \hat{e}_i, \dots e_p]$

note: $i > j$ then $F_i^p \circ F_j^{p-1} = [e_0, \dots, \hat{e}_j, \dots, \hat{e}_i, \dots e_p]$

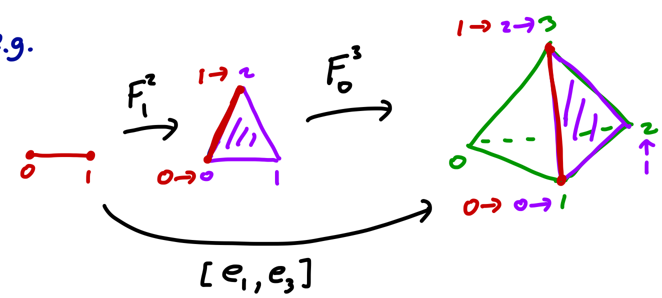
e.g.



$i = 2$
$j = 1$

$i \leq j$ then $F_i^p \circ F_j^{p-1} = [e_0, \dots \hat{e}_i \dots \hat{e}_{j+1} \dots e_p]$

e.g.



$i = 0$
$j = 1$

a singular p-simplex in a space X is a continuous map

$$\sigma : \Delta^p \longrightarrow X$$

the singular group of p-chains in X is

$$C_p(X) = \text{free abelian group generated by singular p-simplices}$$

that is an element of $C_p(X)$ is a finite formal sum

$$\sum_{i=1}^{\ell} n_i \sigma_i \quad \text{where } n_i \in \mathbb{Z} \text{ and } \sigma_i : \Delta^p \to X \quad \forall i$$

exercise: There is an obvious way to add two elements. Show this makes $C_p(X)$ an abelian group.

we call elements of $C_p(X)$ (singular) p-chains

given a singular p-simplex σ we say the i^{th} face of σ is

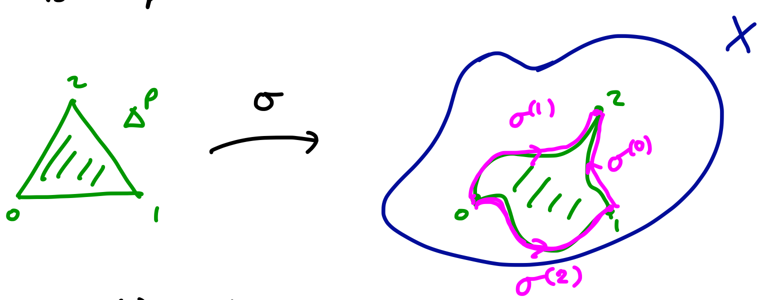
$$\sigma^{(i)} = \sigma \circ F_i^p$$

and the boundary of σ is

$$\partial \sigma = \sum_{i=0}^{p} (-1)^i \sigma^{(i)}$$

note: $\partial\sigma$ is a $(p-1)$-chain!

example:



$$\partial\sigma = \sigma^{(0)} - \sigma^{(1)} + \sigma^{(2)}$$

now define the boundary map

$$\partial_p : C_p(X) \to C_{p-1}(X)$$
$$\sum_{i=1}^{k} n_i \sigma_i \longmapsto \sum_{i=1}^{k} n_i (\partial \sigma_i)$$

lemma 1:

$$\boxed{\text{With notation as above} \quad \partial_{p-1} \circ \partial_p = 0}$$

subscript p is usually omitted so lemma then stated $\underline{\underline{\partial^2 = 0}}$

Proof: $\partial_{p-1} \circ \partial_p \sigma = \partial_{p-1}\left(\sum_{i=0}^{p} (-1)^i \sigma \circ F_i^p\right) = \sum_{i=0}^{p} (-1)^i \sum_{j=0}^{p-1} (-1)^j \sigma \circ F_i^p \circ F_j^{p-1}$

$$= \sum_{0 \le j < i \le p} (-1)^{i+j} \sigma \circ F_i^p \circ F_j^{p-1} + \sum_{0 \le i \le j \le p-1} (-1)^{i+j} \sigma \circ F_i^p \circ F_j^{p-1}$$

$$= \sum_{0\le j<i\le p} (-1)^{i+j}\, \sigma \circ [e_0 \ldots \hat{e}_j \ldots \hat{e}_i \ldots e_p] + \sum_{0\le i\le j\le p-1} (-1)^{i+j}\, \sigma \circ [e_0, \ldots \hat{e}_i \ldots \hat{e}_{j+1} \ldots e_n]$$

$\downarrow$ $k = j+1$

$$= \sum_{0\le j<i\le p} (-1)^{i+j}\, \sigma \circ [e_0 \ldots \hat{e}_j \ldots \hat{e}_i \ldots e_p] + \sum_{0\le i<k\le p} (-1)^{i+k-1}\, \sigma \circ [e_0 \ldots \hat{e}_i \ldots \hat{e}_k \ldots e_n]$$

note $\downarrow$ relable back to j

$$= \sum_{j<i} (-1)^{i+j}\, \sigma \circ [e_0 \ldots \hat{e}_j \ldots \hat{e}_i \ldots e_p] - \sum_{i<j} (-1)^{i+j}\, \sigma \circ [e_0 \ldots \hat{e}_i \ldots \hat{e}_j \ldots e_n]$$

$$= 0$$

note: lemma $\Rightarrow$ image $\partial_p \subset$ kernel ∂_{p-1}

we define the p^{th} homology group of X to be

$$H_p(X) = \ker \partial_p / \operatorname{im} \partial_{p+1}$$

an element of $\ker \partial_p$ is called a (singular) p-cycle

an element of $\operatorname{im} \partial_{p+1}$ is called a (singular) p-boundary

we say two chains $c_1, c_2 \in C_p(X)$ are homologous if $\exists$ some $d \in C_{p+1}(X)$ s.t. $c_1 - c_2 = \partial d$, we write $c_1 \sim c_2$

this is an equivalence relation if c is a p-cycle

let $[c]$ be its equivalence class

$H_p(X) = \{[c] \mid c \text{ a p-cycle}\}$

Remark: There are no singular p-simplicies for $p<0$ so $C_p(X) = \{0\}$

$\therefore$ $H_p(X) = 0 \quad \forall\, p<0$

lemma 2:

let X = one point space

then $H_p(X) = \begin{cases} \mathbb{Z} & p=0 \\ 0 & p\neq 0 \end{cases}$

Proof: for each $p \ge 0$ $\exists!$ map $\sigma_p : \Delta^p \to X$

$\therefore$ $C_p(X) = \mathbb{Z}$ generated by σ_p

now $\partial_p \sigma_p = \sum_{i=0}^{p} (-1)^i \sigma_p^{(i)} = \sum_{i=0}^{p} (-1)^i \sigma_{p-1}$

so
$$\partial_p \sigma_p = \begin{cases} \sigma_{p-1} & p \text{ even} \quad p>0 \\ 0 & p \text{ odd or } p=0 \end{cases}$$

$\therefore$ p odd
$$H_p(X) = \frac{\ker \partial_p}{\operatorname{im} \partial_{p+1}} = \frac{C_p(X)}{C_p(X)} = \{0\}$$

p even, $p>0$
$$H_p(X) = \frac{\ker \partial_p}{\operatorname{im} \partial_{p+1}} = \frac{\{0\}}{\{0\}} = \{0\}$$

$p=0$
$$H_0(X) = \frac{\ker \partial_0}{\operatorname{im} \partial_1} = \frac{C_0(X)}{\{0\}} = \frac{\mathbb{Z}}{\{0\}} \cong \mathbb{Z}$$

Thm 3:

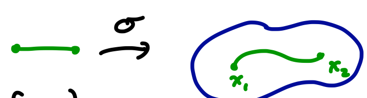
$$\boxed{\begin{array}{l} H_0(X) = \text{free abelian group generated by path components of } X \\ \cong \bigoplus_n \mathbb{Z} \quad \text{where } X \text{ has } n \text{ path components} \end{array}}$$

Proof: an element $c \in C_0(X)$ is $c = \sum_{i=1}^{k} n_i x_i$ for points $x_i \in X$, $n_i \in \mathbb{Z}$

define $\varepsilon : C_0(X) \to \mathbb{Z}$
$$\sum n_i x_i \mapsto \sum n_i$$

It is easy to check ε is a homomorphism (do it!)

