IV Poincaré Duality

A Statement and Consequences

a <u>manifold of dimension n</u> is a topological space M that is

<u>Hausdorff</u> and <u>locally Euclidean</u>

points can be separated by disjoint open sets

each point $x \in M$ has an open neighborhood homeomorphic to $\mathbb{R}^n$, such a nbhd called a <u>coordinate chart</u>

<u>note</u>: we don't require M to be second countable as some definitions do.

a <u>manifold with boundary of dimension n</u> is a space M that is Hausdorff and every point has an open neighborhood homeomorphic to $\mathbb{R}^n$ or $\mathbb{R}^n_+ = \{(x_1, \dots x_n) \mid x_n \geq 0\}$

$\partial M = \{x \in M$ that don't have nbhd homeo to $\mathbb{R}^n\}$

$\text{int } M = \{x \in M$ that do have nbhd homeo to $\mathbb{R}^n\}$

<u>exercise</u>: $\partial(\partial M) = \emptyset$

$\text{int}(\partial M) = \partial M$

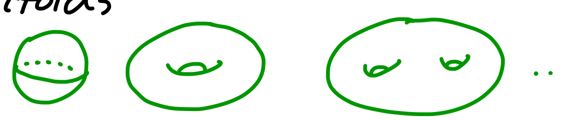
$\partial(\text{int } M) = \emptyset$

∂M is an $(n-1)$ dimensional manifold

we say M is closed if M is compact and $\partial M = \emptyset$

<u>examples</u>:

1) Surfaces are 2-manifolds



2) $S^n \subset \mathbb{R}^{n+1}$ is an n-manifold

3) products of manifolds are manifolds: eg $S^n \times S^m$

4) $\mathbb{R}P^n = \mathbb{R}^{n+1} - \{(0, \dots 0)\} / \mathbb{R} - \{0\}$ is a closed n-manifold

$\mathbb{C}P^n = \mathbb{C}^{n+1} - \{(0, \dots 0)\} / \mathbb{C} - \{0\}$ is a closed $2n$-manifold

Th^m 1:

let R be a ring

1) M a closed connected manifold of dimension n

$$M \text{ is } R\text{-orientable iff } H_n(M;R) \cong R$$

2) M a compact connected n-manifold with boundary

$$M \text{ is } R\text{-orientable iff } H_n(M,\partial M;R) \cong R$$

Remarks: 1) we will define R-orientations and prove th^m in next section

2) all manifolds are $\mathbb{Z}/_2$ -orientable

3) the "standard" definition of orientable (say from differential topology) is equivalent to $\mathbb{Z}$ -orientable

4) a choice of generator for $H_n(M;R)$ is called a fundamental class of M, is denoted $[M]$, and determines an orientation

similarly for a generator $[M,\partial M]$ of $H_n(M,\partial M;R)$

Th^m 2:

Poincaré Duality: if M is a closed connected R-oriented n-mainfold with fundamental class $[M]$, then

$$H^p(M;R) \xrightarrow[{[M]\cap\cdot}]{} H_{n-p}(M;R)$$

is an isomorphism.

Poincaré-Lefschetz duality: if M is a compact connected R-oriented n-manifold with boundary and $[M,\partial M]$ is a fundamental class, then

$$\partial[M,\partial M] = [\partial M]$$

where $\partial : H_n(M,\partial M;R) \to H_{n-1}(\partial M;R)$ comes from the long exact sequence of the pair $(M,\partial M)$

moreover

$$\begin{tikzcd}
\cdots \arrow[r] & H^{p-1}(M) \arrow[r] \arrow[d, "{[M,\partial M]\cap\cdot}"] & H^{p-1}(\partial M) \arrow[r] \arrow[d, "{[\partial M]\cap\cdot}"] & H^p(M,\partial M) \arrow[r] \arrow[d, "{[M,\partial M]\cap\cdot}"] & H^p(M) \arrow[r] \arrow[d, "{[M,\partial M]\cap\cdot}"] & \cdots \\
\cdots \arrow[r] & H_{n-p+1}(M,\partial M) \arrow[r] & H_{n-p}(\partial M) \arrow[r] & H_{n-p}(M) \arrow[r] & H_{n-p}(M,\partial M) \arrow[r] & \cdots
\end{tikzcd}$$

commutes (up to sign) and vertical maps are isomorphisms.

we prove this later, for now we consider some consequences

Cor 3:

let M be a closed compact oriented n-manifold the cup product pairing

$$\left(H^p(M)/tor\right) \times \left(H^{n-p}(X)/tor\right) \to \mathbb{Z}$$
$$(\alpha, \beta) \longmapsto \alpha \cup \beta\,([M])$$

is nondegenerate and if α generator, then $\exists\, \beta$ st. $\alpha \cup \beta$ a generator of $\mathbb{Z} \cong H^n(X)$

(i.e. $(\alpha \cup \beta)[M] = 0 \ \forall \beta \Rightarrow \alpha = 0$)

Proof: Universal Coefficients Theorem says

$$0 \to Tor(H_{p-1}(M), \mathbb{Z}) \to H^p(M;\mathbb{Z}) \xrightarrow{\phi} Hom(H_p(M), \mathbb{Z}) \to 0$$
$$\alpha \longmapsto \phi(\alpha)(\sigma) = \alpha(\sigma)$$

so $H^p(M)/tor \overset{\phi}{\cong} Hom(H_p(M), \mathbb{Z}) \cong Hom(H_p(M)/tor, \mathbb{Z})$

Poincare Duality says

$$H^p(M)/tor \cong H_{n-p}(M)/tor$$
$$\alpha \longmapsto [M] \cap \alpha$$

$\therefore \quad H^p(M)/tor \xrightarrow{\Phi} Hom(H^{n-p}(M)/tor\, ; \mathbb{Z})$ an isomorphism

(composition of ϕ and P.D. $\cong$)

$$\alpha \longmapsto (H^{n-p}(M)/tor \to \mathbb{Z})$$
$$\beta \longmapsto \phi(\alpha)([M] \cap \beta) = \alpha([M] \cap \beta)$$
$$= \beta \cup \alpha([M])$$

so $\beta \cup \alpha([M]) = 0 \ \forall \beta \Rightarrow \Phi(\alpha) = 0 \Rightarrow \alpha = 0$

now if α a generator of $H^p(X)/tor$ then $\exists$ a homomorphism $\phi : H^p(X) \to \mathbb{Z}$ st. $\phi(\alpha) = 1$ by nondegeneracy of $\cup$, $Hom(H^p(X), \mathbb{Z}) \cong H^{n-p}(X)$ so $\exists\, \beta$ st. $\phi(\gamma) = \beta \cup \gamma([M])$ so $\beta \cup \alpha$ generates $H^n(X)$ ▦

Cor 4:

> the cohomology of $\mathbb{C}P^n$ is
> $$H^*(\mathbb{C}P^n;\mathbb{Z}) \cong \mathbb{Z}[x]/\langle x^{n+1}\rangle \quad \text{where } \deg x = 2$$

