Math 6452 - Fall 2014
Homework 5

Work all these problems and talk to me if you have any questions on them, but carefully write up and turn in **only** *problems 6, 7, 10, 14, 16.* **Due: In class on November 24.**

1. For any finite dimensional vector space show that there are canonical isomorphisms $V \otimes \mathbb{R} \cong V \cong \mathbb{R} \otimes V$.
2. For finite dimensional vector spaces V and W show there is a canonical isomorphism $V^* \otimes W \cong Hom(V, W)$.
3. Let $\omega^1, \ldots, \omega^k$ be covectors in V^*. Show they are linearly dependent if and only if $\omega^1 \wedge \cdots \wedge \omega^k = 0$.
4. If $\{\omega^1, \ldots, \omega^k\}$ and $\{\eta^1, \ldots, \eta^n\}$ are linearly independent covectors in V^*, then show they have the same span if and only if $\omega^1 \wedge \cdots \wedge \omega^k = c\eta^1 \wedge \cdots \wedge \eta^k$ for some real number c.
5. Define the 1-form ω on $\mathbb{R}^2 - \{(0,0)\}$ by

$$\omega(x, y) = \left(\frac{-y}{x^2 + y^2}\right) dx + \left(\frac{x}{x^2 + y^2}\right) dy$$

   (a) Compute $\int_C \omega$ where C is a circle of radius r about the origin.

   (b) Compute $\int_C \omega$ where C is any circle in ω on $\mathbb{R}^2 - \{(0,0)\}$.

   (c) Is ω the differential of a function on $\mathbb{R}^2 - \{(0,0)\}$? Explain why or why not.

6. Prove that a 1-form α on S^1 is the differential of a function if and only if $\int_{S^1} \alpha = 0$.
7. Prove that the first De Rham cohomology of S^1 is $H^1_{DR}(S^1) \cong \mathbb{R}$.
   Hint: Show that is show that if α is a fixed 1-form on S^1 such that $\int_{S^1} \alpha \neq 0$ then for any other 1-form ω there is a real number c such that $\omega = c\alpha + df$ for some function f.
8. Given a vector space V and a vector $v \in V$ define the interior product

$$\iota_v : \Lambda^k(V) \to \Lambda^{k-1}(V)$$

   as follows: given $\omega \in \Lambda^k(V)$ define $\iota_v\omega$ to be the $(k-1)$ form:

$$\iota_v\omega(v_1, \ldots, v_{k-1}) = \omega(v, v_1, \ldots, v_{k-1}).$$

   If $\omega \in \Lambda^k(V)$ and $\eta \in \Lambda^l(V)$ then show that

$$\iota_v(\omega \wedge \eta) = (\iota_v\omega) \wedge \eta + (-1)^k \omega \wedge (\iota_v\eta).$$

9. On $\mathbb{R}^{2n}$ with coordinates $(x^1, y^1, \ldots, x^n, y^n)$ define the 1-form $\lambda = \frac{1}{2}\sum(x^i\, dy^i - y^i\, dx^i)$. Compute $d\lambda$ and $(d\lambda)^n$ (this means take the wedge product of $d\lambda$ with itself n times, for example $(d\lambda)^3 = (d\lambda) \wedge (d\lambda) \wedge (d\lambda)$). The 2-form $d\lambda$ is called the standard symplectic form on $\mathbb{R}^{2n}$.
10. Let $a : S^n \to S^n$ be the antipodal map, that is the map $a(x) = -x$ when we think of S^n as the unit sphere in $\mathbb{R}^n$. Show that a is orientation preserving if and only if n is odd.
11. Show that $\mathbb{R}P^n$ is orientable if and only if n is odd.
12. Suppose that M and N are oriented manifolds and $f : M \to N$ is a local diffeomorphism. If M is connected then show that f is either orientation preserving or orientation reversing.

13. On $\mathbb{R}^n - \{0\}$ consider the $(n-1)$-form

$$\omega = \frac{1}{\|x\|^n} \sum_{i=1}^{n} (-1)^{i-1} x^i \, dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^n.$$

Compute $d\omega$.

14. Let S^2 be the unit sphere in $\mathbb{R}^3$ and ω the 2-form from the the previous exercise. If $i : S^2 \to \mathbb{R}^3$ is the inclusion map then compute

$$\int_{S^2} i^* \omega.$$

Is there and 1-form η on $\mathbb{R}^3 - \{0\}$ such that $d\eta = \omega$? Explain why or why not. Notice that this and the previous exercise imply that $H^2_{DR}(\mathbb{R}^3 - \{0\}) \neq 0$.
If you feel like it maybe try to work this problem again for S^{n-1} (this is not required to be turned in).

15. Use Stokes theorem to prove the classical Green's formula: Give a region R in $\mathbb{R}^2$ with smooth boundary $\partial R = \gamma$ then show

$$\int_\gamma f \, dx + g \, dy = \int_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dxdy.$$

16. Given any embedding $f : T^2 \to S^3$ show that for any closed 2-form ω on S^3 we have

$$\int_{T^2} f^* \omega = 0.$$

Hint: Show that there is a smooth homotopy $H : S^2 \times [0,1] \to S^3$ from f to a constant map. Now use Stokes theorem.

17. Show there is some embedding $f : T^2 \to T^3$ and a closed 2-form ω on T^3 such that

$$\int_{T^2} f^* \omega \neq 0.$$

Notice that this problem together with the previous one implies that S^3 is not diffeomorphic to T^3.