If σ is a singular 1-simplex then



$$\partial \sigma = x_1 - x_2 \quad \text{(end points of } \sigma)$$

so $\varepsilon(\partial\sigma) = 0$

if $d \in C_1(X)$ then $d = \sum_{i=1}^{k} n_i \sigma_i$ so $\varepsilon(\partial d) = 0$

i.e. $\operatorname{im} \partial_1 \subset \ker \varepsilon$

so ε induces a homomorphism

$$\varepsilon_* : H_0(X) \to \mathbb{Z} \qquad \text{called an } \underline{\text{augmentation}} \text{ (so is } \varepsilon)$$
$$[c] \mapsto \varepsilon(c)$$

Claim: if X is path connected then ε_* is an isomorphism

Pf: clearly ε_* is surjective $(\varepsilon_*([x]) = 1)$

fix $x_0 \in X$, then for any $x \in X$ we can take $\lambda_x : [0,1] \to X$ s.t. $\lambda_x(0) = x_0$, $\lambda_x(1) = x$

so $\partial \lambda_x = x - x_0$

now given any $c = \sum n_i x_i$ such that $\varepsilon(c) = 0$

let λ_{x_i} be path for x_i

note $c - \partial \sum n_i \lambda_{x_i} = \sum n_i x_i - \sum n_i (x_i - x_0) = \sum n_i x_0 = (\sum n_i) x_0 = 0$

so $c = \partial \sum n_i \lambda_{x_i}$ and $[c] = 0$ in $H_0(X)$ ✓

exercise: If the path components of X are $X_\alpha, \alpha \in A$

then $C_p(X) = \bigoplus_{\alpha \in A} C_p(X_\alpha)$

and $H_p(X) = \bigoplus_{\alpha \in A} H_p(X_\alpha)$

note lemma now follows ∎

Remark: if we set $\tilde{\partial}_0 = \varepsilon$ and $\tilde{\partial}_i = \partial_i$ for $i \geq 1$ then the proof

shows $\tilde{\partial}_i \circ \tilde{\partial}_{i+1} = 0 \quad \forall i \geq 0$

so we can define $\tilde{H}_p(X) = \dfrac{\ker \tilde{\partial}_p}{\operatorname{im} \tilde{\partial}_{p+1}}$

Clearly $H_p(X) = \tilde{H}_p(X) \ \forall p \geq 1$ and

$H_0(X) = \tilde{H}_0(X) \oplus \mathbb{Z}$

$\tilde{H}_p(X)$ is called the reduced homology of X.

notice that if $\gamma : [0,1] \to X$ is a loop based at x_0

then γ is also a singular 1-simplex and

$\partial \gamma = 0$ so $[\gamma] \in H_1(X)$

this gives a map

$\phi : \pi_1(X, x_0) \to H_1(X)$ called the Hurewitz map

(we check it's well-defined below)

Thm 4:

If X path connected, then the Hurewitz map induces an isomorphism

$$\phi_* : (\pi_1(X, x_0))^{ab} \longrightarrow H_1(X)$$

where $(\pi_1(X, x_0))^{ab}$ is the abelianization of $\pi_1(X, x_0)$

the abelianization G^{ab} of a group G is the largest abelian quotient of G.

that is if A is any abelian group and $f: G \to A$ a homomorphism, then $\exists\, \tilde{f}: G^{ab} \to A$ s.t.

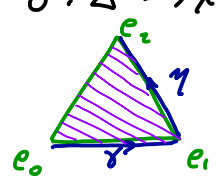
$$\begin{tikzcd} G \arrow[rr, "f"] \arrow[dr] & \circ & A \\ & G^{ab} \arrow[ur, "\tilde{f}"'] & \end{tikzcd}$$

<u>exercise:</u> $G^{ab} = G/[G,G]$ where $[G,G]$ is the smallest normal subgroup of G containing $\{[g,h] : g,h \in G\}$

<u>Proof:</u> we will denote equiv. classes in $\pi_1(X,x_0)$ by $[\gamma]$ and equiv. classes in $H_1(X)$ by $[\![\gamma]\!]$

<u>note:</u> 1) If γ, η paths in X with $\gamma(1) = \eta(0)$, then $\overbrace{\gamma * \eta - \gamma - \eta}^{\text{1-chain}}$ is a boundary

indeed, define $\sigma: \Delta^2 \to X$ by



constant on purple lines

so on $[e_0 e_2]$ have $\gamma * \eta$

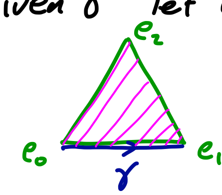
now $\partial\sigma = \gamma - \gamma*\eta + \eta$ ✓

2) If γ a path in X, then $\gamma + \bar{\gamma}$ is a boundary (and constant path a boundary)

indeed, if $\sigma: \Delta^2 \to X$ a constant map,

then $\partial\sigma = \underbrace{\sigma^{(0)} - \sigma^{(1)} + \sigma^{(2)}}_{\text{each a constant path } c} = c$

now given γ let $\sigma': \Delta^2 \to X$ be



constant on pink lines

so σ' on $[e_1, e_2]$ is $\bar{\gamma}$

on $[e_0, e_2]$ is some constant path c

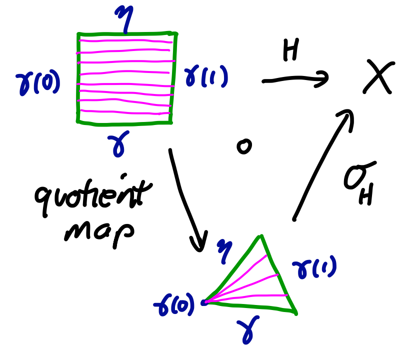
let σ be singular 2-simplex with $\partial\sigma = c$

so $\partial(\sigma' - \sigma) = \gamma + \bar{\gamma}$

3) if γ and η are homotopic rel end points then $\gamma - \eta$ is a boundary

indeed, let $H:[0,1]\times[0,1]\to X$ be the homotopy

then H induces a singular 2-simplex σ_H



now $\partial\sigma = \gamma - \eta +$ constant path

since a constant path is a boundary so is $\gamma - \eta$ ✓

now: 3) $\Rightarrow$ ϕ is well defined

1) $\Rightarrow$ ϕ a homomorphism:

$$\phi([\gamma]\cdot[\eta]) = \phi([\gamma * \eta]) = [\![\gamma * \eta]\!] \overset{(1)}{=} [\![\gamma]\!] + [\![\eta]\!] = \phi([\gamma]) + \phi([\eta])$$

since $H_1(X)$ abelian we get

$$\phi_*: (\pi_1(X,x_0))^{ab} \longrightarrow H_1(X)$$

we construct an inverse for ϕ_*

for each point $x \in X$ let γ_x be a path x_0 to x

given a singular 1-simplex σ let

$$\hat{\sigma} = \gamma_{\sigma(0)} * \sigma * \overline{\gamma}_{\sigma(1)}$$ (choose γ_{x_0} just to be constant path e_{x_0})

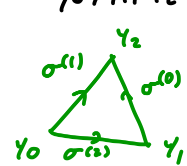
this is a loop in X based at x_0

now define $\psi(\sigma) = [\hat{\sigma}]$ since $(\pi_1(X,x_0))^{ab}$ is abelian and $C_1(X)$ a free abelian group this defines

$$\psi: C_1(X) \to (\pi_1(X,x_0))^{ab}$$

note $\psi \circ \phi_*([\gamma]) = [\underbrace{e_{x_0}}_{\text{constant path}} * \gamma * \overline{e}_{x_0}] = [\gamma]$

if σ is a 2-simplex then let y_0, y_1, y_2 be vertecies



$$\psi(\partial_2\sigma) = \psi(\sigma^{(0)} - \sigma^{(1)} + \sigma^{(2)}) = \psi(\sigma^{(2)})\,\psi(\sigma^{(0)})\,\psi(\sigma^{(1)})^{-1}$$
$$= [\gamma_{y_0} * \sigma^{(2)} * \bar{\gamma}_{y_1} * \gamma_{y_1} * \sigma^{(0)} * \bar{\gamma}_{y_2} * \gamma_{y_2} * \overline{\sigma^{(1)}} * \bar{\gamma}_{y_0}]$$
$$= [\gamma_{y_0} * \sigma^{(2)} * \sigma^{(0)} * \overline{\sigma^{(1)}} * \bar{\gamma}_{y_0}] = [e_{x_0}]$$

since loop bounds disk σ!

So $\operatorname{im}\partial_2 \subset \ker\psi$ $\therefore$ ψ induces a map
$$\psi_* : H_1(X) \to (\pi_1(X,x_0))^{ab}$$

from above we clearly have $\psi_* \circ \phi_* = id$

now if $\llbracket c \rrbracket \in H_1(X)$ with $c = \sum_i n_i \sigma_i$

concatenate n_i copies

$$\phi_* \circ \psi_*(\llbracket c \rrbracket) = \phi_*\Big(\Big[\mathop{*}_i \big(\gamma_{\sigma_i(0)} * \sigma_i * \bar{\gamma}_{\sigma_i(1)}\big)^{n_i}\Big]\Big)$$

concatenate

$$= \sum n_i \llbracket \gamma_{\sigma_i(0)} * \sigma_i * \bar{\gamma}_{\sigma_i(1)} \rrbracket$$
$$= \sum n_i \big(\llbracket \gamma_{\sigma_i(0)} \rrbracket + \llbracket \sigma_i \rrbracket - \llbracket \gamma_{\sigma_i(1)} \rrbracket\big)$$
$$= \sum n_i \llbracket \sigma_i \rrbracket = \llbracket \textstyle\sum n_i \sigma_i \rrbracket = \llbracket c \rrbracket$$

since $\partial c = 0$ for each $\sigma_i(0)$, $\exists$ a j st. $\sigma_j(1) = \sigma_i(0)$

so $\phi_* \circ \psi_* = id$

Remark: For any n can similarly define a map
$$\phi_n : \pi_n(X,x_0) \to H_n(X)$$
and can show for the first k for which $H_k(X) \neq 0$
ϕ_k is an isomorphism if $k > 1$

B Intro to homological algebra and maps on homology

a sequence of abelian groups C_* and maps
denotes general index
$$\partial_n : C_n \to C_{n-1}$$
is called a chain complex if $\partial_{n-1} \circ \partial_n = 0$ for all n

the homology of the complex is $H_n(C_*, \partial) = \ker\partial_n / \operatorname{im}\partial_{n+1}$

Homological algebra is the study of general chain complexes

when a definition or theorem is purely about homologic algebra I'll denote it by (HA) such results are true for <u>any</u> chain complex

in particular, they will apply to the singular chain groups.