Proof: earlier we saw $\mathbb{C}P^n = (0\text{-cell}) \cup (2\text{-cell}) \cup \dots \cup (2n\text{-cell})$

so $H^k(\mathbb{C}P^n;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & k=0,2,\dots 2n \\ 0 & \text{otherwise} \end{cases}$

we have the inclusion $\imath : \mathbb{C}P^{n-1} \to \mathbb{C}P^n$

the long exact sequence of a pair gives

$\mathbb{C}P^n/\mathbb{C}P^{n-1} \cong S^{2n}$

$$k<2n \qquad 0 = H^k(\mathbb{C}P^n, \mathbb{C}P^{n-1}) \to H^k(\mathbb{C}P^n) \xrightarrow{\imath^*} H^k(\mathbb{C}P^{n-1}) \to H^{k+1}(\mathbb{C}P^n, \mathbb{C}P^{n-1}) = 0$$

so $\imath^*$ an isomorphism on H^k $\forall\ k<2n$

statement in theorem clearly true for $n=1$: $H^*(\mathbb{C}P^1) \cong \mathbb{Z}[x]/\langle x^2\rangle$

now if true for $\mathbb{C}P^{n-1}$ then

$$x \in H^2(\mathbb{C}P^{n-1}) \text{ s.t. } x^k \text{ generates } H^{2k}(\mathbb{C}P^{n-1}) \ \forall\ k=1,2\dots,n-1$$

so $\imath^*(\alpha)^k$ generates $H^{2k}(\mathbb{C}P^n)$ $\forall\ k<n$

$\therefore$ by Cor 3, $\imath^*(\alpha) \cup \imath^*(\alpha)^{n-1}$ must generate $H^{2n}(\mathbb{C}P^n)$ ∎

Cor 5:

> any homotopy equivalence $\mathbb{C}P^{2n} \to \mathbb{C}P^{2n}$ preserves orientation

Proof: Such an f induces an isomorphism on $H^2(\mathbb{C}P^{2n}) \cong \mathbb{Z}$

so $f^*(x) = \pm x$

$$\therefore f^*(x^{2n}) = (f^*(x))^{2n} = (\pm x)^{2n} = x^{2n}$$

so f^* takes a fundamental class to itself $\therefore$ preserves or$^{\underline{n}}$ ∎

(by universal coeff. theorem)

Cor 6:

> M a closed oriented n-manifold
> $$\text{Free } H_{n-k}(M) \cong \text{Free } H_k(M)$$
> $$\text{Tor } H_{n-k}(M) \cong \text{Tor } H_{k-1}(M)$$
> if n is odd then $\chi(M) = 0$ ← Euler characteristic
>
> if $n = 4m+2$ then $\chi(M)$ even

Proof: 1^{st} part is just Poincaré duality and Universal Coefficients

if $\dim M = 2m+1$, then

$$\chi(M) = \sum_{i=0}^{2m+1} (-1)^i \underbrace{\text{rank } H_i}_{b_i} = \sum_{i=0}^{m} (-1)^i b_i + \sum_{i=m+1}^{2m+1} (-1)^i b_i$$

$$= \sum_{i=0}^{m} (-1)^i b_i + \sum_{i=0}^{m} (-1)^{2m+1-i} b_{2m+1-i}$$

$$= \sum_{i=0}^{m} (-1)^i b_i + \sum_{i=0}^{m} (-1)^{1-i} b_i = 0$$ since $\text{Free } H_k = \text{Free } H_{2m+1-k}$

if $\dim M$ even then same computation gives

$$\chi(M) = b_{n/2} + \text{even number}$$

if $\dim M = 4m+2$ then $\chi(M)$ even $\Leftrightarrow$ b_{2m+1} even

Cor 3 $\Rightarrow$ $H^{2m+1}(M)/tor \times H^{2m+1}(M)/tor \to \mathbb{Z}$

a non-degenerate skew-symmetric pairing

linear algebra fact:

If V an k-dimensional vector space

$$q : V \times V \to \mathbb{R}$$

is a non-degenerate skew-symmetric pairing

then k is even

exercise: Prove this

hint: if W subspace of V

and $W^\perp = \{v \in V : q(v,w) = 0, \forall w \in W\}$

then $\dim V = \dim W + \dim W^\perp$

$(W^\perp)^\perp = W$

so fact $\Rightarrow$ $\chi(M)$ even. ∎

Cor 7:

$$\boxed{\begin{array}{l}\text{let } M^{2n} = \partial V^{2n+1} \text{ with } V \text{ compact, orientable, and } M \text{ connected} \\ \text{then } \operatorname{rank}(H^n(M)) \text{ is even and} \\ \dim(\ker i_* : H_n(M) \to H_n(V)) = \dim(\operatorname{im}(i^* : H^n(V) \to H^n(M))) = \tfrac{1}{2}\dim H^n(M) \\ \text{moreover any two classes in image } i^* \text{ cup to zero}\end{array}}$$

Proof:

$$\begin{tikzcd} H^n(V) \arrow[r, "\imath^*"] & H^n(M) \arrow[r, "\delta^*"] \arrow[d, "{[M]\cap \cdot}"', "\cong"] & H^{n+1}(V,M) \arrow[d, "\cong"', "{[V,\partial V]\cap \cdot}"] \\ & H_n(M) \arrow[r, "\imath_*"] & H_n(V) \end{tikzcd}$$

by Poincaré-Lefschetz duality

so $[M]\cap(\operatorname{im}\imath^*) = [M]\cap(\ker\delta_*) = \ker\imath_*$

and $\operatorname{rank}\imath^* = \dim(\operatorname{im}\imath^*) = \dim(\ker\imath_*) = \dim H_n(M) - \operatorname{rank}\imath_*$

$= \dim H_n(M) - \operatorname{rank}\imath^*$

since $\langle \imath^*\alpha, c\rangle = (\imath^*\alpha)(c) = \alpha(\imath_* c) = \langle\alpha, \imath_* c\rangle$

so $\imath^*, \imath_*$ are adjoints $\therefore$ have same rank (i.e. rank of matrix and transpose are equal.)

