D. Degree and Cellular Homology

for a map $f: S^n \to S^n$

we get $f_*: \tilde{H}_n(S^n) \to \tilde{H}_n(S^n)$ $\quad$ ($n \neq 0$ could just use $H_n(S^n)$)

$$\begin{array}{ccc} \tilde{H}_n(S^n) & \to & \tilde{H}_n(S^n) \\ \wr\wr & & \wr\wr \\ \mathbb{Z} & & \mathbb{Z} \end{array}$$

define the degree of f to be $\deg(f) = f_*(1) \in \mathbb{Z}$

note: 1) $\deg(\mathrm{id}_{S^n}) = 1$

2) $\deg(f)$ only depends on f upto homotopy

3) if f is not surjective, then $\deg f = 0$

since if f misses a point $x \in S^1$

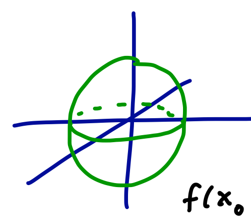
then
$$\begin{tikzcd} S^n \arrow[r, "f"] \arrow[dr, "\tilde{f} = f"'] & S^n \\ & S^n - \{x\} \arrow[u, "i"'] \end{tikzcd}$$

but $\tilde{f}_*(1) = 0 \in H_n(S^n - \{x\}) = 0$

so $f_*(1) = i_*(\tilde{f}_*(1)) = i_*(0) = 0.$

4) $\deg(f \circ g) = \deg(f)\deg(g)$

5) if f is reflection then $\deg f = -1$



$f(x_0, x_1 \dots x_n)$
$= (-x_0, x_1, \dots x_n)$

indeed: $n = 0$ $\quad S^0 = \{-1, 1\}$ and

$f(\pm 1) = \mp 1$

$H_0(S^0) \cong H_0(\{-1\}) \oplus H_0(\{1\})$

$f_*(a, b) = (b, a)$

recall to compute reduced homology we consider

$$C_1(S^0) \xrightarrow{\partial = 0} C_0(S^0) \xrightarrow{\varepsilon} \mathbb{Z}$$
$$\sum m_i x_i \mapsto \sum m_i$$

so $\tilde{H}_0(S^0) \cong \ker(\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}) \cong \mathbb{Z}$ gen by $(1, -1)$
$\qquad (a, b) \mapsto a + b$

now $f_*(1, -1) = (-1, 1) = -(1, -1)$

so $\deg f = -1$

now suppose result for S^k with $k<n$

let $D^n_\pm = \{(x_0, \dots x_n) \in S^n \mid \pm x_n \geq 0\}$

note f preserves $D^n_\pm$

$$\begin{tikzcd}
\tilde{H}_n(S^n) \arrow[r, "\cong"] \arrow[d, "f_*"] \arrow[dr, phantom, "\circ"] & H_n(S^n, D^n_+) \arrow[d, "f_*"] \arrow[dr, phantom, "\circ"] & H_n(D^n_-, \partial D^n_-) \arrow[l, "\cong"'] \arrow[r, "\cong"] \arrow[d, "f_*"] \arrow[dr, phantom, "\circ"] & \tilde{H}_{n-1}(\partial D^n_-) \arrow[d, "f_*"] \\
\tilde{H}_n(S^n) \arrow[r, "\cong", "\text{good pair Th}^{\underline{m}}\,12"'] & H_n(S^n, D^n_+) & H_n(D^n_-, \partial D^n_-) \arrow[l, "\cong", "\text{excision}"'] \arrow[r, "\cong"] & \tilde{H}_{n-1}(\partial D^n_-)
\end{tikzcd}$$

$\partial D^n_- = S^{n-1}$

so all vertical maps are multiplication by -1

$$H_n(D^n_-) \to H_n(D^n_-, \partial D^n_-) \xrightarrow[\cong]{\partial} H_{n-1}(\partial D^n_-) \to H_{n-1}(D^n_-)$$
$H_n(D^n_-) = 0$, $H_{n-1}(D^n_-) = 0$ for $n>1$

if $n=1$

$$H_1(D^1_-) \to H_1(D^1_-, \partial D^1_-) \to H_0(S^0) \xrightarrow{i} H_0(D^1) \to 0$$
$H_1(D^1_-) = 0$, $H_0(S^0) \cong \mathbb{Z}\oplus\mathbb{Z} \to H_0(D^1) \cong \mathbb{Z} \to 0$