<u>def^{n} (HA)</u>: given two chain complexes (C_*, ∂) and (C_*', ∂') a <u>chain map</u> is a sequence of homomorphisms $f_n : C_n \to C_n'$ such that

$$\partial_n' \circ f_n = f_{n-1} \circ \partial_n$$

$$\begin{tikzcd} \cdots \arrow[r] & C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] \arrow[dr, phantom, "\circ"] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\ \cdots \arrow[r] & C_n' \arrow[r, "\partial_n'"'] & C_{n-1}' \arrow[r] & \cdots \end{tikzcd}$$

<u>lemma 5 (HA)</u>:

> A chain map $\{f_n\} : (C_*, \partial) \to (C_*', \partial')$ induces homomorphisms
> $$(f_n)_* : H_n(C_*, \partial) \to H_n(C_*', \partial')$$

<u>Proof</u>: note if $[h] \in H_n(C_*, \partial)$, then $\partial_n h = 0$

so $\partial_n'(f_n(h)) = f_{n-1}(\partial_n h) = f_{n-1}(0) = 0$

so $f_n(h)$ gives a class in $H_n(C_*', \partial')$

define $(h_n)_* : H_n(C_*, \partial) \to H_n(C_*', \partial')$

$$[h] \longmapsto [f_n(h)]$$

note, $(h_n)_*$ is well-defined since if $[h] = [h']$ then $\exists k$ st.

$$h - h' = \partial_{n+1} k$$

so we have

$$f_n(h) = f_n(h' + \partial_{n+1} k) = f_n(h') + \partial_{n+1}'(f_{n+1} k)$$

so $[f_n(h)] = [f_n(h')]$

$(f_n)_*$ is a homeomorphism since f_n is (exercise)

Now for singular homology

if $f : X \to Y$ is a continuous map, then define

$$f_n : C_n(X) \to C_n(Y) : \sum m_i \sigma_i \longmapsto \sum m_i (f \circ \sigma_i)$$

this is clearly a homomorphism and

$$f_{n-1}(\partial_n \sigma) = f_{n-1}\left(\sum_{i=0}^{n} (-1)^i \sigma^{(i)}\right) = \sum (-1)^i (f \circ \sigma^{(i)})$$
$$= \sum (-1)^i (f \circ \sigma)^{(i)}$$
$$= \partial_n (f_n(\sigma))$$

so $f_{n-1} \circ \partial_n = \partial_n \circ f_n$

$\therefore$ by lemma we get

$$(f_n)_*: H_n(X) \to H_n(Y)$$

<u>exercise:</u> 1) $(f_n \circ g_n)_* = (f_n)_* \circ (g_n)_*$

2) $(id_X)_n = id_{H_n(X)}$

<u>def^n (HA):</u>

given chain complexes (C_*, ∂) and (C'_*, ∂') and chain maps $\{f_n\}$ and $\{g_n\}$ from (C_*, ∂) to (C'_*, ∂')

a <u>chain homotopy</u> between $\{f_n\}$ and $\{g_n\}$ is a sequence of homeomorphisms

$$P_n: C_n \to C'_{n+1}$$

such that

$$\partial'_{n+1} \circ P_n + P_{n-1} \circ \partial_n = f_n - g_n$$

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, shift right, "f_{n+1}"'] \arrow[d, shift left, "g_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[dl, "P_n"'] \arrow[d, shift right, "f_n"'] \arrow[d, shift left, "g_n"] & C_{n-1} \arrow[r] \arrow[dl, "P_{n-1}"'] \arrow[d, shift right, "f_{n-1}"'] \arrow[d, shift left, "g_{n-1}"] & \cdots \\
\cdots \arrow[r] & C'_{n+1} \arrow[r, "\partial'_{n+1}"'] & C'_n \arrow[r, "\partial'_n"'] & C'_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

<u>lemma 6 (HA):</u>

If there is a chain homotopy between $\{f_n\}$ and $\{g_n\}$

then $(f_n)_* = (g_n)_* : H_n(C_*, \partial) \to H_n(C'_*, \partial)$

Proof:

If $h \in C_n$ and $\partial_n h = 0$, then $f_n(h) - g_n(h) = \partial'_{n+1} \circ P_n(h) + P_{n-1}(\partial_n h)$
$$= \partial'_{n+1}(P_n(h))$$

so $[f_n(h)] = [g_n(h)]$ ▦

Th$^{\underline{m}}$ 7:

> let $f, g : X \to Y$ be homotopic maps
> Then $\{f_n\}$ and $\{g_n\}$ are chain homotopic maps $(C_*(X), \partial)$ to $(C_*(Y), \partial)$
> and thus $(f_n)_* = (g_n)_* : H_n(X) \to H_n(Y)$

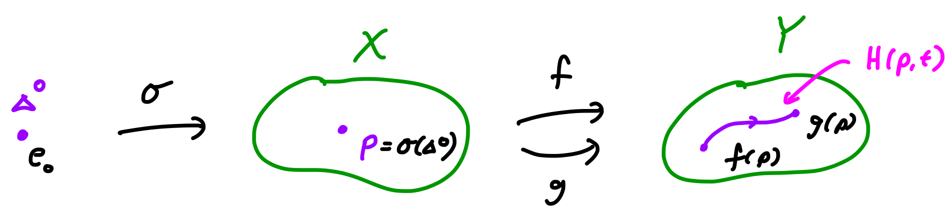
Proof: let $H : X \times [0,1] \to Y$ be the homotopy

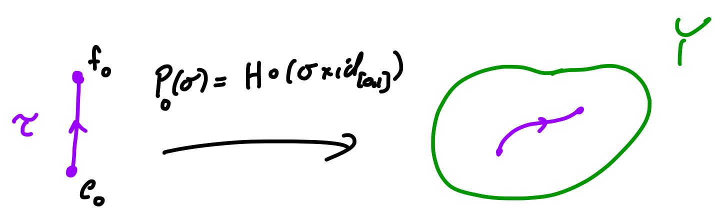
so $H(x,0) = f(x)$ and $H(x,1) = g(x)$

given a simplex $\sigma : \Delta^n \to X$
we define $P(\sigma)$ by $H \circ (\sigma \times id_{[0,1]}) : \Delta^n \times [0,1] \to Y$

to make sense of this we need to see $\Delta^n \times [0,1]$
as a union on $(n+1)$-simplicies

e.g. $\underline{n=0}$

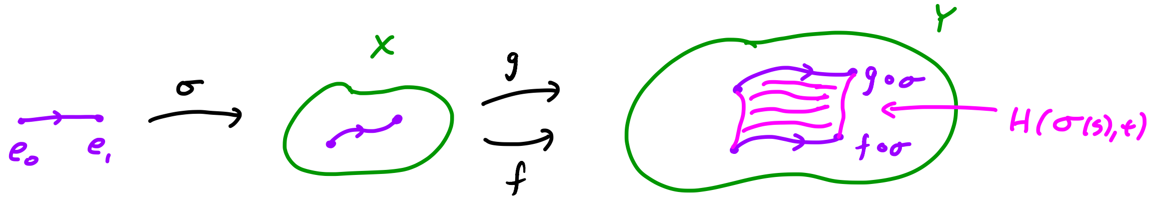


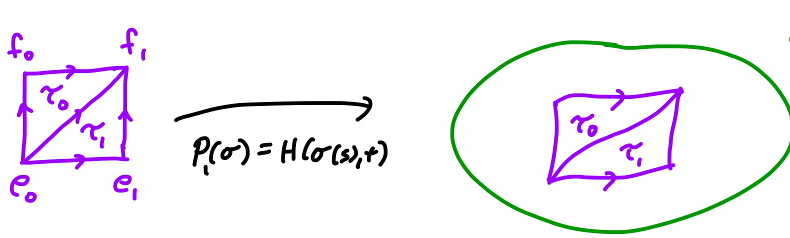


$$P_0(\sigma) = H \circ (\sigma \times id_{[0,1]}) = H(\sigma(e_0), t)$$
$$\partial P_0(\sigma) = \partial(H(\sigma(e_0), t)) = H(\sigma(e_0), 1) - H(\sigma(e_0), 0)$$
$$= g \circ \sigma - f \circ \sigma$$

(let $P_{-1} = 0$)

$\underline{n=1}$





$\tau_1 = [e_0, e_1, f_1]$

$\tau_0 = [e_0, f_0, f_1]$

define $P_1(\sigma) = \overbrace{H\circ(\sigma\times id_{[0,1]})}^{H_\sigma}\big|_{\tau_0} - H\circ(\sigma\times id)\big|_{\tau_1}$

$$\partial(P_1(\sigma)) = H_\sigma|_{[f_0 f_1]} - \cancel{H_\sigma|_{[e_0, f_1]}} + H_\sigma|_{[e_0 f_0]} - H_\sigma|_{[e_1 f_1]} + \cancel{H_\sigma|_{[e_0 f_1]}} - H_\sigma|_{[e_0 e_1]}$$

$$= g\circ\sigma + \underbrace{H_\sigma|_{[e_0 f_0]} - H_\sigma|_{[f_0 f_1]}} - f\circ\sigma$$