$\therefore \dim H_n(M) = 2\operatorname{rank}\imath^*$

now if $\alpha,\beta \in H^n(V)$ then

since $\delta^*\circ\imath^* = 0$

$$\delta^*(\imath^*(\alpha)\cup\imath^*(\beta)) = \delta^*\circ\imath^*(\alpha\cup\beta) = 0$$

but

$$\begin{tikzcd} H^{2n}(M) \arrow[r, "\delta^*"] \arrow[d, "\cong"] & H^{2n+1}(V,M) \arrow[d, "\cong"] \\ H_0(M) \arrow[r, "\imath_*"] & H_0(V) \end{tikzcd}$$

and $\imath_*$ injective $\therefore \delta^*$ injective

so $\imath^*(\alpha)\cup\imath^*(\beta) = 0$ ▦

Cor 8:

$$\boxed{\text{If } M=\partial V \text{ connected and } V \text{ compact and orientable then } \chi(M) \text{ even}}$$

Proof: if $\dim M$ odd then $\chi(M)=0$ ✓

if $\dim M = 4m+2$ then $\chi(M)$ even by Cor 6

if $\dim M = 4m$, then proof of Cor 6 $\Rightarrow$ ($\chi(M)$ even $\Leftrightarrow b_{2m}$ even)

but Cor 7 says it is even ▦

Cor 9: $\boxed{\mathbb{C}P^{2n} \text{ is not the boundary of a compact oriented } (4n+1)\text{-manifold.}}$

B. Fundamental classes of manifolds

let M be a manifold and R a ring with identity (usually $\mathbb{Z}$ or $\mathbb{Z}/2$)

if $x \in M$ and U open nbhd of x that is homeo. to $\mathbb{R}^n$ then by excision

$$H_n(M, M-\{x\}; R) \cong H_n(U, U-\{x\}; R) \cong H_n(\mathbb{R}^n, \mathbb{R}^n - \{x\}; R)$$

abuse of notation, really image of x under homeo.

the long exact sequence of the pair $(\mathbb{R}^n, \mathbb{R}^n - \{x\})$ gives

$n > 1$

$$\underset{\substack{\| \\ 0}}{H_n(\mathbb{R}^n)} \to H_n(\mathbb{R}^n, \mathbb{R}^n - \{x\}; R) \to \underset{\substack{\cong \\ H_{n-1}(S^{n-1}; R) \\ \cong \\ R}}{H_{n-1}(\mathbb{R}^n - \{0\}; R)} \to \underset{\substack{\| \\ 0}}{H_{n-1}(\mathbb{R}^n)}$$

so $\boxed{H_n(M, M-\{x\}; R) \cong R \quad \forall x \in M}$

we call a generator of $H_n(M, M-\{x\}; R)$ a <u>local R-orientation of M at x</u> and denote it by μ_x

<u>note</u>: if $R = \mathbb{Z}$ then every point has <u>two</u> local orientations
if $R = \mathbb{Z}/2$ " " <u>one</u> " "

<u>exercise</u>: If you know another definition of orientation at x show it is equivalent to a $\mathbb{Z}$-orientation at x

now if B is an open ball in a coordinate chart U, then as above

$$H_n(M, M-B; R) \cong R$$

moreover the inclusion $(M, M-\{B\}) \xrightarrow{i} (M, M-\{x\})$ for $x \in B$
induces an isomorphism

$$H_n(M, M-B; R) \xrightarrow{i_*} H_n(M, M-\{x\}; R)$$

thus a generator for either group determines one for the other so if x, y are in a ball B in a coordinate chart U in M then

$$H_n(M, M-\{x\}; R) \cong H_n(M, M-B; R) \cong H_n(M, M-\{y\}; R)$$

and isomorphisms induced by inclusion

so a local orientation at x determines one at y

an R-orientation on M is a choice of local R-orientations μ_x for all $x \in M$ s.t. for all open balls B in coordinate charts of M, $\exists\, \mu_B$ a generator of $H_n(M, M-B; R)$ s.t. $\mu_x = \imath_*(\mu_B)$ $\forall x \in B$ (where $\imath : (M, M-B) \to (M, M-\{x\})$)

(i.e. a consistant choice of local R-orientations)

if an R-orientation exists on M, we say M is R-orientable, if $R = \mathbb{Z}$, we say M is orientable.

exercise: If you know another definition of orientable, show it is equivalent to this definition

lemma 10:

$$\boxed{\text{all manifolds have a unique } \mathbb{Z}/2\text{-orientation}}$$

Proof: $\forall x \in M$, μ_x must be the unique generator of $\mathbb{Z}/2$

similarly μ_B for any open ball in a coordinate chart

$\therefore$ $\imath_*(\mu_B) = \mu_x$ $\forall x \in B$ ▦

lemma 11:

> Suppose M is R-orientable and connected
>
> if two R-orientations agree at some $x \in M$, then they are the same.
>
> (i.e. if M is R-orientable, then an R-orientation is determined by a choice of local R-orientation at any point $x \in M$)

Proof: let $\{\mu_x\}_{x \in M}$ and $\{\tilde{\mu}_x\}_{x \in M}$ be two R-orientations on M.

assume $\exists\, x_0 \in M$ st. $\mu_{x_0} = \tilde{\mu}_{x_0}$

let $S = \{x \in M : \mu_x = \tilde{\mu}_x\}$

$S \neq \emptyset$ since $x_0 \in S$

$\underline{S \text{ is open}}$: $x \in S$ then $\exists$ open ball B s.t. $x \in B \subset U$ ← coord. chart

let μ_B be generator of $H_n(M, M-B; R)$ s.t. $\iota_*(\mu_B) = \mu_x$

$\tilde{\mu}_B$ " " " " $\iota_*(\tilde{\mu}_B) = \tilde{\mu}_x$

since ι_* isomorphism, and $\mu_x = \tilde{\mu}_x$ we have $\mu_B = \tilde{\mu}_B$

now for any $y \in B$ we have $\mu_y = \iota_*(\mu_B) = \iota_*(\tilde{\mu}_B) = \tilde{\mu}_y$

so $B \subset S$

Similarly S is closed

so $S = M$ since M connected, and orientations agree.

for the parenthetical statement:

let $\{\mu_x\}_{x \in M}$ be an R-orientation

let $\tilde{\mu}_{x_0}$ be a choice of generator for $H_n(M, M-\{x_0\}; R)$

so $\exists r \in R$ s.t. $\tilde{\mu}_{x_0} = r\mu_{x_0}$ and r a unit

$\therefore \{r\mu_x\}_{x \in M}$ an R-orientation on M determined by $\tilde{\mu}_{x_0}$ ▦

Cor 12

> If M is orientable and connected, then M has exactly two orientations.

Proof: $\mathbb{Z}$ has two units $+1$ and -1 ▦

Th^m 13:

> let M be a closed connected n-manifold
>
> 1) if M is R-orientable then the map $\iota: (M, \emptyset) \to (M, M-\{x\})$ induces an isomorphism
>
> $$\iota_*: H_n(M; R) \to H_n(M, M-\{x\}; R) \cong R$$
>
> for all $x \in M$
>
> 2) if M is <u>not</u> R-orientable the inclusion above induceses an <u>injective</u> map
>
> $$\iota_*: H_n(M; R) \to H_n(M, M-\{x\}; R)$$
>
> with image $= \{r \in R : 2r = 0\}$ for all $x \in M$

3) $H_i(M;R)=0 \quad \forall i>n$

an element $[M] \in H_n(M;R)$ whose image in $H_n(M, M-\{x\};R)$ is a generator for all $x \in M$ is called a fundamental class of M with coefficients in R.

note: by lemma 11, for connected M, the fundamental classes of M are in one-to-one correspondence with R-orientations.

for R-orientable manifolds M a choice of generator for $H_n(M;R)$ is sometimes called an R-orientation on M.