$H_1(D^1_-, \partial D^1_-) \cong \ker i \cong \tilde{H}_0(S^0)$

6) f = antipodal map $= -id_{S^n}$

then $\deg(f) = (-1)^{n+1}$

this follows from <u>exercise</u>: f = composition of $(n+1)$ reflections

<u>Some nice applications of degree:</u>

<u>lemma 21:</u>

$$\boxed{\begin{array}{l} \text{let } f, g : X \to S^n \subseteq \mathbb{R}^{n+1} \\ \text{if } f(x) \neq -g(x) \ \forall x \in X, \text{ then } f \simeq g \end{array}}$$

<u>Proof:</u>

$$H : X \times [0,1] \to S^1$$
$$(x,t) \longmapsto \frac{(1-t)f(x) + t g(x)}{\|(1-t)f(x) + t g(x)\|}$$

is the homotopy (note OK since $f(x) \neq -g(x)$) ▦

<u>Cor 22:</u>

$$\boxed{\begin{array}{l} \text{let } f : S^n \to S^n \\ (1) \text{ if } f \text{ has } \underline{\text{no}} \text{ fixed point, then } \deg f = (-1)^{n+1} \\ (2) \text{ if there is } \underline{\text{no}}\ x \in S^n \text{ s.t. } f(x) = -x, \text{ then } \deg f = 1 \end{array}}$$

<u>Proof:</u> (1) apply lemma 21 to f and antipodal map and use homotopy invariance

(2) same as above but for f and id_{S^n} ▦

Cor 23: $\boxed{\text{If } n \text{ is even, then any map } f: S^n \to S^n \text{ has a fixed point or an antipodal point } (x \text{ s.t. } f(x) = -x)}$

Proof: If not then $\deg f = 1$ and -1 ⊠ ▦

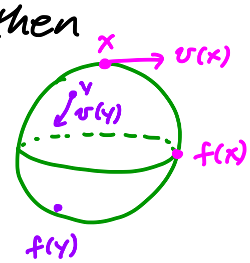
Cor 24:

$$\boxed{S^n \text{ has a nonzero vector field} \iff n \text{ is odd}}$$

Proof: If n is even then any vector field must have a zero since if v a vector field with no zero then

$$f: S^n \to S^n : x \mapsto \frac{v(x)}{\|v(x)\|}$$

has no fixed points or antipodal points ⊠



$v(x)$ lies in plane $\perp$ to x

if $n = 2k+1$, then

$$v(x_0, x_1, \dots, x_{2k}, x_{2k+1}) = (x_1, -x_0, \dots, x_{2k+1}, -x_{2k})$$

an nonzero vector field.

Remark: Its actually true that maps $\boxed{f, g: S^n \to S^n \text{ are homotopic} \iff \deg f = \deg g}$

<u>How to compute degree:</u>

Suppose $f: S^n \to S^n$ $n > 0$
and $\exists y \in S^n$ s.t. $f^{-1}(y) =$ finite set of points $x_1, \dots, x_k$

<u>note</u>:

$$\underset{0}{\underset{\shortparallel}{H_n(S^n - \{y\})}} \to H_n(S^n) \xrightarrow{i_*} H_n(S^n, S^n - \{y\}) \to \underset{0}{\underset{\shortparallel}{H_{n-1}(S^n - \{y\})}}$$

$n > 1$, think about $n = 1$ case

so i_* an isomorphism

similarly

$$j_*: H_n(S^n) \to H_n(S^n, S^n - \{x_i\}) \text{ an isomorphism too}$$

let V be a neighborhood of y and

U_i be neighborhoods of the x_i

s.t. $f(U_i) \subset V$ and

$x_j \notin U_i \quad \forall i \neq j$

by excision $H_n(S^n, S^n - \{y\}) \cong H_n(S^n - (S^n - V), (S^n - \{y\}) - (S^n - V))$

$$= H_n(V, V - \{y\})$$

Similarly for $H_n(U_i, U_i - \{x_i\})$

so we get

$$\begin{tikzcd} f_* : H_n(U_i, U_i - \{x_i\}) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & H_n(V, V - \{y\}) \arrow[d, phantom, "\cong" rotate=90] \\ \mathbb{Z} & \mathbb{Z} \\ 1 \arrow[r, mapsto] & d \end{tikzcd}$$

note: $\cong$ with $\mathbb{Z}$ comes from $\cong$ to $H_n(S^n)$ and can fix generator there so d well-def

we define the local degree of f at x_i to be

$$\deg(f, x_i) = f_*(1) \text{ above}$$

note: • If we change V get same number

• " " U_i " " (as long as $x_j \notin U_i$!)