$$= g\circ\sigma - f\circ\sigma - P_0(\partial\sigma)$$

in general let $e_0 \dots e_n$ be the vertices of Δ^n in $\mathbb{R}^{n+1}$

think of $\mathbb{R}^{n+1} \subset \mathbb{R}^{n+2}$ as $x_{n+2} = 0$

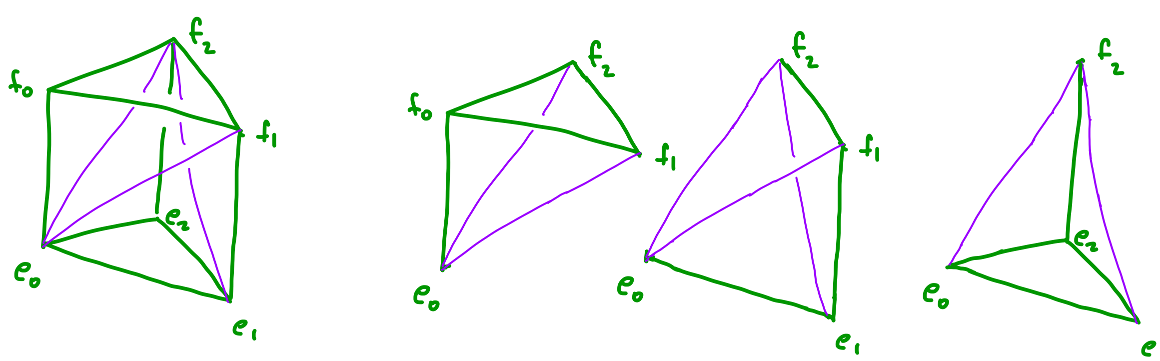
let f_i be the points in $\mathbb{R}^{n+2}$ above e_i with x_{n+2} coord $= 1$

so $[e_0 \dots e_n]$ describes $\Delta^n \times \{0\}$ and

$[f_0 \dots f_n]$ " " $\Delta^n \times \{1\}$

and $\Delta^n \times [0,1]$ is the union of the $(n+1)$-simplicies

$[e_0, \dots e_i, f_i, \dots f_n]$



now define

$$P_n : C_n(X) \to C_{n+1}(Y) \text{ by}$$

$$P_n(\sigma) = \sum_{i=0}^{n} (-1)^i \, H \circ (\sigma \times id_{[0,1]})\Big|_{[e_0 \dots e_i f_i \dots f_n]}$$

$$\partial P_n(\sigma) = \sum_{j \le i} (-1)^i (-1)^j \, H \circ (\sigma \times id_{[0,1]})\Big|_{[e_0 \dots \hat{e}_j \dots e_i f_i \dots f_n]}$$

$$+ \sum_{i \le j} (-1)^i (-1)^{j+1} H \circ (\sigma \times id_{[0,1]})\Big|_{[e_0 \dots e_i f_i \dots \hat{f}_j \dots f_n]}$$

the $i=j$ terms cancel except for

$[\hat{e}_0 f_0 \dots f_n] = [f_0 \dots f_n]$ term which is $g \circ \sigma$

and

$[e_0 \dots e_n \hat{f}_n] = [e_0 \dots e_n]$ term which is $-f \circ \sigma$

<u>exercise</u>: the $i \neq j$ terms give $-P_{n-1}(\partial \sigma)$

<u>Cor 8</u>: If $f : X \to Y$ is a homotopy equivalence then $f_* : H_n(X) \to H_n(Y)$ is an isomorphism for all n.

<u>Remark</u>: If X is a contractible space, then

$$H_n(X) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n \neq 0 \end{cases}$$

C. Relative Homology and Excision

let A be a subspace of X

so $C_n(A) \subset C_n(X)$

note $C_n(A)$ is in the kernel of

$$\begin{tikzcd} C_n(X) \arrow[r, "\partial_n"] \arrow[rr, bend right] & C_{n-1}(X) \arrow[r, "\text{quotient}"] & C_{n-1}(X)/C_{n-1}(A) \end{tikzcd}$$

so ∂_n induces a map

$$\partial_n : C_n(X)/C_n(A) \to C_{n-1}(X)/C_{n-1}(A)$$

define $C_n(X,A) = C_n(X)/C_n(A)$ and

$\partial_n : C_n(X,A) \to C_{n-1}(X,A)$ as above

clearly $\partial_{n-1} \circ \partial_n = 0$

the relative (singular) homology of (X,A) is

$$H_n(X,A) = \ker \partial_n / \operatorname{im} \partial_{n+1}$$

note: 1) an element in $H_n(X,A)$ is represented by a relative cycle

i.e. $\alpha \in C_n(X)$ s.t. $\partial_n \alpha \in C_{n-1}(A)$

2) a relative cycle α is trivial in $H_n(X,A)$ if it is a relative boundary

i.e. $\exists\, \beta \in C_{n+1}(X)$ and $\gamma \in C_n(A)$ s.t.

$$\alpha = \partial\beta + \gamma$$

Recall a sequence of homomorphisms

$$A \xrightarrow{\phi} B \xrightarrow{\psi} C$$

is called exact at B if $\operatorname{im}\phi = \ker\psi$

a longer sequence is exact if it is exact at every group

lemma 9 (HA):

If (A_*, ∂_A), (B_*, ∂_B), and (C_*, ∂_C) are 3 chain complexes and $\{\phi_n\} : (A_*, \partial_A) \to (B_*, \partial_B)$ and $\{\psi_n\} : (B_*, \partial_B) \to (C_*, \partial_C)$ are chain maps such that

$$0 \to A_n \xrightarrow{\phi_n} B_n \xrightarrow{\psi_n} C_n \to 0$$

is exact $\forall n$

then there is a long exact sequence

$$\dots \to H_n(A_*, \partial_A) \xrightarrow{(\phi_n)_*} H_n(B_*, \partial_B) \xrightarrow{(\psi_n)_*} H_n(C_*, \partial_C) \xrightarrow{\partial_n} H_{n-1}(A_n, \partial_A) \to \dots$$

some times this is written as an exact triangle

$$\begin{tikzcd} H_*(A,\partial_A) \arrow[rr, "\phi_*"] & & H_*(B,\partial_B) \arrow[dl, "\psi_*"] \\ & H_*(C,\partial_C) \arrow[ul, "\partial_*"] & \end{tikzcd}$$

ϕ_*, ψ_* preserve degree

∂_* reduces degree by 1

before proving this we note an important consequence

Thm 10:

let A be a subspace of X

1) let $i : A \to X$ be the inclusion map and

$q_n : C_n(X) \to C_n(X,A)$ the quotient map

Then

$$0 \to C_n(A) \xrightarrow{i_n} C_n(X) \xrightarrow{q_n} C_n(X,A) \to 0$$

is exact

2) So $\exists$ a long exact sequence

$$\dots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{q_*} H_n(X,A) \xrightarrow{\partial_*} H_{n-1}(A) \to \dots$$

called <u>long exact sequence of a pair.</u>

<u>Proof</u>: 2) follows from 1) and lemma 9

1) clearly $i_n : C_n(A) \to C_n(X)$ is injective

and $q_n : C_n(X) \to C_n(X,A)$ is surjective

and $\ker q_n = \operatorname{im} i_n$ $\blacksquare$

<u>exercise</u>: verify that ∂_* in the exact sequence is the map

$h \in H_n(X,A)$, choose $\alpha \in C_n(X)$ s.t. $\partial_n \alpha \in C_{n-1}(A)$ and $\alpha \in h$

then $\partial_* h = [\partial_n \alpha] \in H_{n-1}(A)$

<u>Proof of lemma 9</u>: consider

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\dots \arrow[r] & A_n \arrow[r, "\partial_A"] \arrow[d, "\phi_n"] & A_{n-1} \arrow[r, "\partial_A"] \arrow[d, "\phi_{n-1}"] & A_{n-2} \arrow[r] \arrow[d, "\phi_{n-2}"] & \dots \\
\dots \arrow[r] & B_n \arrow[r, "\partial_B"] \arrow[d, "\psi_n"] & B_{n-1} \arrow[r, "\partial_B"] \arrow[d, "\psi_{n-1}"] & B_{n-2} \arrow[r] \arrow[d, "\psi_{n-2}"] & \dots \\
\dots \arrow[r] & C_n \arrow[r, "\partial_B"] \arrow[d] & C_{n-1} \arrow[r, "\partial_C"] \arrow[d] & C_{n-2} \arrow[r] \arrow[d] & \dots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

given $c \in C_n$ with $\partial_C c = 0$

$\exists\, b \in B_n$ s.t. $\psi_n(b) = c$

<u>note</u>: $\psi_{n-1}(\partial_B b) = \partial_C(\psi_n(b))$
$= \partial_C(c) = 0$

so $\partial_B b \in \ker \psi_{n-1} = \operatorname{im} \phi_{n-1}$

and $\exists\, a \in A_{n-1}$ s.t. $\phi_{n-1}(a) = \partial_B b$

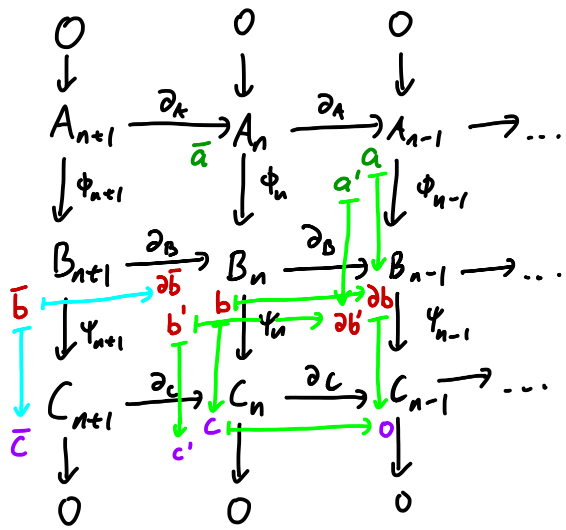
<u>note</u>: $\phi_{n-2}(\partial_A a) = \partial_B(\phi_{n-1}(a)) = \partial_B^2 b = 0$

since ϕ_{n-2} injective $\partial_A a = 0$

define: $\partial : H_n(C_*, \partial_C) \to H_{n-1}(A_*, \partial_A)$

$[c] \longmapsto [a]$ where a is constructed above

Claim: $\partial[c]$ is well-defined



note: only 2 choices: ① $c \in [c]$ and ② b s.t. $\psi_n(b) = c$

now given $c, c' \in [c]$

$\exists\ \bar{c} \in C_{n+1}$ s.t. $\partial_C \bar{c} = c - c'$

choose any $b, b' \in B_n$ and $\bar{b} \in B_{n+1}$ s.t.

