

## Math 6452 - Fall 2014

### Homework 4

Work all these problems and talk to me if you have any questions on them, but carefully write up and turn in **only** problems 4, 5, 6, 8, 12. **Due: In class on November 5.**

1. (Problem 2 from Section 1.5 in Guillemin and Pollack) Which of the following spaces intersect transversely?
  - The  $xy$ -plane and the  $z$ -axis in  $\mathbb{R}^3$ .
  - The  $xy$ -plane and the plane spanned by  $(3, 2, 0)$  and  $(0, 4, -1)$  in  $\mathbb{R}^3$ .
  - The spaces  $\mathbb{R}^k \times \{0\}$  and  $\{0\} \times \mathbb{R}^l$  in  $\mathbb{R}^n$ . (This depends on  $k, l$ , and  $n$ .)
  - The spaces  $\mathbb{R}^k \times \{0\}$  and  $\mathbb{R}^l \times \{0\}$  in  $\mathbb{R}^n$ . (This depends on  $k, l$ , and  $n$ .)
  - The spaces  $V \times \{0\}$  and the diagonal in  $V \times V$ , where  $V$  is a vector space.
  - The symmetric ( $A^t = A$ ) and skew-symmetric ( $A^t = -A$ ) matrices in  $M(n)$ .
2. For which values of  $r$  does the sphere  $x^2 + y^2 + z^2 = r$  and  $x^2 + y^2 - z^2 = 1$  intersect transversely? Draw the intersection for representative values of  $r$ .
3. A space  $X$  is called *contractible* if the identity map is homotopic to a constant map (that is there is some point  $p \in X$  such that the map  $id : X \rightarrow X : x \mapsto x$  is homotopic to the map  $c : X \rightarrow X : x \mapsto p$ ). Show that if  $X$  is contractible then for any space  $Y$  any two maps  $Y \rightarrow X$  are homotopic. Also show that  $\mathbb{R}^n$  is contractible for any  $n$ .
4. A space  $X$  is called *simply connected* if every map from  $S^1$  to  $X$  is homotopic to a constant map. Show a contractible space is simply connected. Moreover show that the  $n$ -sphere  $S^n$  is simply connected if  $n > 1$ .  
 Hint: Given a smooth map  $S^1 \rightarrow S^n$  use Sard's theorem to say it misses a point and then think about stereographic projection.
5. Show that  $S^n \times S^1$  is not simply connected for  $n \geq 0$ .  
 Hint: Consider the submanifold  $S = S^n \times \{p\}$  for some  $p \in S^1$  and the map  $f : S^1 \rightarrow S^n \times S^1 : \theta \mapsto (x, \theta)$  for some  $x \in S^n$ .  
 Notice that problems 4 and 5 imply that  $S^3$  and  $S^1 \times S^2$ , which are both  $S^1$  bundles over  $S^2$ , are not diffeomorphic.
6. If  $M$  and  $N$  are submanifolds of  $\mathbb{R}^n$  then show that for almost every  $x \in \mathbb{R}^n$  the translate  $M + x$  is transverse to  $N$ . (Here *almost everywhere* means “off of a set of measure zero” and  $M + x = \{y + x : y \in M\}$ .)
7. Suppose that  $f : M \rightarrow N$  is transverse to the submanifold  $S$  in  $N$ . Show that  $T_p f^{-1}(S)$  is given by  $(df_p)^{-1}(T_{f(p)} S)$ . In particular if  $S_1$  and  $S_2$  are submanifolds of  $N$  and they intersect transversely then  $T_p(S_1 \cap S_2) = (T_p S_1) \cap (T_p S_2)$ .
8. If  $f : M \rightarrow N$  has  $p$  as a regular value and  $S = f^{-1}(p)$  show that the normal bundle to  $S$  in  $M$  is trivial.
9. Let  $M$  and  $N$  be manifolds of the same dimensions with  $M$  compact and  $N$  connected. Prove that if  $f : M \rightarrow N$  has  $\deg_2(f) \neq 0$  then  $f$  is surjective.
10. Let  $f : M \rightarrow \mathbb{R}$  be a smooth function. A critical point of  $f$  is a point  $p \in M$  such that  $df_p = 0$ . We say that  $p$  is non-degenerate in the coordinate chart  $\phi : U \rightarrow V$  if the matrix

$$H = \left( \frac{\partial^2 F}{\partial x^i \partial x^j}(q) \right)$$

is non-singular where  $F = f \circ \phi^{-1}$  and  $\phi(p) = q$ . Show that a critical point is non-degenerate in one coordinate chart if and only if it is non-degenerate in any coordinate chart. Thus it makes sense to talk about non-degenerate critical points independent of coordinate charts.

Note: The matrix  $H$  is not well-defined independent of the coordinate chart, but whether it is non-singular or not is.

11. Show that non-degenerate critical points of a function  $f : M \rightarrow \mathbb{R}$  are isolated (that is each such critical point has a neighborhood containing no other critical points).

Hint: Work in local coordinate so the function is of the form  $f : \mathbb{R}^k \rightarrow \mathbb{R}$  and one can then think of  $df$  as a function  $df : \mathbb{R}^k \rightarrow \mathbb{R}^k$ . Prove  $df$  is a local diffeomorphism near a non-degenerate critical point.

A function  $f : M \rightarrow \mathbb{R}$  is called a *Morse function* if all of its critical points are non-degenerate.

12. Show that the function  $\mathbb{R}^{n+1} \rightarrow \mathbb{R} : (x^1, \dots, x^{n+1}) \mapsto x^{n+1}$  restricted to  $S^n$  is a Morse function with exactly two critical points. (This function is sometimes called the *height function*.)
13. Suppose that  $M$  is a submanifold of  $\mathbb{R}^{k+1}$ . The set of  $v \in S^k$  for which the map  $f_v : M \rightarrow \mathbb{R} : x \mapsto v \cdot x$  is not a Morse function has measure zero. (So every manifold has a lot of Morse functions.)
14. Suppose that  $M$  is a submanifold of  $\mathbb{R}^{k+1}$ . The set of points  $p \in \mathbb{R}^{k+1}$  for which the map  $f_p : M \rightarrow \mathbb{R} : x \mapsto \|x - p\|^2$  is not a Morse function has measure zero.