

Math 6452 - Fall 2017

Homework 3

Work all these problems and talk to me if you have any questions on them, but carefully write up and turn in **only** problems 3, 5, 6, 9, 11, 12. **Due: In class on October 5.**

1. Prove that TS^1 is diffeomorphic to $S^1 \times \mathbb{R}$.
2. If S^k is the unit sphere in \mathbb{R}^{k+1} show that S^k has a non-zero vector field if k is odd.
Hint: For $k = 1$ you can use the vector field $v(x^1, x^2) = (-x^2, x^1)$. Here we are thinking of $S^1 \subset S^2$ and $T_x S^1 \subset T_x \mathbb{R}^2 = \mathbb{R}^2$.
3. A rank k vector bundle $p : E \rightarrow M$ is called trivial if $E \cong M \times \mathbb{R}^k$. (Here “rank k ” means the fiber of the bundle is a k dimensional vector space). Show that E is trivial if and only if there are k sections $\sigma_1, \dots, \sigma_k$ of E such that at each point $x \in M$ the vectors $\sigma_1(x), \dots, \sigma_k(x)$ form a basis for $p^{-1}(x)$.
4. Suppose E and \widehat{E} are two rank k vector bundles over M . Suppose that $\{U_\alpha\}_{\alpha \in A}$ is an open cover of M such that both E and \widehat{E} have local trivializations over the U_α and their transition functions are $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow GL(k, \mathbb{R})$ and $\widehat{\tau}_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow GL(k, \mathbb{R})$, respectively. Show that there is a smooth bundle isomorphism

$$\begin{array}{ccc} E & \xrightarrow{f} & \widehat{E} \\ \downarrow p & & \downarrow \widehat{p} \\ M & \xrightarrow{Id_M} & M \end{array}$$

if and only if there are smooth maps $\sigma_\alpha : U_\alpha \rightarrow GL(k, \mathbb{R})$ for all $\alpha \in A$ such that

$$\widehat{\tau}_{\alpha\beta}(x) = \sigma_\alpha^{-1}(x) \tau_{\alpha\beta}(x) \sigma_\beta(x)$$

for all $x \in U_\alpha \cap U_\beta$.

5. Stereographic coordinates provide a local trivialization of the tangent bundle of S^2 . Compute the transition functions for the tangent bundle to S^2 using the trivializations determined by stereographic coordinates.
6. Let v be a vector field on M and $f : M \rightarrow \mathbb{R}$ a positive function. If $\gamma : \mathbb{R} \rightarrow M$ is a flow line of v show there is a function g with positive derivative such that the reparameterization $\gamma \circ g$ of γ is a flow line of fv .
7. Let A be an $n \times n$ symmetric real matrix and $b \in \mathbb{R}$ a nonzero real number. Show that

$$M = \{x \in \mathbb{R}^n : x^t A x = b\}$$

is a manifold of dimension $n - 1$.

8. Let

$$H(m, n) = \{(z, w) \in \mathbb{C}P^m \times \mathbb{C}P^n : \sum_{i=0}^m z^i w^i = 0\}$$

is a manifold of dimension $2(m + n - 1)$, where $m \leq n$ and $z = [z^1 : \dots : z^m]$ and $w = [w^1 : \dots : w^n]$ are homogeneous coordinates.

9. Given a submanifold N of M we say a smooth map $f : W \rightarrow M$ is transverse to N if for every $p \in f^{-1}(N)$ we have $T_{f(p)}M$ being spanned by vectors in $T_{f(p)}N$ and the image of df_p . Note: a point $p \in M$ is a regular value of f if and only if f is transverse to p . So this notion of transversality generalized the notion of a regular value.
Show that if $f : W \rightarrow M$ is transverse to the submanifold N of M then $f^{-1}(N)$ is a submanifold of W whose codimension is the same as the codimension of N in M .
10. If S_1 and S_2 are two submanifolds of M then we say they are transverse if for all $p \in S_1 \cap S_2$ we have T_pM spanned by T_pS_1 and T_pS_2 . Note if $I_i : S_i \rightarrow M$ is the inclusion map then S_1 is transverse to S_2 if and only if I_1 is transverse to S_2 if and only if I_2 is transverse to S_1 . If S_1 and S_2 are transverse submanifolds in M show that $S_1 \cap S_2$ is a submanifold of dimension $\dim(S_1) + \dim(S_2) - \dim(M)$ (said another way the codimension of $S_1 \cap S_2$ is the sum of the codimensions of S_1 and S_2).
11. Let $f : M \rightarrow \mathbb{R}^k$ be a smooth map and $N \subset \mathbb{R}^k$ a submanifold. Show that for any $\epsilon > 0$ there is a vector v with $\|v\| < \epsilon$ such that the map $M \rightarrow \mathbb{R}^k : x \mapsto f(x) + v$ is transverse to N .
Hint: consider the map $M \times N \rightarrow \mathbb{R}^k : (x, y) \mapsto y - f(x)$.
12. Given a function $f : M \rightarrow \mathbb{R}$ and a vector field v on M , show that $\mathcal{L}_v f = 0$ if and only if f is constant on the flow lines of v . (Here $\mathcal{L}_v f$ means the Lie derivative of f along v .)