$\psi_n(b) = c$, $\psi_n(b') = c'$, and $\psi_{n+1}(\bar{b}) = \bar{c}$

so $\psi_n(\partial_B \bar{b} - b + b') = \partial_C \psi_{n+1}(\bar{b}) - c + c' = 0$

and hence $\exists\ \bar{a} \in A_n$ s.t. $\phi_n(\bar{a}) = \partial_B \bar{b} - b + b'$

as above $\exists!\ a, a' \in A_{n-1}$ s.t. $\phi_{n-1}(a) = \partial_B b$ and $\phi_{n-1}(a') = \partial_B b'$

now $\phi_{n-1}(\partial_A \bar{a} + a - a') = \partial_B \phi_n(\bar{a}) + \partial_B b - \partial_B b'$

$= \partial_B(\partial_B \bar{b} - b + b') + \partial_B b - \partial_B b' = 0$

ϕ_{n-1} injective $\Rightarrow$ $a' = a + \partial_A \bar{a}$

and hence $[a] = [a']$ i.e. $\partial[c]$ well-defined ✓

exercise:
1) Show ∂ a homomorphism
2) $\mathrm{im}(\phi_n)_* = \ker(\psi_n)_*$
3) $\mathrm{im}(\psi_n)_* = \ker \partial_n$
4) $\mathrm{im}\, \partial_n = \ker(\phi_{n-1})_*$

example: $H_n(X, x_0) = \tilde{H}_n(X)$ ← reduced homology

indeed $H_n(x_0) \to H_n(X) \to H_n(X, x_0) \to H_{n-1}(x_0)$

for $n \geq 2$: $H_n(x_0) = 0$, $H_{n-1}(x_0) = 0$

so $H_n(X) \to H_n(X, x_0)$ is injective and surjective

$\therefore H_n(X) = H_n(X, x_0)$

and
$$\begin{tikzcd}[column sep=small] H_1(x_0) \arrow[r] & H_1(X) \arrow[r] & H_1(X,x_0) \arrow[r] & H_0(x_0) \arrow[r] & H_0(X) \arrow[r] & H_0(X,x_0) \arrow[r] & 0 \\ 0 \arrow[r] & H_1(X) \arrow[r] & H_1(X,x_0) \arrow[r] & \mathbb{Z} \arrow[r] & \bigoplus_k \mathbb{Z} \arrow[r] & H_0(X,x_0) \arrow[r] & 0 \end{tikzcd}$$

\# path components

isomorphism onto $\mathbb{Z}$ given by path component of X containing x_0

$\therefore$ 0 map

so $H_1(X,x_0) = H_1(X)$ and

$H_0(X,x_0) = \bigoplus_{k-1} \mathbb{Z}$

exercises:

1) If $f:(X,A) \to (Y,B)$ is continuous then
$$f_*: H_n(X,A) \to H_n(Y,B)$$

2) if $f,g:(X,A) \to (Y,B)$ are homotopic through maps taking A to B then $f_* = g_*$

3) $A \subset B \subset X$ then you get a long exact sequence
$$\ldots \to H_n(B,A) \to H_n(X,A) \to H_n(X,B) \to H_{n-1}(B,A) \to \ldots$$

Th$^{\underline{m}}$ 11 (Excision):

let $Z \subset A$ be subspaces of X
assume $\bar{Z} \subset \operatorname{int} A$
Then the inclusion map
$$\imath:(X-Z, A-Z) \to (X,A)$$
induces an isomorphism on homology
$$\imath_*: H_n(X-Z, A-Z) \to H_n(X,A)$$

We give the proof later

a pair $A \subset X$ is called good if A is non-empty, closed, and has a neighborhood U in X s.t. A is a deformation retract of U (i.e. $\exists\, H: U \times [0,1] \to U$ s.t. $H(x,0)=x$, $H(x,1) \in A$ $\forall x \in U$, and $H(x,t)=x$ $\forall x \in A$.)

examples: if A is a submanifold of a manifold X, then (X,A) a good pair

if X is built from A by attaching cells, then (X,A) is a good pair

Th^m 12:

$\boxed{\begin{array}{l}\text{If } (X,A) \text{ is a good pair, then the quotient map} \\ \qquad q:(X,A)\to(X/A,A/A) \\ \text{induces an isomorphism} \\ \qquad q_*: H_n(X,A)\to H_n(X/A,A/A)\cong H_n(X/A,pt)\cong \tilde{H}_n(X/A)\end{array}}$

Proof: let U be a neighborhood of A that def retracts to A

we have

$$\begin{tikzcd}
H_n(X,A) \arrow[r] \arrow[d, "q_*"] & H_n(X,U) \arrow[d, "q_*"] & H_n(X-A,U-A) \arrow[l, "\cong" ', "\text{by excision (Th}^m\text{11)}"] \arrow[d, "q_*"] \\
H_n(X/A,A/A) \arrow[r] & H_n(X/A,U/A) & H_n(X/A-A/A,U/A-A/A) \arrow[l, "\cong"']
\end{tikzcd}$$

<u>note:</u> $q:(X-A,U-A)\to(X/A-A/A,U/A-A/A)$ a homeomorphism! (the "identity map")

so right most q_* an isomorphism

so middle q_* an isomorphism by excision

now we have $(A,A)\xrightarrow{i}(U,A)\xrightarrow{h_1}(A,A)$ where $h_1(x)=H(x,1)$ and H is def. retraction U to A

and $h_1\circ i=id_{(A,A)}$

$i\circ h_1\simeq id_{(U,A)}$ by h_t

so $i_*:H_n(A,A)\to H_n(U,A)$ an isomorphism

thus the long exact sequence of $A\subset U\subset X$ gives

$$\begin{array}{ccccccc}
H_n(U,A) & \to & H_n(X,A) & \to & H_n(X,U) & \to & H_{n-1}(U,A) \\
\cong & & & & & & \cong \\
H_n(A,A) & & & & & & H_{n-1}(A,A) \\
\parallel & & & & & & \parallel \\
0 & & & & & & 0
\end{array}$$

and $H_n(X,A)\to H_n(X,U)$ an isomorphism.

we also know U/A deformation retracts to A/A (H induces def retract)

so the same argument shows

$$H_n(X/A,A/A)\to H_n(X/A,U/A) \text{ is an isomorphism}$$

$\therefore$ left most q_* above is an isomorphism as claimed ▦

Prop 13:

$$\boxed{\begin{array}{l} H_k(S^n) = \begin{cases} \mathbb{Z} & k = 0, n \\ 0 & k \neq 0, n \end{cases} \\ \text{and} \\ H_k(D^n, \partial D^n) = \begin{cases} \mathbb{Z} & k = n \\ 0 & k \neq n \end{cases} \end{array}}$$

Proof: note

$$H_k(S^0) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & k = 0 \\ 0 & k \neq 0 \end{cases}$$

and

$$H_k(D^n) = \begin{cases} \mathbb{Z} & k = 0 \\ 0 & k \neq 0 \end{cases}$$

n=1 case: Long exact sequence of (D^1, S^0) (note $D^1/S^0 \cong S^1$)

$$\begin{tikzcd} H_k(S^0) \arrow[r] \arrow[d, equal] & H_k(D^1) \arrow[r] \arrow[d, equal] & H_k(D^1, S^0) \arrow[r] & H_{k-1}(S^0) \arrow[d, equal] \\ 0 & 0 & & 0 \end{tikzcd} \qquad k \geq 2$$

so $H_k(S^1) \cong H_k(D^1/S^0) \cong H_k(D^1, S^0) = 0$ for $k \geq 2$

$$\begin{tikzcd} H_1(S^0) \arrow[r] \arrow[d, equal] & H_1(D^1) \arrow[r] \arrow[d, equal] & H_1(D^1, S^0) \arrow[r] & H_0(S^0) \arrow[r] \arrow[d, "\cong"] & H_0(D^1) \arrow[r] \arrow[d, "\cong"] & H_0(D^1, S^0) \arrow[d, "\cong"] \\ 0 & 0 & & \mathbb{Z} \oplus \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[dr] & \tilde{H}_0(S^1) \arrow[d, equal] \\ & & & & & 0 \end{tikzcd}$$

$$\Rightarrow H_1(S^1) \cong H_1(D^1, S^0) \cong \mathbb{Z}$$

Induction: assume Prop is true for S^{n-1} $(n \geq 1)$

Consider (D^n, S^{n-1}):

$$\begin{tikzcd} H_n(D^n) \arrow[r] \arrow[d, equal] & H_n(D^n, S^{n-1}) \arrow[r] & H_{n-1}(S^{n-1}) \arrow[r] \arrow[d, "\cong"] & H_{n-1}(D^n) \arrow[d, equal] \\ 0 & & \mathbb{Z} & 0 \end{tikzcd}$$

so $H_n(S^n) \cong H_n(D^n, S^{n-1}) \cong \mathbb{Z}$

for $k \neq n, 0, 1$

$$\begin{tikzcd} H_k(D^n) \arrow[r] \arrow[d, equal] & H_k(D^n, S^{n-1}) \arrow[r] & H_{k-1}(S^{n-1}) \arrow[r] \arrow[d, equal] & H_{k-1}(D^n) \arrow[d, equal] \\ 0 & & 0 & 0 \end{tikzcd}$$

so $H_k(S^n) \cong H_k(D^n, S^{n-1}) = 0$ $\quad k \neq n, 1.$

now

$$\begin{tikzcd} H_1(D^n) \arrow[r] \arrow[d, equal] & H_1(D^n, S^{n-1}) \arrow[r] & H_0(S^{n-1}) \arrow[r] \arrow[d, equal] & H_0(D^n) \arrow[r] \arrow[d, "\cong"] & H_0(D^n, S^{n-1}) \arrow[d, equal] \\ 0 & & \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$

so $H_1(S^n) = H_1(D^n, S^{n-1}) = 0$

Cor 14:

$\boxed{\begin{array}{l}\partial D^n \text{ is not a retract of } D^n \text{ and} \\ \text{any map } f: D^n \to D^n \text{ has a fixed point.}\end{array}}$

Proof:

If $r: D^n \to \partial D^n$ is a retraction then $r \circ \imath = id_{\partial D^n}$ where $\imath: \partial D^n \to D^n$ is inclusion

thus $r_* \circ \imath_* : \underset{\mathbb{Z}}{\underset{\cong}{H_{n-1}(\partial D^n)}} \to \underset{\mathbb{Z}}{\underset{\cong}{H_{n-1}(\partial D^n)}}$ is an isomorphism

but $r_*: H_{n-1}(D^n) \to H_{n-1}(\partial D^n)$ the trivial map! since $H_{n-1}(D^n) = 0$ ※

$\therefore$ r does not exist!

the second statement follows from the first as in proof of Cor I.14 ▦

Cor 15:

$\boxed{\begin{array}{l}\text{If } U \subset \mathbb{R}^n \text{ and } V \subset \mathbb{R}^m \text{ are open sets that are homeomorphic} \\ \quad \text{then } n = m\end{array}}$

Remark: this is called "invariance of domain" and implies that any n manifold is not homeomorphic to an m manifold for $n \neq m$

Proof: for any $x \in U$, we have $\mathbb{R}^n - U \subset \mathbb{R}^n - \{x\} \subset \mathbb{R}^n$, so excision says

$$H_k(\mathbb{R}^n, \mathbb{R}^n - \{x\}) \cong H_k(\mathbb{R}^n - (\mathbb{R}^n - U), (\mathbb{R}^n - \{x\}) - (\mathbb{R}^n - U)) = H_k(U, U - \{x\})$$

the exact sequence for $(\mathbb{R}^n, \mathbb{R}^n - \{x\})$ says

$$\underset{0}{\underset{\parallel}{H_k(\mathbb{R}^n)}} \to H_k(\mathbb{R}^n, \mathbb{R}^n - \{x\}) \to H_{k-1}(\mathbb{R}^n - \{x\}) \to \underset{0}{\underset{\parallel}{H_{k-1}(\mathbb{R}^n)}}$$

so $H_k(\mathbb{R}^n, \mathbb{R}^n - \{x\}) \cong H_{k-1}(\mathbb{R}^n - \{x\}) \cong H_{k-1}(S^{n-1})$

$\mathbb{R}^n - \{x\} \simeq S^{n-1}$

and $H_k(U, U - \{x\}) \cong \begin{cases} \mathbb{Z} & k = n \\ 0 & k \neq n \end{cases}$

similarly $H_k(V, V - \{y\}) \cong \begin{cases} \mathbb{Z} & k = m \\ 0 & k \neq 0 \end{cases} \quad \forall y \in V$

if $h: U \to V$ a homeomorphism then $H_k(U, U - \{x\}) \cong H_k(V, V - \{h(x)\}) \ \forall k$

$\therefore m = n$ ▦

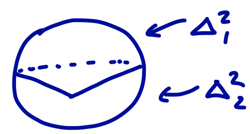
later it will be useful to know the generators of $H_n(S^n)$ and $H_n(D^n, \partial D^n)$

Prop 16:

1) We can identify $(D^n, \partial D^n)$ with $(\Delta^n, \partial\Delta^n)$
under this identification
$$H_n(D^n, \partial D^n) \cong \mathbb{Z}$$
is generated by the identity map $\Delta^n \to \Delta^n$

2) We can identify S^n with $\Delta_1^n \cup \Delta_2^n$ with $\partial\Delta_1^n$ glued to $\partial\Delta_2^n$ by a homeomorphism that preserves the order of the verticies and is the identity on faces

e.g.



if $f_1 : \Delta_1^n \to S^n$ is the inclusion map
then $f_1 - f_2$ is the generator of $H^n(S^n) \cong \mathbb{Z}$

Proof: 1) we induct on n

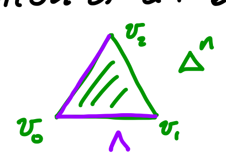
<u>$n=0$:</u> $H_0(D^0, \partial D^0) = H_0(D^0, \emptyset) = \mathbb{Z}$
pt

$H_0(D^0)$ is the free abelian group gen. by path components (by Thm 3)
so generator is $D^0 = \Delta^0$

<u>induction:</u> assume we know $H_{n-1}(D^{n-1}, \partial D^{n-1})$ generated by the inclusion
$$f : \Delta^{n-1} \to \Delta^{n-1} \cong D^{n-1}$$

clearly the inclusion $g : \Delta^n \to \Delta^n$ is a relative cycle $(\partial_n g \in C_{n-1}(\partial\Delta^n))$
so $\partial g = 0$ in $C_{n-1}(D^n, \partial D^n)$

let Λ be the union of all but one of the $(n-1)$-dim'l faces of Δ^n



consider the long exact sequence for $(\Delta^n, \partial\Delta^n, \Lambda)$

$$H_n(\Delta^n, \Lambda) \to H_n(\Delta^n, \partial\Delta^n) \xrightarrow{\partial} H_{n-1}(\partial\Delta^n, \Lambda) \to H_{n-1}(\Delta^n, \partial\Delta^n)$$

$H_n(\Delta^n, \Lambda) \cong \tilde{H}_n(\Delta^n/\Lambda) = 0$, where $\Delta^n/\Lambda \simeq \Delta^n \simeq pt$

$H_{n-1}(\Delta^n, \partial\Delta^n) = 0$ same

so ∂ is an isomorphism

note $\partial\Delta^n = \Lambda \cup \Delta^{n-1}_1$

$(\partial\Delta^n, \Lambda)$ a good pair, and

$\partial\Delta^n/\Lambda \overset{\psi}{\cong} \Delta^{n-1}/\partial\Delta^{n-1}$ induced by inclusion $i: \Delta^{n-1} \to \partial\Delta^n$

so $H_{n-1}(\partial\Delta^n, \Lambda) \underset{\cong}{\xrightarrow{\psi_*}} H_{n-1}(\Delta^{n-1}/\partial\Delta^{n-1}) \cong H_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1})$

by induction the inclusion of Δ^{n-1} into Δ^{n-1} generates $H_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1})$

$\therefore$ inclusion Δ^{n-1} into $\partial\Delta^n$ generates $H_{n-1}(\partial\Delta^n, \Lambda)$

now $\partial[g] = [\partial g] = [\partial\Delta^n]$ in $H_{n-1}(\partial\Delta^n, \Lambda)$

but $[\partial\Delta^n] = [\Delta^{n-1}]$ in $H_{n-1}(\partial\Delta^n, \Lambda)$

$\therefore \partial[g]$ is a generator of $H_{n-1}(\partial\Delta^n, \Lambda)$

and $[g]$ a generator of $H_n(\Delta^n, \partial\Delta^n)$ since ∂ an isomorphism

<u>now for S^n:</u> note $S^n/\Delta^n_2 \cong \Delta^n_1/\partial\Delta^n_1$

this homeomorphism comes from

$$\begin{tikzcd} \Delta^n_1 \arrow[r, "\text{inc.}"] \arrow[rr, bend right, "f"'] & S^n \arrow[r, "\text{quot.}"] & S^n/\Delta^n_2 \end{tikzcd}$$

this decends to $h: \Delta^n_1/\partial\Delta^n_1 \to S^n/\Delta^n_2$ a homeomorphism

$\therefore$ we get an isomorphism $h_*: H_n(\Delta^n_1, \partial\Delta^n_1) \to H_n(S^n, \Delta^n_2)$

so Δ^n_1 maps to a generator $[\Delta^n_1]$ of $H_n(S^n, \Delta^n_2)$

<u>note:</u> $H_n(\Delta^n_2) \to H_n(S^n) \xrightarrow{\cong} H_n(S^n, \Delta^n_2) \to H_{n-1}(\Delta^n_2)$

$H_n(\Delta^n_2) = 0$, $H_{n-1}(\Delta^n_2) = 0$

induced by inclusion $i: (S^n, \varnothing) \to (S^n, \Delta^n_2)$

so $H_n(S^n) \cong \mathbb{Z}$ and it is generated by a cycle in $C_n(S^n)$ that maps by i to Δ^n_1

now $\sigma = \Delta_1^n - \Delta_2^n$ is a cycle (exercise if not obvious)

and $\sigma = \Delta_1^n$ in $C_n(S^n)/C_n(\Delta_2^n)$

$\therefore\ i_*([\sigma]) = [\Delta_1^n] \in H_n(S^n, \Delta_2^n)$

and hence $[\sigma]$ generates $H_n(S^n)$ ▦

When making computations it is useful to know the long exact sequences "respect" maps between spaces. This is called naturality.