Cor 14:

1) if M is a closed, connected, orientable n-manifold then

$$H_n(M;\mathbb{Z}) \cong \mathbb{Z}$$
$$H_n(M;\mathbb{Z}/_2) \cong \mathbb{Z}/2$$

2) if M is a closed, connected n-manifold that is not-orientable then

$$H_n(M;\mathbb{Z}) = 0$$
$$H_n(M;\mathbb{Z}/_2) \cong \mathbb{Z}/2$$

Proof: clear from lemma 10 and theorem 13 ▦

to prove theorem we need some preliminary work

let $M_R = \{\alpha_x \mid x \in M, \ \alpha_x \in H_n(M, M-\{x\};R)\}$

we put a topology on M_R as follows

for each open ball B in a coordinate chart of M and each $\alpha \in H_n(M, M-B;R)$

let $U(\alpha, B) = \{ i^x_*(\alpha) \}_{x \in B}$ where $i^x : (M, M-B) \to (M, M-\{x\})$ is inclusion

exercise: 1) Show this is a basis for a topology on M_R

2) $M_R \xrightarrow{\pi} M : \alpha_x \longmapsto x$ is a covering map (M_R might be disconnected)

3) if $\sigma: M \to M_R$ is continuous s.t. $\pi \circ \sigma = id_M$
(we call such a map a section of M_R)
and $\forall x$, $\sigma(x)$ is a generator of $H_n(M, M-\{x\}; R)$
then σ defines an R-orientation on M
similarly an R-orientation on M gives a σ as above.

lemma 15:

let M be an n-manifold and $A \subset M$ a compact subset.

1) if $\sigma: M \to M_R$ is a section of M_R, then $\exists!$ class $\alpha_A \in H_n(M, M-A; R)$ whose image in $H_n(M, M-\{x\}; R)$ is $\sigma(x)$ $\forall x \in A$.

2) $H_q(M, M-A; R) = 0$ $\forall q > n$

Proof of Th^{m} 13:

If $A = M$ in lemma 15 then Th^{m} 13 part 3) follows from lem 15 part 2)

for part 1) of Th^{m} 13

let $\Gamma_R = \{$sections of $M_R\}$

note: 1) sum of two sections is a section

2) if σ a section and $r \in R$, then $r\sigma$ a section

so Γ_R is an R-module

lemma 15 part 1) $\Rightarrow \exists$ a well-defined map of R-modules
$$\Gamma_R \xrightarrow{\phi} H_n(M; R)$$

Claim: ϕ an isomorphism

indeed, if $\alpha \in H_n(M; R)$, then define $\sigma_\alpha(x) = i_*^x(\alpha)$
where $i^x: M \to (M, M-\{x\})$

exercise: σ_α a section and $\phi(\sigma_\alpha) = \alpha$

$\therefore \phi$ onto.

now if $\sigma \in \Gamma_R$ and $\phi(\sigma) = 0 \in H_n(M; R)$
then $\sigma(x) = 0$ $\forall x \in M$, $\therefore \sigma = 0$ in Γ_R
so ϕ injective ✓

just as in the proof of lemma 11, if M connected, then two sections of M_R are the same if they agree at one point.

$\therefore$ if we fix $x_0 \in M$ the map

$$\Gamma_R \longrightarrow R = \pi^{-1}(x_0) = H_n(M, M-\{x\}; R)$$
$$\sigma \longmapsto \sigma(x_0)$$

is injective

if M is R-orientable, $\exists$ a section σ, s.t. $\sigma(x_0)$ a generator of $H_n(M, M-\{x_0\}; R)$

$\therefore$ above map onto.

and $H_n(M; R) \cong \Gamma_R \cong \underline{R}$

for part 2) of Th$^{\underline{m}}$4 see Hatcher (or work it out yourself!)

Proof of lemma 15:

Claim 1: If lemma true for A and B and $A \cap B$, then true for $A \cup B$

Claim 2: If lemma true for $M = \mathbb{R}^n$, then true for all manifolds

Claim 3: lemma is true for $\mathbb{R}^n$

Clearly lemma follows from claims.

Proof of Claim 1: note $(M, M-(A\cup B)) = (M, (M-A)\cap(M-B))$

so Mayer-Vietoris gives

$$H_{i+1}(M, M-(A\cap B)) \to H_i(M, M-(A\cup B)) \to H_i(M, M-A) \oplus H_i(M, M-B)$$

$i > n$ $\quad$ $H_{i+1}(M, M-(A\cap B)) = 0$, $\quad H_i(M, M-A) = 0$, $\quad H_i(M, M-B) = 0$

so $H_i(M, M-(A\cup B)) = 0 \quad i > n$

for $i = n$

$$0 \to H_n(M, M-(A\cup B)) \xrightarrow{\Phi} H_n(M, M-A) \oplus H_n(M, M-B) \xrightarrow{\Psi} H_n(M, M-(A\cap B))$$

where $\Psi(\alpha, \beta) = \alpha - \beta$ and $\Phi(\alpha) = (\alpha, \alpha)$

now suppose σ is a section of M_R

by assumption $\exists!\ \alpha_A \in H_n(M, M-A)$ and

$\alpha_B \in H_n(M, M-B)$

s.t. $\iota_x^*(\alpha_A) = \sigma(x) = \iota_x^*(\alpha_B)$ $\quad \forall x \in A$ or B

so $\Psi(\alpha_A, \alpha_B)$ is the class in $H_n(M, M-(A\cap B))$ corresponding to the section $\tilde{\sigma}$ that is always 0

so it is 0

by exactness $\exists\ \alpha_{A\cup B} \in H_n(M, M-(A\cup B))$ s.t. $\Phi(\alpha_{A\cup B}) = (\alpha_{A\cup B}, \alpha_{A\cup B})$

$\therefore\ \iota_x^*(\alpha_{A\cup B}) = \sigma(x) \quad \forall x \in A\cup B$ $\qquad = (\alpha_A, \alpha_B)$

to see $\alpha_{A\cup B}$ unique, note that if $\tilde{\alpha}$ was another such class, then $\iota_x^*(\alpha_{A\cup B} - \tilde{\alpha}) = 0 \quad \forall x \in A\cup B$

$\therefore\ \alpha_{A\cap B} - \tilde{\alpha}$ as a class in $H_n(M, M-A)$ or $H_n(M, M-B)$ also has this property