• If $f|_{U_i} : U_i \to f(U_i)$ a homeomorphism then replace V by $f(U_i)$

and

$$\begin{tikzcd} f_* : H_n(U_i, U_i - \{x_i\}) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & H_n(V, V - \{y\}) \arrow[d, phantom, "\cong" rotate=90] \\ \mathbb{Z} & \mathbb{Z} \\ 1 \arrow[r, mapsto] & \pm 1 \end{tikzcd}$$

so $\deg(f, x_i) = \pm 1$

i.e. f local homeomorphism near x_i, then $\deg(f, x_i) = \pm 1$

lemma 25:

> with $f : S^n \to S^n$, y and $x_1, \dots x_k$ as above
>
> $$\deg(f) = \sum_{i=1}^{k} \deg(f, x_i)$$

Proof:

Choose all U_i disjoint

set $Z = S^n - \bigcup_{i=1}^{k} U_i$

$$H_n(S^n, S^n - f^{-1}(y)) = H_n(S^n, S^n - \{x_1, \dots, x_k\}) \overset{\text{excision}}{\cong} H_n(S^n - Z, S^n - \{x_1 \dots x_k\} - Z)$$
$$= H_n\Big(\bigcup_{i=1}^{k} U_i, \bigcup_{i=1}^{k} (U_i - \{x_i\})\Big)$$
$$\cong \bigoplus_{i=1}^{k} H_n(U_i, U_i - \{x_i\})$$

now

$$\begin{tikzcd}
H_n(S^n) \arrow[r, "f_*"] \arrow[d, "i_*"] \arrow[dd, bend right, "g"'] \arrow[dr, phantom, "\circ"] & H_n(S^n) \arrow[d, "i_* \cong"] \\
H_n(S^n, S^n - f^{-1}(y)) \arrow[r, "f_*"] \arrow[d, "\cong"] \arrow[dr, phantom, "\circ"] & H_n(S^n, S^n - \{y\}) \arrow[d, "\cong"] \\
\bigoplus_{i=1}^{k} H_n(U_i, U_i - \{x_i\}) \arrow[r, "\oplus (f|_{U_i})_*"'] & H_n(V, V - \{y\})
\end{tikzcd}$$

$$1 \longmapsto \deg(f)$$

note: $H_n(S^n) \longrightarrow H_n(U_i, U_i - \{x_i\})$

$$\begin{array}{ccc} H_n(S^n) & \longrightarrow & H_n(U_i, U_i - \{x_i\}) \\ \cong & & \cong \\ \mathbb{Z} & & \mathbb{Z} \\ 1 & \longmapsto & 1 \end{array}$$

so $g(1) = (1, 1, \dots 1)$

and $\deg f = f_*(1) = \big(\oplus (f|_{U_i})_*\big) \circ g(1) = \oplus (f|_{U_i})_*(1) = \sum_{i=1}^{k} \deg(f, x_i)$ ■

Remark: if you know differential topology then given a smooth map $f: S^n \to S^n$ we can homotop f so y is a regular value $\Rightarrow f^{-1}(y)$ finite and f local homeomorphism

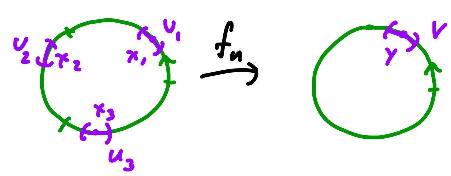
$$df_{x_i}: \underset{\mathbb{R}^n}{\overset{\cong}{T_{x_i} S^n}} \to \underset{\mathbb{R}^n}{\overset{\cong}{T_y S^n}} \quad \text{isomorphism}$$

$$\deg(f, x_i) = \begin{cases} +1 & df_{x_i} \text{ orientation preserving} \\ -1 & df_{x_i} \text{ orientation reversing} \end{cases}$$

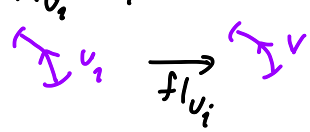
examples:

1) $f_n: S^1 \to S^1$
$\quad z \mapsto z^n$

$n > 0$



can choose so $f|_{U_i}: U_i \to V$ a homeo.