Th^{m} 17:

If $f:(X,A)\to(Y,B)$ is a map of pairs, then the following diagram is commutative

$$\begin{array}{ccccccccc}
\cdots \to & H_n(A) & \to & H_n(X) & \to & H_n(X,A) & \to & H_{n-1}(A) & \to \cdots \\
& \downarrow f_* & \circ & \downarrow f_* & \circ & \downarrow f_* & \circ & \downarrow f_* & \\
\cdots \to & H_n(B) & \to & H_n(Y) & \to & H_n(Y,B) & \to & H_{n-1}(B) & \to \cdots
\end{array}$$

similarly for the long exact sequence of a triple.

Proof: note

$$\begin{array}{ccccccccc}
0 \to & C_n(A) & \to & C_n(X) & \to & C_n(X,A) & \to 0 \\
& \downarrow f_* & & \downarrow f_* & & \downarrow f_* & \\
0 \to & C_n(B) & \to & C_n(Y) & \to & C_n(Y,B) & \to 0
\end{array}$$

is clearly commutative so result follows from homological algebra lemma below ▦

lemma 18 (HA):

If we have chain complexes and chain maps s.t.

$$\begin{array}{ccccccc}
0 \to & A_* & \xrightarrow{i} & B_* & \xrightarrow{j} & C_* & \to 0 \\
& \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & \\
0 \to & A_*' & \xrightarrow{i'} & B_*' & \xrightarrow{j'} & C_*' & \to 0
\end{array}$$

is commutative and rows are exact

then

$$\begin{array}{ccccccccc}
\cdots \to & H_n(A_*) & \xrightarrow{i_*} & H_n(B_*) & \xrightarrow{j_*} & H_n(C_*) & \xrightarrow{\partial} & H_{n-1}(A_*) & \to \cdots \\
& \downarrow \alpha_* & & \downarrow \beta_* & & \downarrow \gamma_* & & \downarrow \alpha_* & \\
\cdots \to & H_n(A_*') & \xrightarrow{i_*'} & H_n(B_*') & \xrightarrow{j_*'} & H_n(C_*') & \xrightarrow{\partial} & H_{n-1}(A_*') & \to \cdots
\end{array}$$

is commutative

Proof: since $\beta \circ \imath = \imath' \circ \alpha \Rightarrow \beta_* \circ \imath_* = \imath_*' \circ \alpha_*$

similarly for $\gamma_* \circ \jmath_* = \jmath_*' \circ \beta_*$

now recall $\partial[c] = [a]$ where $a \in A_{n-1}$ s.t.

$\imath(a) = \partial b$ for some $b \in B_n$ s.t. $\jmath(b) = c$

so $\partial[\gamma(c)] = [\alpha(a)]$ since $\gamma(c) = \gamma(\jmath(b)) = \jmath'(\beta(b))$

and $\imath'(\alpha(a)) = \beta(\imath(a)) = \beta(\partial b) = \partial\beta(b)$ β chain map

$\therefore \partial \circ \gamma_*[c] = \alpha_* \circ \partial[c] \quad \forall [c]$

Here is a long exact sequence that generalizes Van Kampen to homology theory

Th^m 19 (Mayer-Vietoris):

let $A, B \subset X$ be subspaces s.t. $X = (\text{int} A) \cup (\text{int} B)$

let

$$\begin{tikzcd} & A \arrow[dr, "j_A"] & \\ A \cap B \arrow[ur, "\imath_A"] \arrow[dr, "\imath_B"'] & & X \\ & B \arrow[ur, "j_B"'] & \end{tikzcd}$$

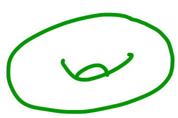
be inclusion maps

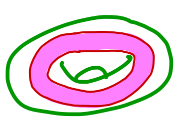
then $$\cdots \to H_n(A \cap B) \xrightarrow{\phi} H_n(A) \oplus H_n(B) \xrightarrow{\psi} H_n(X) \xrightarrow{\partial} H_{n-1}(A \cap B) \to \cdots$$

is exact where $\phi = (\imath_A)_* \oplus (\imath_B)_*$

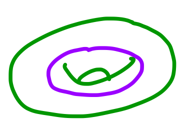
$\psi([a],[b]) = (\jmath_A)_*([a]) - (\jmath_B)_*([b])$ and

$\partial[z] = [\partial a]$ where $z = a + b$ for $a \in C_*(A)$ and $b \in C_*(B)$

example: $T^2 = S^1 \times S^1$ 

$A =$  $= (0, \frac{\pi}{2}) \times S^1 \simeq S^1$

$$H_n(A) = H_n(B) \cong \begin{cases} \mathbb{Z} & n = 0,1 \\ 0 & n \neq 0,1 \end{cases}$$

$B = T^2 - C \simeq S^1$ where  $C = \{\frac{\pi}{4}\} \times S^1$

$$H_n(A \cap B) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & n = 0,1 \\ 0 & n \neq 0,1 \end{cases}$$

$\simeq S^1 \cup S^1$

$$H_2(A)\oplus H_2(B) \to H_2(T^2) \to H_1(A\cap B) \to H_1(A)\oplus H_1(B) \to H_1(T^2) \to H_0(A\cap B) \to H_0(A)\oplus H_0(B) \to H_0(T^2) \to 0$$

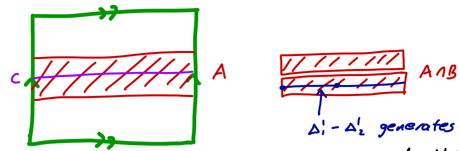
$$\begin{array}{ccccccccccccccccc} \shortparallel & & & & \cong & & \cong & & & & \cong & & \cong & & \cong & & \shortparallel \\ 0 & \longrightarrow & H_2(T^2) & \to & \mathbb{Z}\oplus\mathbb{Z} & \xrightarrow{\phi_1} & \mathbb{Z}\oplus\mathbb{Z} & \xrightarrow{\psi_1} & H_1(T^2) & \xrightarrow{\partial} & \mathbb{Z}\oplus\mathbb{Z} & \xrightarrow{\phi_0} & \mathbb{Z}\oplus\mathbb{Z} & \xrightarrow{\psi_0} & \mathbb{Z} & & 0 \end{array}$$

since path connected

note $\phi_0(1,0)=(1,1)$ each path component of $A\cap B$
$\phi_0(0,1)=(1,1)$ maps into a path component of A by 1_A and B by 1_B

$\phi_1(1,0)=(1,1)$

$\phi_1(0,1)=(1,1)$



$\Delta_1' - \Delta_2'$ generates one factor of $H_1(A\cap B)$ and $H_1(A)$

similarly for other component of $A\cap B$

so $\ker\phi_1 \cong \mathbb{Z}$ generated by $(1,-1)$

$\therefore\ H_2(T^2)\cong\mathbb{Z}$

$\ker\psi_1 = \operatorname{im}\phi_1 \cong \mathbb{Z}$ generated by $(1,1)$

$\therefore\ \operatorname{im}\psi_1 = \mathbb{Z}\oplus\mathbb{Z}/\ker\psi_1 \cong \mathbb{Z}$

$\operatorname{im}\partial \cong \ker\phi_0 \cong \mathbb{Z}$ generated by $(1,-1)$

so $\mathbb{Z}\cong\operatorname{im}\partial \cong H_1(T^2)/\ker\partial \cong H_1(T^2)/\operatorname{im}\psi_1 \cong H_1(T^2)/\mathbb{Z}$

Claim: $H_1(T^2)\cong\mathbb{Z}\oplus\mathbb{Z}$

let $a+K$ generate $\mathbb{Z}\cong H_1(T^2)/K$ where $K\cong\mathbb{Z}$

if $e\in H_1(T^2)$ then $e+K = na+K$ some $n\in\mathbb{Z}$

now $e-na\in K$ so if b generates K, then

$e-na=mb$ some $m\in\mathbb{Z}$

so each $e\in H_1(T^2)$ is $na+mb$

you can easily argue that $na+mb=n'a+m'b \iff n=n', m=m'$

exercise: $H_n(T^2)=0\ \forall n\geq 3$

so $H_n(T^2)\cong\begin{cases}\mathbb{Z} & n=0,2\\ \mathbb{Z}\oplus\mathbb{Z} & n=1\\ 0 & \text{otherwise}\end{cases}$

to establish Mayer-Vietoris we need

lemma 20 (HA):

given two long exact sequences and maps between them as below so that the diagram commutes

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r] \arrow[d, "\gamma_{n+1}"] & A_n \arrow[r, "f_n"] \arrow[d, "\alpha_n"] & B_n \arrow[r, "g_n"] \arrow[d, "\beta_n"] & C_n \arrow[r, "h_n"] \arrow[d, "\gamma_n"] & A_{n-1} \arrow[r] \arrow[d, "\alpha_{n-1}"] & B_{n-1} \arrow[r] \arrow[d, "\beta_{n-1}"] & \cdots \\
\cdots \arrow[r] & C'_{n+1} \arrow[r] & A'_n \arrow[r, "f'_n"] & B'_n \arrow[r, "g'_n"] & C'_n \arrow[r, "h'_n"] & A'_{n-1} \arrow[r] & B'_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