$\therefore$ by uniqueness for A and B $\alpha_{A\cup B} - \tilde{\alpha} = 0$ in $H_n(M, M-A)$ and $H_n(M, M-B)$

thus injectivity of $\Phi \Rightarrow \alpha_{A\cup B} - \tilde{\alpha} = 0$ ✓

Proof of Claim 2: if $A \subset M$ compact, then we can write $A = A_1 \cup .. \cup A_k$ where A_i are compact and each is in a coordinate chart U_i

$$H_i(M; M-A_\ell) \cong H_i(U_\ell, U_\ell - A_\ell) \cong H_i(\mathbb{R}^n, \mathbb{R}^n - A_\ell)$$

excision

so if lemma true for compact subsets of $\mathbb{R}^n$ then true for (M, A_i) and $(M, A_i \cap A_j)$

still in $\mathbb{R}^n$

$\therefore$ by Claim 1 true for $(M, A_1 \cup A_2)$

since $(A_1 \cup A_2) \cap A_3 \subset U_3$ can continue inductively

so lemma true for (M, A) ✓

Proof of Claim 3: if A is convex then $\mathbb{R}^n - A$ and $\mathbb{R}^n - \{x\}$ both retract onto a sphere centered at x

$$\therefore\ H_i(\mathbb{R}^n, \mathbb{R}^n - A) \cong H_{i-1}(\mathbb{R}^n - A) \cong H_{i-1}(S^{n-1}) \cong H_{i-1}(\mathbb{R}^n - \{x\})$$
$$\cong H_i(\mathbb{R}^n, \mathbb{R}^n - \{x\})$$

so part 2) of lemma clear

<u>exercise</u>: $\mathbb{R}^n_R = \mathbb{R}^n \times R$ (R has discrete topology)

so sections of $\mathbb{R}^n_R$ are constant and $\therefore$ 1) also true.

by Claim 1, lemma now true for A = finite unions of convex sets

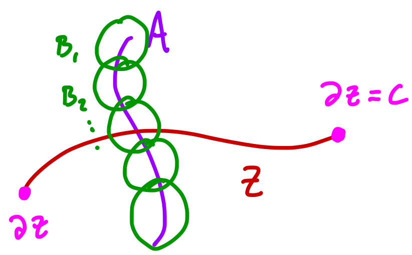
now let A be any compact set in $\mathbb{R}^n$

let z be a cycle that represents $\alpha \in H_i(\mathbb{R}^n, \mathbb{R}^n - A; R)$

thus $\partial z \in C_{i-1}(\mathbb{R}^n - A)$

let C = union of images of simplicies in ∂z

since C, A are compact $\exists$ some r s.t. $d(x,y) > r$ $\forall x \in C$ any $y \in A$



by compactness of A we can find finitely many closed r-balls $B_1, \dots, B_k$ that cover A and $C \cap B_i = \emptyset$

let $K = \cup B_i$

note z defines an element $\alpha_K \in H_i(\mathbb{R}^n, \mathbb{R}^n - K)$ that maps to $\alpha \in H_i(\mathbb{R}^n, \mathbb{R}^n - A)$ by inclusion

since B_i are convex, if $i > n$, then $\alpha_K = 0$ $\therefore$ $\alpha = 0$

if $i = n$ and σ a section of $\mathbb{R}^n_R$ then $\exists\, \alpha_K \in H_n(\mathbb{R}^n, \mathbb{R}^n - K)$

s.t. $i^x_*(\alpha_K) = \sigma(x)$ $\forall x \in K$

but
$$\begin{tikzcd} H_n(\mathbb{R}^n, \mathbb{R}^n - K) \arrow[r, "i_*"] \arrow[rr, bend right, "i^x_*"'] & H_n(\mathbb{R}^n, \mathbb{R}^n - A) \arrow[r, "i^x_*"] & H_n(\mathbb{R}^n, \mathbb{R}^n - \{x\}) \end{tikzcd}$$

so $\alpha = i_*(\alpha_K)$ is desired element now suppose α, α' are two such elements

then $\iota_*^x(\alpha - \alpha') = 0 \quad \forall x \in A$

if $y \in K$ then $\exists$ some B_i and $x \in A \cap B_i$ s.t. $y \in B_i$

then

$$\begin{tikzcd}
H_n(\mathbb{R}^n, \mathbb{R}^n - \{x\}; R) & & H_n(\mathbb{R}^n, \mathbb{R}^n - \{y\}; R) \\
& H_n(\mathbb{R}^n, \mathbb{R}^n - B_i; R) \arrow[ul, "\iota_*^x", "\cong"'] \arrow[ur, "\cong", "\iota_*^y"']
\end{tikzcd}$$

so $\quad \iota_*^y(\alpha - \alpha') = \iota_*^y(\iota_*^x)^{-1}(0) = 0$

$\therefore \quad \iota_*^y(\alpha - \alpha') = 0 \quad \forall y \in K$

$\therefore$ from above $\alpha - \alpha' = 0$ and we have uniqueness

Remark: a fundamentol class $[M, \partial M]$ can similarly be considered for compact manifolds with boundary

C. Algebraic limits and Proof of Duality

a set I is a <u>directed set</u> if $\exists$ a partial order $\iota \leq \iota'$ defined on certain pairs in I s.t. $\forall \iota, \iota' \in I, \exists \iota'' \in I$ s.t. $\iota \leq \iota''$ and $\iota' \leq \iota''$

examples: 1) I = subsets of a set X

$\leq$ given by inclusion

2) $I = \mathbb{Z}$ with $\leq$ standard inequality

now suppose $\{M_\iota\}_{\iota \in I}$ is a family of R-modules indexed by a directed set I s.t. $\forall \iota \leq \iota'$, $\exists$ a homomorphism

$$\phi_{\iota', \iota} : M_\iota \to M_{\iota'}$$

s.t. $\phi_{\iota'', \iota'} \circ \phi_{\iota', \iota} = \phi_{\iota'', \iota}$ if $\iota \leq \iota' \leq \iota''$

and $\phi_{\iota, \iota} = id_{M_\iota}$

this is called a <u>directed system of modules</u>

the <u>direct limit</u> of $\{M_\iota\}_{\iota \in I}$ is a module M together with homomorphisms

$$\phi_\iota : M_\iota \longrightarrow M$$

s.t. $\quad \phi_{i'} \circ \phi_{i',i} = \phi_i \quad \forall\, i \leq i'$

and for any module N and maps $\psi_i : M_i \to N$ satisfying $\psi_{i'} \circ \phi_{i,i'} = \psi_i$

$\exists!$ homeomorphism $\psi : M \to N$ s.t. $\psi_i = \psi \circ \phi_i$

$$\begin{tikzcd} & & N & & \\ & M_1 \arrow[ur, "\psi_1"] \arrow[r, "\phi_{21}"] \arrow[dr, "\phi_1"'] & M_2 \arrow[u, "\psi_2"] \arrow[r, "\phi_{32}"] \arrow[d, "\phi_2"] & M_3 \arrow[ul, "\psi_3"'] \arrow[r] \arrow[dl, "\phi_3"] & \cdots \\ & & M & \cdots & \end{tikzcd} \Rightarrow \exists!\ \psi : M \to N$$

exercise: any two direct limits are isomorphic

we denote the direct limit by $\varinjlim M_i$

lemma 16: $\boxed{\text{direct limits exist}}$

Proof: let $M^+ = \bigoplus M_i$

and $\phi_i^+ : M_i \to M^+$

$x \mapsto$ I-tuple with i^{th} cpt $= x$ others 0

let $J =$ submodule of M^+ generated by $\{\phi_{i'}^+ \circ \phi_{i',i}(x) - \phi_i^+(x)\}$ $\forall x \in M_i$ and $i, i' \in I$

set $M = M^+/J$

and $\phi_i = \pi \circ \phi_i^+$ where $\pi : M^+ \to M$ is the quotient map

exercise: check (M, ϕ_i) is the direct product ▦

exercises:

1) if M_i are all submodules of M and $i \leq i' \Rightarrow \phi_{i,i'} : M_i \to M_{i'}$ is inclusion

then $\varinjlim M_i = \bigcup M_i$

2) if $\exists m \in I$ s.t. $i \leq m$ $\forall i \in I$, then $\phi_m : M_m \to \varinjlim M_i$ is an isomorphism

3) suppose $\forall i \in I$, $M_i = N_i \oplus P_i$ and $\phi_{i,i} = \psi_{i,i} \oplus \rho_{i,i}$ $\forall i \leq i'$

let $N = \varinjlim N_i$, $P = \varinjlim P_i$, $M = \varinjlim M_i$

then we get $\psi : N \to M$ and $\rho : P \to M$ s.t.