can extend $f|_{U_i}$ to a homeo $g_i: S^1 \to S^1$ that preserves or^n

such homeos are isotopic to id_{S^1}

$\therefore \quad 1 = \deg g_i = \deg(g_i, x_i) = \deg(f_n, x_i)$

so $\deg f_n = n$

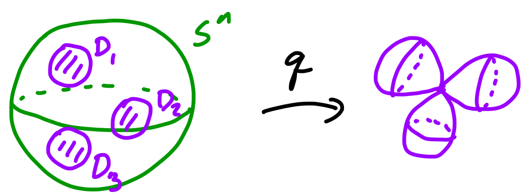
if $n<0$, then $f_n = f_{-n} \circ r$ ↙ reflection

so $\deg f_n = \deg(f_{-n}) \deg r$
$= (-n)(-1) = n$

2) let $D_1, \dots, D_k$ be disjoint D^n in S^n $\quad n>1$

$C = S^n - \bigcup_{i=1}^{k} D_i$

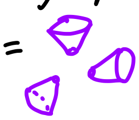
then $S^n/C \cong \overbrace{S^n \vee S^n \vee \dots \vee S^n}^{X}$ ↙ wedge of k, n-spheres



$q : S^n \to X$

let U be a nbhd of wedge pt. in X

let $V = X -$ wedge point

<u>note</u>: $U \cap V =$  $\cong \coprod_{i=1}^{k} (S^{n-1} \times (0,1))$

$$\begin{array}{ccccccccc} H_n(U) \oplus H_n(V) & \to & H_n(X) & \to & H_{n-1}(U\cap V) & \to & H_{n-1}(U) \oplus H_{n-1}(V) \\ \| & & & & \cong & & \| \\ 0 & & & & \bigoplus_{i=1}^{k} \mathbb{Z} & & 0 \end{array}$$

so $H_n(X) \cong \bigoplus_{i=1}^{k} \mathbb{Z}$

let $f_i : X \to S^n$ collapse all but i^{th} S^n in X

<u>Claim</u>: $(f_i)_* : H_n(X) \to H_n(S^n)$

$$\begin{array}{ccc} H_n(X) & \longrightarrow & H_n(S^n) \\ \cong & & \cong \\ \bigoplus \mathbb{Z} & & \mathbb{Z} \\ (m_1, \dots m_k) & \longmapsto & m_i \end{array}$$

indeed let $S^n \xrightarrow{j_i} X \xrightarrow{f_j} S^n$

↑ inc S^n to i^{th} sphere

note $(f_j \circ j_i)_* (1) = 0$ if $i \neq j$

and $(f_i \circ j_i)_* (1) = \pm 1$

since $f_i \circ j_i$ a homeomorphism

so we must have $(j_i)_*(1) = (0,\dots,0,1,0,\dots,0)$ ← j^{th} slot

and so $(f_i)_*(0,\dots,0,1,0,\dots,0) = \pm 1$ if -1 comp w/ refl.

now set $f : X \to S^n$ to be f_i on i^{th} sphere

so $f_*(m_1, \dots m_k) = m_1 + \dots + m_k$

as above $q_* : H_n(S^n) \to H_n(X)$

$1 \longmapsto (1,1,\dots,1)$ <u>exercise:</u> prove this if not clear (consider ex 1) above)

set $g_k = f \circ q : S^n \to S^n$

clearly $\deg(g_k) = k$

Cellular Homology

let X be a CW complex

set $C_n^{cw}(X)$ = free abelian group generated by n-cells $e_1^n, \dots e_{l_n}^n$

let $f_i^n : \partial e_i^n \to X^{(n-1)}$ the attaching map for e_i^n

given e_i^n and e_j^{n-1}, $n \geq 2$, consider

$$\begin{tikzcd} S^{n-1} = \partial e_i^n \arrow[r, "f_i^n"] \arrow[rrrr, bend right, "g_{ij}"'] & X^{(n-1)} \arrow[r, "\text{quotient map}"'] & X^{(n-1)}/X^{(n-2)} \cong \bigvee_{i=1}^{l_{n-1}} S^{n-1} \arrow[r, "p_j"] & S^{n-1} \end{tikzcd}$$

quotient onto j^{th} S^{n-1}

let d_{ij} = degree g_{ij}

define $\partial_n^{cw} : C_n^{cw}(X) \longrightarrow C_{n-1}^{cw}(X)$

$$e_i^n \longmapsto \sum_{j=1}^{l_{n-1}} d_{ij} e_j^{n-1}$$

for $n=1$ define:

$$\partial_1^{cw} : C_1^{cw}(X) \to C_0^{cw}(X)$$

$$e_i^1 \longmapsto \partial e_i^1$$

singular boundary since $e_i^1 \hookrightarrow X$ is a sing 1-simplex

<u>note:</u> if $X^{(0)} = \{\text{one point}\}$, then $\partial_1^{cw} e_i^1 = 0 \quad \forall i$