If the γ_n are isomorphisms, then

$$\cdots \to A_n \xrightarrow{\Phi_n} A'_n \oplus B_n \xrightarrow{\Psi_n} B'_n \xrightarrow{\Gamma_n} A_{n-1} \to \cdots$$

is exact where $\Phi_n(a) = (\alpha_n(a), f_n(a))$,

$\Psi_n(a', b) = \beta_n(b) - f'_n(a)$, and

$\Gamma_n(b') = h_n \circ \gamma_n^{-1} \circ g'_n(b')$

Proof: easy diagram chase

for example lets check exactness at B'_n

$$\Gamma_n \circ \Psi_n(a', b) = h_n \circ \gamma_n^{-1} \circ g'_n\left(\beta_n(b) - \cancel{f'_n(a)}\right) \quad 0 \text{ since } g'_n \circ f'_n = 0$$

$$= h_n \circ \gamma_n^{-1}(\gamma_n \circ g_n(b)) = h_n \circ g_n(b) = 0$$

so $\operatorname{im} \Psi_n \subset \ker \Gamma_n$

now if $b' \in \ker \Gamma_n$ then $\gamma_n^{-1} \circ g'_n(b') \in \ker h_n$

so $\exists\, b \in B_n$ s.t. $g_n(b) = \gamma_n^{-1} \circ g'_n(b')$

i.e. $g'_n(b') = \gamma_n \circ g_n(b) = g'_n \circ \beta_n(b)$

thus $g'_n(\beta_n(b) - b') = 0$

and so $\exists\, a' \in A'_n$ s.t. $f'_n(a') = \beta_n(b) - b'$

$\therefore\ b' = -f'_n(a') + \beta_n(b) = \Psi_n(a', b)$

thus $\operatorname{im} \Psi_n = \ker \Gamma_n$

exercise: check other cases ▩

Proof of Th$^{\underline{m}}$ 19 (Mayer-Vietoris).

consider the long exact sequences of pairs $(A, A\cap B)$ and (X, B)

$$\begin{array}{ccccccc}
H_n(A\cap B) & \xrightarrow{(\imath_A)_*} & H_n(A) & \xrightarrow{(k_A)_*} & H_n(A, A\cap B) & \to & H_{n-1}(A\cap B) \\
\downarrow (\imath_B)_* & & \downarrow (\jmath_A)_* & & \downarrow I_* & & \downarrow \\
H_n(B) & \xrightarrow[(\jmath_B)_*]{} & H_n(X) & \xrightarrow[(k_B)_*]{} & H_n(X, B) & \longrightarrow & H_{n-1}(B)
\end{array}$$

where k_A, k_B and I are obvious inclusions

<u>note:</u> $A/_{A\cap B} \cong X/_B$ (exercise prove this. almost obvious)

so if $(A, A\cap B)$ and (X, B) are good pairs then I_* is an isomorphism since

$$\begin{array}{ccc}
H_n(A, A\cap B) & \cong & \tilde{H}_n(A/_{A\cap B}) \\
\downarrow I_* & \circ & \downarrow \Phi_* \cong \\
H_n(X, B) & \cong & H_n(X/_B)
\end{array}$$

If pairs not good still expect Φ is an isomorphism, and it is (see book, or try to prove after we prove excision)

the result now follows from lemma (check if not obvious) ▦

Finally back to proof of excision

Proof Th$^{\underline{m}}$ 11 (Excision):

recall the set up: let $Z \subset A$ be subspaces of X with $\bar{Z} \subset \text{int}\, A$

we need to show the inclusion map

$$\imath : (X-Z, A-Z) \to (X, A)$$

induces an isomorphism on homology

$$\imath_* : H_n(X-Z, A-Z) \longrightarrow H_n(X, A)$$

<u>Main idea:</u> (*) for any n and any $\alpha = \sum_{i=1}^{k} m_i \sigma_i \in C_n(X, A) = C_n(X)/_{C_n(A)}$

we can find $\beta \in C_{n+1}(X, A)$ such that

$$\alpha + \partial\beta = \sum_{i=1}^{\ell} m_i \tau_i$$

where $\text{im}\, \tau_i \subset X - Z$ or $\text{im}\, \tau_i \subset A$ $\forall i$

we prove ⊛ later but see how it implies theorem.

take $[\alpha] \in H_n(X,A)$

$\circledast \Rightarrow \exists\, \alpha' \in [\alpha]$ s.t. $\alpha' = \sum_i m_i \tau_i$ as in ⊛

let $\alpha'' = \sum_{\substack{i \text{ s.t.} \\ \operatorname{im}\tau_i \not\subset A}} m_i \tau_i$

<u>note</u>: 1) $\alpha'' = \alpha'$ in $C_n(X)/C_n(A) = C_n(X,A)$

2) $\partial \alpha'' \in C_{n-1}(A)$ in fact in $C_{n-1}(A-Z)$

$\left(\text{since } -\partial\alpha'' = \partial\underbrace{(\alpha' - \alpha'')}_{\subset A} - \underbrace{\partial\alpha'}_{\subset A} \text{ so } \partial\alpha'' \subset A \cap (X-Z)\right)$

3) $\alpha'' \in C_n(X-Z)/C_n(A-Z)$

$\therefore$ α'' defines an element $[\alpha''] \in H_n(X-Z, A-Z)$

clearly $\imath_*([\alpha'']) = [\alpha''] = [\alpha'] = [\alpha]$

so $\imath_*$ is <u>onto</u>

now suppose $[\alpha] \in H_n(X-Z, A-Z)$ and $\imath_*([\alpha]) = 0$

so $\imath \circ \alpha = \alpha = \partial \beta$ some $\beta \in C_{n+1}(X,A)$

$\circledast \Rightarrow \exists\, \gamma \in C_{n+2}(X,A)$ s.t. $\beta + \partial\gamma = \sum m_i \tau_i$ with τ_i as in ⊛

clearly $\alpha = \partial(\beta + \partial\gamma)$

let $\beta' = \sum_{\substack{i \text{ s.t.} \\ \operatorname{im}\tau_i \not\subset A}} m_i \tau_i$

<u>note</u>: 1) $\beta' \in C_n(X-Z)$ and

2) $\partial\beta' = \partial\beta$ − terms with image in $(A \cap X-Z) = A-Z$

so $\partial\beta' = \alpha$ in $C_n(X-Z)/C_n(A-Z)$

$\therefore$ $[\alpha] = 0$ in $H_n(X-Z, A-Z)$

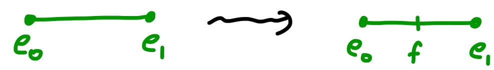
and $\imath_*$ injective ✓

<u>now to prove ⊛:</u>

need barycentric subdivision of a simplex

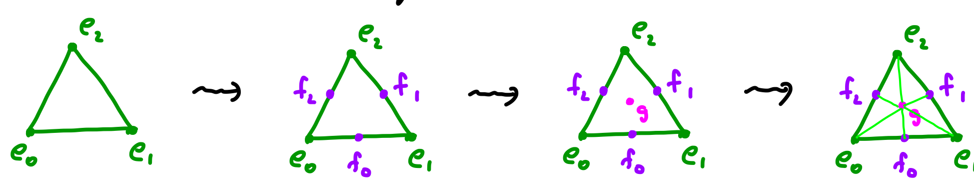
subdivision of 0-simplex : do nothing

" " 1-simplex : break simplex into 2 equal pieces



$[e_0, e_1] = [e_0, f] \cup [f, e_1]$

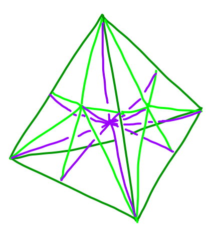
subdivision of 2-simplex: subdivide faces
add center point
add edges to all verticies



$[e_0\, e_1\, e_2] = [g\, e_0\, f_0] \cup [g\, f_0\, e_1] \cup [g\, e_1\, f_1] \cup \ldots$

inductively subdivision of n-simplex: subdivide faces
add center point
add edges to all verticies
add all faces so you have
broken Δ^n into a bunch
of n-simplicies

e.g.



exercise: Show "upto boundaries you can subdivide simplicies"

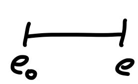
that is if $\sigma : \Delta^n \to X$ a singular n-simplex

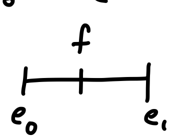
and if $\Delta^n = \Delta^n_1 \cup \ldots \cup \Delta^n_k$ is its barycentric subdivision

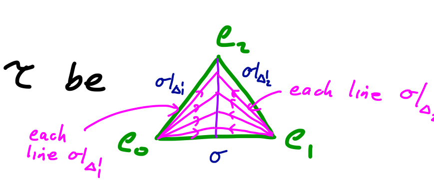
then $\exists$ (n+1)-chain τ such that

$$\sigma + \partial \tau = \sum_{i=1}^{k} \pm \sigma|_{\Delta^n_i}$$

determine sign

Hint: for $\Delta^1 =$ 

$\Delta^1 = \Delta^1_1 \cup \Delta^1_2$ 

let τ be 

$$\text{so } \partial\tau = \tau|_{[e_0 e_1]} - \tau|_{[e_0 e_2]} + \tau|_{[e_0 e_1]}$$
$$= \sigma - \sigma|_{\Delta^1_1} + \sigma|_{\Delta^1_2}$$

<u>note</u>: as you repeatedly barycentrically subdivide a simplex the size of the resulting simplicies goes to zero.

now given a singular n-simplex $\sigma : \Delta^n \to X$

note $\{\text{int}A, X-\bar{Z}\}$ is an open cover for X

$\therefore \{\sigma^{-1}(\text{int}A), \sigma^{-1}(X-\bar{Z})\}$ is an open cover of Δ^n

$\therefore \exists$ a Lebesgue number $\delta > 0$ for the cover s.t. any set of diameter $< \delta$ is mapped by σ to $\text{int}A$ or $X - \bar{Z}$

barycentrically subdivide Δ^n till each simplex has diam $< \delta$

for exercise $\exists \tau \in C_{n+1}(X)$ s.t. $\sigma + \partial\tau = \sum \pm \sigma|_{\text{subsimplices}}$

so we can replace σ with simplicies satisfying (*)

$\therefore$ we can do this for any $\sum m_i \sigma_i \in C_n(X,A)$ ▦