$\psi \circ \psi_i = \phi_i|_N$, $\rho \circ \rho_i = \phi_i|_P$

and $\psi \oplus \rho : N \oplus P \to M$ is an isomorphism

4) a subset $J \subset I$ is called <u>final</u> if $\forall i \in I, \exists j \in J$ s.t. $i \leq j$

applying definition to $\phi_j : M_j \to M$ we get a homomorphism

$$\lambda : \varinjlim_{J} M_j \longrightarrow \varinjlim_{I} M_i$$

Show λ is an isomorphism

5) if $\{A_i\}_{i\in I}, \{B_i\}_{i \in I}, \{C_i\}_{i\in I}$ are directed systems and $\forall i$ we have

$$A_i \xrightarrow{\lambda_i} B_i \xrightarrow{\rho_i} C_i \quad \circledast$$

s.t. $\forall i \leq i'$

$$\begin{tikzcd} A_i \arrow[r, "\lambda_i"] \arrow[d, "\phi^A_{ii'}"'] & B_i \arrow[r, "\rho_i"] \arrow[d, "\phi^B_{ii'}"] & C_i \arrow[d, "\phi^C_{ii'}"] \\ A_{i'} \arrow[r, "\lambda_{i'}"] & B_{i'} \arrow[r, "\rho_{i'}"] & C_{i'} \end{tikzcd}$$

is commutative

then in the limit we get homomorphisms

$$\varinjlim A_i \xrightarrow{\lambda} \varinjlim B_i \xrightarrow{\rho} \varinjlim C_i \quad (**)$$

Show if $\circledast$ is exact at B_i $\forall i$, then $(**)$ is exact

lemma 17:

> let $\{U_\alpha\}$ be a directed system of subsets of X st. any compact set $K \subset X$ is in some U_α
>
> Then $\varinjlim H_i(U_\alpha ; R) \cong H_i(X; R)$

Proof: Clearly we have inclusion maps $H_i(U_\alpha ; R) \to H_i(X;R)$ $\forall \alpha$

$\therefore$ get map $\varinjlim H_i(U_\alpha ; R) \to H_i(X;R)$

if $[\sigma] \in H_i(X;R)$ then $\operatorname{im} \sigma \subset U_{\alpha'}$ some α'

so $H_i(U_{\alpha'} ; R) \to H_i(X;R)$ hits $[\sigma]$

but
$$\begin{tikzcd} H_i(U_\alpha ; R) \arrow[rr] \arrow[dr] & & H_i(X;R) \\ & \varinjlim H_i(U_\alpha ; R) \arrow[ur] & \end{tikzcd}$$

so map surjective

exercise: check injective (similar) ▦

now if M is an n-manifold

let $I = \{$all compact subsets of $M\}$ directed by inclusion

<u>note</u>: $K \leq K' \Rightarrow (M, M-K') \xrightarrow{i} (M, M-K)$ inclusion

$\Rightarrow H^q(M, M-K; R) \xrightarrow{i^*} H^q(M, M-K'; R)$

$\therefore \{ H^q(M, M-K; R) \}$ is a directed system of R-modules

define $H^q_c(M; R) = \varinjlim H^q(M, M-K; R)$

<u>note</u>: 1) if M is compact, then M is final in I

$\therefore H^q(M; R) \cong H^q_c(M; R)$

2) you can think of elements of $H^q_c(M; R)$ as cochains that vanish off of some compact subset of M

so we call $H^q_c(M; R)$ the <u>q-cohomology with compact support</u>

fix an R-orientation on M

recall this means a section $\sigma : M \to M_R$ s.t. $\sigma(x)$ generates $H_n(M, M-\{x\})$

let K be a compact set in M

then lemma 15 gives a class $\alpha_K \in H_n(M, M-K; R)$

s.t. $i^x_*(\alpha_K) = \sigma(x)$ where $i^x : (M, M-K) \to (M, M-\{x\})$

the cap product gives

$$H_n(M, M-K; R) \times H^p(M, M-K; R) \longrightarrow H_{n-p}(M; R)$$

so $\alpha_K \cap \cdot$ gives a map

$$H^p(M, M-K; R) \to H_{n-p}(M; R)$$
$$\gamma \longmapsto \alpha_K \cap \gamma$$

if $K \subset K'$ then

$$\begin{tikzcd} H^p(M, M-K; R) \arrow[dr, "\alpha_K \cap \cdot"] \arrow[d] & \\ H^p(M, M-K'; R) \arrow[r, "\alpha_{K'} \cap \cdot"'] & H_{n-p}(M; R) \end{tikzcd}$$

is commutative

why? think about lem 15

so we get a map $H^p_c(M; R) \xrightarrow{D_M} H_{n-p}(M; R)$

Th^{m} 18 (Poincaré Duality Revised):

> if M is an R-oriented n-manifold, then
> $$D_M : H^p_c(M) \to H_{n-p}(M)$$
> is an isomorphism

clearly Th^{m} 2 part 1) follows from this since if M compact $H^p_c(M;R) \cong H^p(M;R)$ and map is given one since $\alpha_M = [M]$

Proof:

Step I: If th^{m} true for open sets U, V, and $U \cap V$ in M then true for $U \cup V$

Step II: let $\{U_i\}$ be a system of open sets totally ordered by inclusion set $U = \bigcup U_i$. If th^{m} true for all U_i then true for U

Step III: th^{m} true for any open $U \subset$ coordinate chart of M.

once we have established Steps I–III we are done as follows:

recall **Zorn's lemma**: if P is a partially ordered set such that every chain has an upper bound, then P has a maximal element

totally ordered subset

some elt greater than (or equal to) all elts in chain

this is equivalent to the axiom of choice

now by Step II and Zorn's lemma there is a maximal element U in M for which th^{m} is true

if $M \neq U$, then let $x \in M - U$

$\exists$ an open set V s.t. $x \in V \subset X - U$ s.t. V is in a coord. chart $\cong \mathbb{R}^n$