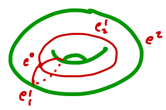
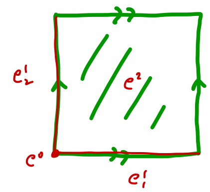
Thm 26:

$$\boxed{\begin{aligned} &\partial_n^{cw} \circ \partial_{n+1}^{cw} = 0 \\ &H_n(X) \cong \ker \partial_n^{cw} / \operatorname{im} \partial_{n+1}^{cw} \end{aligned}}$$

$H_n^{cw}(X) = \ker \partial_n^{cw} / \operatorname{im} \partial_n^{cw}$ is called the cellular homology of X

and thm says $H_n^{cw}(X)$ is isomorphic to singular homology!

example:

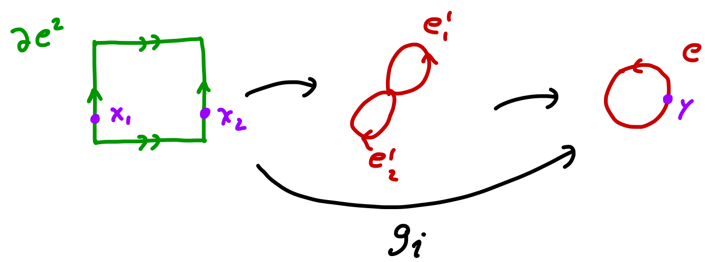
1) $T^2 =$  

$$0 \to \underset{\mathbb{Z}}{\underset{\shortparallel}{C_2^{cw}(T^2)}} \xrightarrow{\partial^{cw}} \underset{\mathbb{Z}\oplus\mathbb{Z}}{\underset{\shortparallel}{C_1^{cw}(T^2)}} \xrightarrow{\partial^{cw}} \underset{\mathbb{Z}}{C_0^{cw}(T^2)} \to 0$$

for $\partial^{cw} e_i^1 = 0$ from above

for $\partial^{cw} e^2$: $\quad \partial e^2 = S^1 \to X^{(1)} \to X^{(1)}/X^{(0)} = X^{(1)} \to S_i^1$ ← corresp to e_i^1

$$\partial e^2 \xrightarrow{g_i} S_i^1$$



note: orientation on ∂e^2 agrees with direction at x_2 but not at x_1

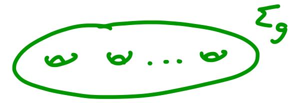
so as discussed above

$$\deg(g_i, x_2) = 1 = -\deg(g_i, x_1)$$

so $\deg(g_i) = 0$ for $i = 0, 1$

$$\therefore \partial_1^{cw} e^2 = 0 e_1^1 + 0 e_2^1 = 0$$

$$\therefore \; H_n(T^2) = \begin{cases} \mathbb{Z} & n=0,2 \\ \mathbb{Z}\oplus\mathbb{Z} & n=1 \\ 0 & \text{otherwise} \end{cases}$$

exercise: If Σ_g is surface of genus g 

then $$H_n(\Sigma_g) = \begin{cases} \mathbb{Z} & n=0,2 \\ \bigoplus_{2g}\mathbb{Z} & n=1 \\ 0 & \text{otherwise} \end{cases}$$

Remarks: 1) $H_k(X)$ has at most $\ell_k = \#$ k-cells generators

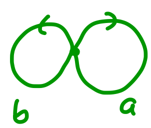
in particular, $H_k(X) = 0$ if no k-cells

2) If X has only cells in even dimensions then $\partial^{cw} = 0$

so $H_n^{cw}(X) = C_n^{cw}(X)$

example: recall $\mathbb{C}P^n = e^0 \cup e^2 \cup e^4 \cup \dots \cup e^{2n}$

so $$H_n^{cw}(\mathbb{C}P^n) = \begin{cases} \mathbb{Z} & n = 0,2,\dots,2n \\ 0 & \text{otherwise} \end{cases}$$

example: let $X =$ 

$\cup\; 2$ (2-cells)

e_1^2 attached along $a^5 b^{-3}$

e_2^2 " " $b^3 (ab)^{-2}$

arguing as above we have

$$\begin{array}{ccccccccc} 0 \to & C_2^{cw}(X) & \xrightarrow{\partial_2^{cw}} & C_1^{cw}(X) & \xrightarrow{\partial_1^{cw}} & C_0^{cw}(X) & \to 0 \\ & \parallel\!\wr & & \parallel\!\wr & & \parallel\!\wr & \\ & \mathbb{Z}\oplus\mathbb{Z} & & \mathbb{Z}\oplus\mathbb{Z} & & \mathbb{Z} & \end{array}$$

