

# Math 6452 - Fall 2017

## Homework 5

Work all these problems and talk to me if you have any questions on them, but carefully write up and turn in **only** problems 2, 5, 6, 7, 8, 12 **Due: In class on November 9.**

1. For any finite dimensional vector space show that there are canonical isomorphisms  $V \otimes \mathbb{R} \cong V \cong \mathbb{R} \otimes V$ .
2. For finite dimensional vector spaces  $V$  and  $W$  show there is a canonical isomorphism  $V^* \otimes W \cong \text{Hom}(V, W)$ .
3. Let  $\omega^1, \dots, \omega^k$  be covectors in  $V^*$ . Show they are linearly dependent if and only if  $\omega^1 \wedge \dots \wedge \omega^k = 0$ .
4. If  $\{\omega^1, \dots, \omega^k\}$  and  $\{\eta^1, \dots, \eta^n\}$  are linearly independent covectors in  $V^*$ , then show they have the same span if and only if  $\omega^1 \wedge \dots \wedge \omega^k = c\eta^1 \wedge \dots \wedge \eta^k$  for some real number  $c$ . Show that  $c = \det(A)$  where  $A$  is the matrix  $(a_{i,j})$  and the  $a_{i,j}$  are determined by  $\omega_i = \sum_j a_{i,j} \eta_j$ .
5. Let  $M$  be a smooth manifold and let  $\omega \in \Gamma(T^k M)$  be a tensor field. Consider the map

$$\Psi_\sigma : (\mathcal{X}(M) \times \dots \times \mathcal{X}(M)) \rightarrow C^\infty(M) : (v_1, \dots, v_k) \mapsto \omega(v_1, \dots, v_k)$$

here  $\mathcal{X}(M)$  is the set of vector fields. Show that this map is multilinear over  $C^\infty(M)$ . Moreover show that given any multilinear map over  $C^\infty(M)$

$$\Psi : (\mathcal{X}(M) \times \dots \times \mathcal{X}(M)) \rightarrow C^\infty(M)$$

it is induced from some tensor field by the above construction.

6. Define the 1-form  $\omega$  on  $\mathbb{R}^2 - \{(0, 0)\}$  by

$$\omega(x, y) = \left( \frac{-y}{x^2 + y^2} \right) dx + \left( \frac{x}{x^2 + y^2} \right) dy$$

- (a) Compute  $\int_C \omega$  where  $C$  is a circle of radius  $r$  about the origin.
- (b) Is  $\omega$  the differential of a function on  $\mathbb{R}^2 - \{(0, 0)\}$ ? Explain why or why not.
7. Prove that a 1-form  $\alpha$  on  $S^1$  is the differential of a function if and only if  $\int_{S^1} \alpha = 0$ .
8. Prove that the first De Rham cohomology of  $S^1$  is  $H_{DR}^1(S^1) \cong \mathbb{R}$ .  
Hint: Show that if  $\alpha$  is a fixed 1-form on  $S^1$  such that  $\int_{S^1} \alpha \neq 0$  then for any other 1-form  $\omega$  there is a real number  $c$  such that  $\omega = c\alpha + df$  for some function  $f$ .
9. Consider the forms on  $\mathbb{R}^3$

$$f \in \omega^0(\mathbb{R}^3), \quad f dx + g dy + h dz \in \omega^1(\mathbb{R}^3), \quad \text{and}$$

$$f dy \wedge dz + g dz \wedge dy + h dx \wedge dy \in \omega^2(\mathbb{R}^3).$$

Compute their exterior derivatives. Do they look like anything from vector calculus?

10. Given a vector space  $V$  and a vector  $v \in V$  define the interior product

$$\iota_v : \Lambda^k(V) \rightarrow \Lambda^{k-1}(V)$$

as follows: given  $\omega \in \Lambda^k(V)$  define  $\iota_v \omega$  to be the  $(k-1)$  form:

$$\iota_v \omega(v_1, \dots, v_{k-1}) = \omega(v, v_1, \dots, v_{k-1}).$$

If  $\omega \in \Lambda^k(V)$  and  $\eta \in \Lambda^l(V)$  then show that

$$\iota_v(\omega \wedge \eta) = (\iota_v \omega) \wedge \eta + (-1)^k \omega \wedge (\iota_v \eta).$$

11. On  $\mathbb{R}^{2n}$  with coordinates  $(x^1, y^1, \dots, x^n, y^n)$  define the 1-form  $\lambda = \frac{1}{2} \sum (x^i dy^i - y^i dx^i)$ . Compute  $d\lambda$  and  $(d\lambda)^n$  (this means take the wedge product of  $d\lambda$  with itself  $n$  times, for example  $(d\lambda)^3 = (d\lambda) \wedge (d\lambda) \wedge (d\lambda)$ ). The 2-form  $d\lambda$  is called the standard symplectic form on  $\mathbb{R}^{2n}$ .
12. Suppose  $V$  is an  $n$ -dimensional vector space. Given a linear map

$$L : V \rightarrow V$$

there is an induced linear map

$$L^* : \wedge^n(V) \rightarrow \wedge^n(V).$$

Since  $\wedge^n(V)$  is 1-dimensional this map is simply multiplication by a constant. We will denote this constant  $\det(L)$ . Prove the following

- (a) If you choose a basis  $e_1, \dots, e_n$  for  $V$  then  $L$  can be written as a matrix  $A_L$ ,  $\det(L) = \det(A)$ , where the determinant of a matrix has the usual definition.
- (b)  $\det(L \circ S) = \det(L) \det(S)$ .
- (c) If  $L$  is the identity map then  $\det(L) = 1$ .
- (d)  $L$  is an isomorphism if and only if  $\det(L) \neq 0$  and in this case  $\det(L^{-1}) = (\det(L))^{-1}$ .