$\therefore$ th^{m} true for $U \cup V$ by Step I $\times$ maximality of U

$\therefore$ $U = M$ and we are done

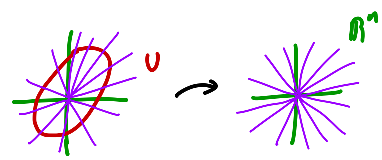
Step III is heart of proof

Proof of Step III: suffices to prove for open set in $\mathbb{R}^n$

Case A: let U be convex open set in $\mathbb{R}^n$

exercise: U homeomorphic to $\mathbb{R}^n$

Hint:



so by naturality of everything just need to check for $\mathbb{R}^n$

let K_r be the closed (compact) ball of radius r in $\mathbb{R}^n$ (centered at 0)

$\{K_r\}_{r\in(0,\infty)}$ is final in all compact sets in $\mathbb{R}^n$

$\therefore \quad H^p_c(\mathbb{R}^n) \cong \varinjlim_{K_r} H^p(\mathbb{R}^n, \mathbb{R}^n - K_r)$

and each $H^p(\mathbb{R}^n, \mathbb{R}^n - K_r) \cong 0 \quad \forall p \neq n$

$\therefore \quad H^p_c(\mathbb{R}^n) = 0$ for $p \neq n$

also $H_{n-p}(\mathbb{R}^n) = 0 \quad \forall\ p \neq n \quad \therefore$ th$^{\underline{m}}$ true if $p \neq n$

for $p = n$ we get $H^n_c(\mathbb{R}^n) \cong R$ and $H_0(\mathbb{R}^n) \cong R$

now consider $\alpha_{K_r} \cap \cdot : H^n(\mathbb{R}^n, \mathbb{R}^n - K_r) \to H_0(\mathbb{R}^n)$

recall $H_n(\mathbb{R}^n, \mathbb{R}^n - K_r) \times H^n(\mathbb{R}^n, \mathbb{R}^n - K_r) \to H_0(\mathbb{R}^n)$

$(\alpha, \beta) \longmapsto \beta({}_n\alpha) = \beta(\alpha)$

eval on front n-face

now α_{K_r} is a generator of $H_n(\mathbb{R}^n, \mathbb{R}^n - K_r)$

(or couldn't map to generator $\sigma(x)$ of $H_n(\mathbb{R}^n, \mathbb{R}^n - \{x\})$)

$\therefore$ its dual β in $\mathrm{Hom}(H_n(\mathbb{R}^n, \mathbb{R}^n - K_r); R) \cong H^n(\mathbb{R}^n, \mathbb{R}^n - K_r)$

evaluates to 1 on α_{K_r}

so β generates $H^n(\mathbb{R}^n, \mathbb{R}^n - K_r)$

and $\alpha_{K_r} \cap \cdot$ is an isomorphism $\therefore D$ an isomorphism

Case B: General open $U \subset \mathbb{R}$

let $\{b_i\}$ be a countable dense set in U

let U_i be balls centered at b_i contained in U

so $U = \bigcup U_i$

set $V_1 = U_1$ and $V_i = V_{i-1} \cup U_i \quad \forall\, i > 1$

th$^{\underline{m}}$ true for each V_i by following claim

Claim: th$^{\underline{m}}$ true for any finite union of convex sets

Proof of Claim: induct on number of convex sets

true for 1 set by Case A

assume true for union of any q convex sets for $q<k$

now given $A_1, \dots, A_k$

we know th^{m} true for $A_1 \cup \dots \cup A_{k-1}$ by induction

A_k by Case A

and $A_k \cap (A_1 \cup \dots \cup A_{k-1}) = \underbrace{(A_k \cap A_1)}_{} \cup \dots \cup \underbrace{(A_k \cap A_{k-1})}_{}$

each convex

only $k-1$ sets

so by induction

$\therefore$ Step I $\Rightarrow$ th^{m} true for $A_1 \cup \dots \cup A_k$ ✓

now th^{m} true for U by Step II ✓

Proof of Step II:

from lemma 17 $\varinjlim H_{n-p}(U_i) \to H_{n-p}(U)$ induced by inclusion is an isomorphism

similarly if $U_i \subset U_j$ and $K \subset U_i$ compact, then excision gives an isomorphism

$$H^p(U_j, U_j - K) \to H^p(U_i, U_i - K)$$

the inverse gives maps $H^p(U_i, U_i - K) \to H^p(U_j, U_j - K) \to H^p_c(U_j)$

$\therefore$ we get a map $H^p_c(U_i) \to H^p_c(U_j)$

Claim: $\varinjlim_i H^p_c(U_i) = H^p_c(U)$ and $D_U = \varinjlim_i D_{U_i}$

note that given the claim since $D_{U_i} : H^p_c(U_i) \to H_{n-p}(U_i)$ an isomorphism $\forall i$, $D_U : H^p_c(U) \to H_{n-p}(U)$ an isomorphism too ✓

Proof of Claim: as above we get maps $H^p_c(U_i) \to H^p_c(U)$

$\therefore$ we have a map $\varinjlim_i H^p_c(U_i) \xrightarrow{G} H^p_c(U)$

now for any compact set $K \subset U$ $\exists j$ s.t. $K \subset U_i$ $\forall i \geq j$

so we get a map $H^p(U, U-K) \to H^p(U_i, U_i - K) \to H^p_c(U_i)$

$\to \varinjlim_i H^p_c(U_i)$

$\therefore$ we have a map $H^p_c(U) \xrightarrow{H} \varinjlim H^p_c(U_i)$

exercise: show G and H are inverses of eachother.

also check claim about D_U

Proof of Step I:

let K be any compact set in U

$\quad L$ " " V

set $B = U \cap V$ and $Y = U \cup V$

<u>note:</u> $(Y, Y-(K\cap L)) = (Y, Y-K) \cup (Y, Y-L)$

$(Y, Y-(K\cup L)) = (Y, Y-K) \cap (Y, Y-L)$

so Mayer-Viétoris for $(Y, U-K)$ and $(Y, V-L)$ gives

$$\begin{tikzcd}
H^p(Y,Y-(K\cap L)) \arrow[r] \arrow[d, "\cong"] & H^p(Y,Y-K)\oplus H^p(Y,Y-K) \arrow[r] \arrow[d, "\cong"] & H^p(Y,Y-(K\cup L)) \arrow[r, "\delta"] \arrow[dd, "\alpha_{K\cup L}\cap\cdot"] & H^{p+1}(Y,Y-(K\cup L)) \arrow[d, "\cong"] \\
H^p(B,B-(K\cap L)) \arrow[r] \arrow[d, "\alpha_{K\cap L}\cap\cdot"] & H^p(U,U-K)\oplus H^p(V,V-L) \arrow[d, "\alpha_K\cap\cdot\,\oplus\,\alpha_L\cap\cdot"] & & H^{p+1}(B,B-(K\cap L)) \arrow[d, "\alpha_{K\cap L}\cap\cdot"] \\
H_{n-p}(B) \arrow[r] & H_{n-p}(U)\oplus H_{n-p}(V) \arrow[r] & H_{n-p}(Y) \arrow[r, "\partial"] & H_{n-p-1}(B)
\end{tikzcd}$$

exercise: 1) first two squares commute (easy since all maps are inclusions or cap products)

2) last square commutes upto sign

<u>Hint:</u> a) recall ∂ is defined as follows:

Th$^{\underline{m}}$ II.19 given $z \in H_{n-p}(Y)$ you can write $z = a+b$ for $a \in C_{n-p}(U)$, $b \in C_{n-p}(V)$

then $\partial[z] = [\partial a]$

b) as in proof of Th$^{\underline{m}}$ II.11 can use Lebesgue number and barycentric subdivision to find chains

$\alpha_K, \alpha_L, \alpha_{K\cap L}$ s.t. $\alpha_{K\cup L} = \alpha_K + \alpha_L + \alpha_{K\cap L}$

you can now compute $\partial \circ (\alpha_{K\cup L}\cap\cdot)$

similarly compute $(\alpha_{K\cap L}\cap\cdot)\circ\delta$

note any compact set in $B = U\cap V$ is $K\cap L$ for some K & L as above and similarly for $Y = U\cup V$

so above gives the following diagram commutes upto sign

$$\begin{tikzcd}
H^p_c(B) \arrow[r, "\Phi"] \arrow[d, "\cong"', "D_B"] & H^p_c(U)\oplus H^p_c(V) \arrow[r, "\Psi"] \arrow[d, "\cong"', "D_U\oplus D_V"] & H^p_c(Y) \arrow[r, "\delta"] \arrow[d, "D_Y"] & H^{p+1}_c(B) \arrow[d, "\cong"', "D_B"] \\
H_{n-p}(B) \arrow[r, "\Phi'"] & H_{n-p}(U)\oplus H_{n-p}(V) \arrow[r, "\Psi'"] & H_{n-p}(Y) \arrow[r, "\partial"] & H_{n-p+1}(B)
\end{tikzcd}$$

isomorphism by assumption

Claim: D_Y is isomorphism

indeed if $\alpha \in H^p_c(Y)$ and $D_Y \alpha = 0$

then $0 = \partial D_Y \alpha = D_B \delta \alpha \Rightarrow \delta\alpha = 0$

$\therefore \exists\ (a,b)$ s.t. $\Psi(a,b) = \alpha$

and $\Psi'(D_U a, D_V b) = D_Y \Psi(a,b) = D_Y \alpha = 0$

$\therefore \exists\ c$ s.t. $\Phi'(c) = (D_U a, D_V b)$

and c' s.t. $D_B c' = c$

now $D_U \oplus D_V(\Phi(c')) = \Phi'(D_B c') = (D_U a, D_U b)$

and $\Phi(c') = (a,b)$ since $D_U \oplus D_V$ an $\cong$

finally $\alpha = \Psi(a,b) = \Psi(\Phi(c')) = 0$ and D_Y injective

exercise: Show D_Y surjective

Next steps in algebraic topology

I) Homotopy Groups

homotopy classes of based maps

recall $\pi_n(X, x_0) = [S^n, X]_0$

and $f: X \to Y$ induces a homomorphism $f_*: \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$ $\forall n$

- Whitehead Th^{m}: if $f: X \to Y$ is a map between CW complexes and $f_*: \pi_n(X) \to \pi_n(Y)$ an isomorphism $\forall n$ then f is a homotopy equivalence!
- for $n \geq 2$, $\pi_n(X, x_0)$ is an abelian group
- hard to compute in general

eg

$n=$	1	2	3	4	5	6	7	8	9	10	...
$\pi_n(S^2)$	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	...

- Given any abelian group G and integer n, $\exists$ a space $K(G,n)$ such that

$$\pi_k(K(G,n)) \cong \begin{cases} G & k=n \\ 0 & k \neq n \end{cases}$$

for such a space we have

$$H^n(X; G) \cong [X, K(G,n)]$$

Brown representation th^{m} relates homotopy and cohomology!

- Hurewicz Th^{m}: if $\pi_k(X) = 0$ $\forall k<n$, then $\tilde{H}_k(X) = 0$ $\forall k<n$ and $\pi_n(X) \cong H_n(X)$
- a map $p: E \to B$ is a fibration if it has the homotopy lifting property

i.e. if $f_t: X \to B$ is a homotopy and $\tilde{f}_0$ is a lift of f_0 then $\exists$ a lift $\tilde{f}_t$ for all t

all fiber bundles are fibrations

if $p: E \to B$ a fibration, then there is a long exact sequence

$$\dots \to \pi_n(F, x_0) \to \pi_n(E, x_0) \xrightarrow{p_*} \pi_n(B, p(x_0)) \to \pi_{n-1}(F, x_0) \to \dots$$

where $x_0 \in E$, $F = p^{-1}(p(x_0))$

II) Spectral sequences

computing the homology of a fibration is much harder!

a group (or module) is bigradded is a collection of groups $E=\{E_{s,t}\}$ indexed by pairs of integers

a map $d: E \to E$ has bidegree (a,b) if $d(E_{s,t}) \subset E_{s+a,t+b}$ $\forall s,t$

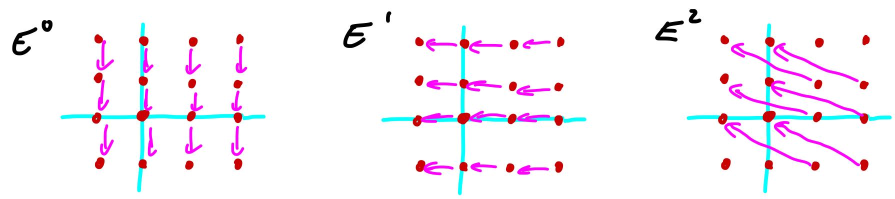
if $d^2=0$, then it is called a differential

and we can consider its homology

$$H_{s,t}(E,d) = \frac{\ker\{d: E_{s,t} \to E_{s+a,t+b}\}}{\operatorname{im}\{d: E_{s-a,t-b} \to E_{s,t}\}}$$

a spectral sequence, is a sequence $\{E^r, d^r\}$ s.t.

1) each E^r is a bimodule, d^r a differential of degree $(-r, r-1)$
2) $E^{r+1} = H(E^r)$



Leray-Hirsh Th^{m} if $p: E \to B$ a fibration
B simply connected CW complex
then $\exists$ a spectral sequence with $E^2_{s,t} = H_s(B; H_t(F))$
and "E^∞" more or less giving $H_*(E)$

can use spectral sequences for many other things too

III Obstruction Theory (and characteristic classes)

given a fibration $p: E \to B$

there are many problems that can be phrased as the existence of a section (e.g. does a manifold have a smooth structure...)

if B is a CW complex then there is a systematic way to try to construct a section skeleta by skeleta

Obstruction theory says: given a section $f: B^{(k-1)} \to E$ there is a cocycle $\sigma(f) \in C^k(B; \pi_{k-1}(F))$ s.t. $\sigma(f)=0$ $\Leftrightarrow$ f extends over $B^{(k)}$ (here F is $p^{-1}(pt)$)

Chern classes: "primary" obstruction to a $\mathbb{C}$ $n-k+1$ frame over $2k$ skeleton of B (here E a $\mathbb{C}^\ell$-bundle)

these are called characteristic classes (also have Stiefel-Whitney, Pontryagin classes...)