$\partial_1^{cw} = 0$

$\partial_2^{cw} = \begin{pmatrix} 5 & -2 \\ -3 & 1 \end{pmatrix}$ note matrix invertable over $\mathbb{Z}$ so $\ker \partial_2^{cw} = 0$

$\operatorname{im} \partial_2^{cw} =$ everything

$$\therefore \; H_n^{cw}(X) \cong \begin{cases} \mathbb{Z} & n=0 \\ 0 & n\neq 0 \end{cases}$$

by Van Kampen $\pi_1(X) \cong \langle a, b \mid a^5 b^{-3}, b^3 (ab)^{-3} \rangle$

one can show this is a group of order 120

so X not contractible

note: example shows π_1 sees things H_n does not

but $\pi_1(S^n) = 0$, $n > 1$ so H_n sees things π_1 does not.

lemma 27:

X a CW complex

1) $H_k(X^{(n)}, X^{(n-1)}) = \begin{cases} \bigoplus_{\ell_n} \mathbb{Z} & n = k \\ 0 & n \neq k \end{cases}$ $\quad \ell_k = \#\, n\text{-cells}$

2) $H_k(X^{(n)}) = 0$ if $k > n$

3) $\imath : X^{(n)} \to X$ induces an isomorphism

$$\imath_* : H_k(X^{(n)}) \to H_k(X) \quad \forall\, k < n$$

Proof: 1) $(X^{(n)}, X^{(n-1)})$ is a good pair so

$$H_k(X^{(n)}, X^{(n-1)}) \cong \tilde{H}_k(X^{(n)}/X^{(n-1)})$$

but $X^{(n)}/X^{(n-1)} \cong \bigvee_{\alpha=1}^{\ell_k} S^n$ for $n \geq 1$

for $n = 0$ also clearly true ✓

2) $H_{k+1}(X^{(n)}, X^{(n-1)}) \to H_k(X^{(n-1)}) \to H_k(X^{(n)}) \to H_k(X^{(n)}, X^{(n-1)})$

$k \neq n, n-1$: the first and last groups are $= 0$

$\therefore H_k(X^{(n-1)}) \cong H_k(X^{(n)}) \quad \forall\, k \neq n, n-1$

so for $k > n$ $\quad H_k(X^{(n)}) \cong H_k(X^{(n-1)}) \cong \dots \cong H_k(X^{(0)}) = 0$ ✓

3) if $k < n$ then

m arbitrary

$$H_k(X^{(n)}) \cong H_k(X^{(n+1)}) \cong \dots \cong H_k(X^{(n+m)})$$

so $H_k(X^{(n)}) \cong H_k(X)$ (clear if X finite dim'l

still true for any X but need

Fact: "Homology commutes with direct limits" (colimits))

Proof of Th^{m} 26:

by lemma 25 we know $C_n^{CW}(X) \cong H_n(X^{(n)}, X^{(n-1)})$

consider the long exact sequence of the tripple $(X^{(n+1)}, X^{(n)}, X^{(n-1)})$

$$\cdots \to H_{n+1}(X^{(n+1)}, X^{(n-1)}) \to H_{n+1}(X^{(n+1)}, X^{(n)}) \xrightarrow{d_{n+1}} H_n(X^{(n)}, X^{(n-1)}) \to \cdots$$

so $d_{n+1} : C_{n+1}^{CW}(X) \to C_n^{CW}(X)$

Claim: $\partial_n^{CW} = d_n$

we prove claim below, but first prove th^{m} given claim

consider 2 long exact sequences of pairs $(X^{(n+1)}, X^{(n)})$ and $(X^{(n)}, X^{(n-1)})$

$$\begin{tikzcd}
 & H_n(X^{(n-1)}) = 0 \arrow[d] & & \\
H_{n+1}(X^{(n+1)}, X^{(n)}) \arrow[r, "\partial_{n+1}"] \arrow[dr, bend right, "j_n \circ \partial_{n+1}"'] & H_n(X^{(n)}) \arrow[r] \arrow[d, "j_n"] & H_n(X^{(n+1)}) \arrow[r] \arrow[d, phantom, "\cong"] & H_n(X^{(n+1)}, X^{(n)}) \arrow[d, phantom, "="] \\
 & H_n(X^{(n)}, X^{(n-1)}) \arrow[d, "\partial_n"] & H_n(X) & 0 \\
 & H_{n-1}(X^{(n-1)}) & &
\end{tikzcd}$$

by lemma 27

exercise: $j_n \circ \partial_{n+1} = d_{n+1}$

(diagram chase, easy to see choices maid to construct ∂_{n+1} can also be used for d_{n+1})

so $d_n \circ d_{n+1} = j_{n-1} \circ \underbrace{\partial_n \circ j_n}_{=0} \circ \partial_{n+1} = 0$

$=0$ since 2 terms in long exact sequence

$\therefore$ can consider $\ker d_n / \operatorname{im} d_{n+1}$

from above $H_n(X) \cong H_n(X^{(n)}) / \operatorname{im} \partial_{n+1}$

note: j_n is injective so

$$\operatorname{im} \partial_{n+1} \cong j_n(\operatorname{im} \partial_{n+1}) = \operatorname{im}(j_n \circ \partial_{n+1}) = \operatorname{im} d_{n+1}$$

and since j_{n-1} is injective too

$$H_n(X^{(n)}) \cong \operatorname{im} j_n = \ker \partial_n \cong \ker(j_{n-1} \circ \partial_n) = \ker d_n$$

$$\therefore\ H_n(X) \cong H_n(X^{(n)}) / \operatorname{im} \partial_{n+1} \underset{\text{given by } j_n}{\cong} \operatorname{im} j_n / \operatorname{im} d_{n+1} = \ker d_n / \operatorname{im} d_{n+1}$$

$$\underset{\text{by claim}}{=} \ker \partial_n^{cw} / \operatorname{im} \partial_{n+1}^{cw}$$

<u>Proof of Claim:</u>

first note: $i : (e_i^n, \partial e_i^n) \to (X^{(n)}, X^{(n-1)})$ given by "inclusion"

induces

$$\begin{array}{ccc} i_* : H_n(e_i^n, \partial e_i^n) & \to & H_n(X^{(n)}, X^{(n-1)}) \\ \text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} \\ \mathbb{Z} & & \bigoplus_{\ell_n} \mathbb{Z} \end{array}$$

is injective and maps $\mathbb{Z}$ to factor corresp to e_i^n

(indeed
$$\begin{tikzcd} (e_i^n, \partial e_i^n) \arrow[r, "i"] \arrow[rrr, bend right, "j"'] & (X^{(n)}, X^{(n-1)}) \arrow[r] & X^{(n)}/X^{(n-1)} \arrow[r] & e_i^n/\partial e_i^n \end{tikzcd}$$

$$\begin{tikzcd} H_n(e_i^n, \partial e_i^n) \arrow[r, "j_*"] \arrow[d, phantom, "\cong"] \arrow[dr, phantom, "\circ"] & H_n(e_i^n/\partial e_i^n) \\ H_n(e_i^n/\partial e_i^n) \arrow[ur, "j_*"'] & \end{tikzcd}$$
identity map)

now
$$\begin{tikzcd} H_n(e_i^n, \partial e_i^n) \arrow[r, "\partial"] \arrow[d, "i_*"] \arrow[dr, phantom, "\circ"] & H_{n-1}(\partial e_i^n) \arrow[d, "(f_i^n)_*"] \\ H_n(X^{(n)}, X^{(n-1)}) \arrow[r, "\partial_n"] \arrow[dr, "d_n"'] & H_{n-1}(X^{(n-1)}) \arrow[d, "j_{n-1}"] \arrow[dl, phantom, "\circ *"] \\ & H_{n-1}(X^{(n-1)}, X^{(n-2)}) \end{tikzcd}$$

<u>exercise</u> check *
(same exercise as above)

so generator in $H_n(X^{(n)}, X^{(n-1)})$ corresponding to e_i^n

maps under d_n to

$$j_{n-1} \circ (f_i^n)_*(1) \ \text{ in } \ H_{n-1}(X^{(n-1)}, X^{(n-2)})$$

but $H_{n-1}(X^{(n-1)}, X^{(n-2)}) \cong \bigoplus_{\ell_{n-1}} \mathbb{Z}$

and by definition $(j_{n-1} \circ f_i^n)_*(1) = (d_{i1}, \ldots, d_{i\ell_{n-1}})$

$$\therefore\ d_n(\text{gen. corresp to } e_i^n) = \partial_n^{cw}(e_i^n)$$

If X and Y are CW complexes a continuous map

$$f : X \to Y$$

is called <u>cellular</u> if $f(X^{(i)}) \subset Y^{(i)}$

Fact:

> any map between CW complexes is homotopic to a cellular map

Remark: this is not hard to prove with differential topology, and can be proven without it, see Hatcher.

now given an n-cell σ of X and n-cell τ of Y consider

$$\begin{tikzcd}
\underset{\sigma}{\overset{D^n}{\shortparallel}} \arrow[r] \arrow[d] & X^{(n)} \arrow[r, "f"] \arrow[d, "q"] & Y^{(n)} \arrow[d, "q'"] & \\
S^n = D^n/S^{n-1} \arrow[r] \arrow[rrr, bend right, "f_{\sigma,\tau}"'] & X^{(n)}/X^{(n-1)} \arrow[r, "\tilde{f}"] & Y^{(n)}/Y^{(n-1)} = \vee S^n \arrow[r, "\text{quotient all but } \tau"] & S^n
\end{tikzcd}$$

Th^{m} 28:

> given a cellular map $f : X \to Y$ then

$$f_* : H_n^{cw}(X) \to H_n^{cw}(Y)$$
is given by
$$f_*([\sigma]) = \sum \deg(f_{\sigma,\tau})[\tau]$$

Proof: similar to the discussion above, exercise

E. Homology with different coefficients

given an abelian group G
and a space X

let $C_n(X,G) = \{ \sum_{i=1}^{k} g_i \sigma_i \mid g_i \in G, \sigma_i \text{ a singular } n\text{-simplex} \}$

$$\partial_n\left(\sum_{i=1}^{k} g_i\sigma_i\right) = \sum_{i=1}^{k} g_i \partial\sigma_i = \sum_{i=1}^{k}\sum_{j=0}^{n} g_i(-1)^j \sigma_i^{(j)}$$ j^{th} face of σ_i

as before $\partial_n \circ \partial_{n+1} = 0$

so we define the homology of X with coefficients in G to be
$$H_n(X;G) = \ker\partial_n / \operatorname{im}\partial_{n+1}$$ (note: for $G=\mathbb{Z}$ get orig def^n)

can also define $H_n(X,A;G)$ using $C_n(X,A;G) = C_n(X;G)/C_n(A;G)$

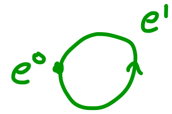
all th^{m}s we proved above work for these homologies too

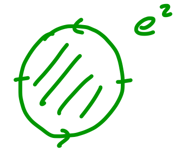
similarly if X a CW complex let
$$C_n^{cw}(X;G) = \bigoplus_{\ell_n} G \qquad \ell_n = \#\ n\text{-cells}$$
and
$$\partial_n^{cw}\left(\sum_{i=1}^{\ell_n} g_i e_i^n\right) = \sum_{i=1}^{\ell_n}\sum_{j=1}^{\ell_{n-1}} g_i(\deg h_{ij}) e_j^{n-1}$$

where
$$\partial e_i^n \to X^{(n-1)} \to X^{(n-1)}/X^{(n-2)} \to S^{n-1} \text{ corresp to } e_j^{n-1}$$
$$\underbrace{\partial e_i^n \to \cdots \to S^{n-1}}_{h_{ij}}$$

again this gives $H_n(X;G)$

example: $\mathbb{RP}^2$





use $\mathbb{Z}$ coeff:

$$0 \to C_2(\mathbb{RP}^2) \to C_1(\mathbb{RP}^2) \to C_0(\mathbb{RP}^2) \to 0$$
$$\begin{array}{ccccc} \cong & & \cong & & \cong \\ \mathbb{Z} & \xrightarrow{\times 2} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} \end{array}$$

$$H_n(\mathbb{RP}^2) = \begin{cases} \mathbb{Z} & n=0 \\ \mathbb{Z}/2 & n=1 \\ 0 & n \neq 0,1 \end{cases}$$

use $\mathbb{Z}/2$ coeff:

$$0 \to C_2(\mathbb{RP}^2; \mathbb{Z}/2) \to C_1(\mathbb{RP}^2; \mathbb{Z}/2) \to C_0(\mathbb{RP}^2; \mathbb{Z}/2) \to 0$$
$$\begin{array}{ccccc} \cong & & \cong & & \cong \\ \mathbb{Z}/2 & \xrightarrow{0} & \mathbb{Z}/2 & \xrightarrow{0} & \mathbb{Z}/2 \end{array}$$

$$H_n(\mathbb{RP}^2; \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & n=0,1,2 \\ 0 & n \neq 0,1,2 \end{cases}$$