Math 6452 Midterm Sample Questions Fall 2017

1. Prove that $\mathbb{R}^m$ is diffeomorphic to $\mathbb{R}^n$ if and only if $n = m$.
   Hint: Tangent spaces and differentials.

2. Given a smooth function $f : M \to N$ and a point $p \in M$, we say that f has a *local section* through p if there is a neighborhood U of $f(p)$ in N and a smooth map $g : U \to M$ such that $g(f(p)) = p$ and $f \circ g$ is the identity function on U. Show that a smooth surjective map $f : M \to N$ is a submersion if and only if it has a local section through each point of M.

3. Show that the set of 2×2 matrices of rank 1 is a manifold of dimension 3.
   Hint: Consider the determinate function on the set of non-zero 2×2 matrices.

4. Suppose $f : M \to \mathbb{R}$ has 0 as a regular value. Show that

   $$F(M \times \mathbb{R}) \to \mathbb{R} : (x, t) \mapsto f(x) - t^2$$

   also has 0 as a regular value. (10 points)
   Extra Credit: Describe $F^{-1}(0)$ in terms of f. (1 extra point, so don't spend too much time on this until you finish the exam.)

5. Let V be a vector space and $L : V \to V$ be a linear map. Consider the diagonal $\Delta = \{(v, w) \in V \times V : v = w\}$ and the graph $\Gamma_L = \{(v, w) \in V \times V : w = Lv\}$. Show that Δ is transverse to Γ_L if and only if L does not have $+1$ as an eigenvalue. (10 points)
   Remark: You can interpret V and the other spaces above to be manifolds and consider as manifolds whether Δ is transverse to Γ_L or you can interpret them just as vector spaces. If you do the second (which is probably slightly easier) then let me recall (since this was not explicitly done in class) that two vector sub-spaces U and W of V are transverse if the span of $U \cup W$ is all of V.

6. Answer the following questions **T**rue or **F**alse.

   (a) Any smooth function $f : M \to N$ can be represented in local coordinates as a linear map.

   (b) A function $f : M \to N$ is smooth if and only if there is a smooth atlas $\{\phi_\alpha : U_\alpha \to V_\alpha\}_{\alpha \in A}$ for M and $\{\psi_\beta : U_\beta \to V_\beta\}_{\beta \in B}$ of N, such that $\psi_\beta \circ f \circ \phi_\alpha^{-1}$ is a smooth function between open sets in Euclidean spaces for all $\alpha \in A$ and $\beta \in B$.

   (c) A function $f : M \to N$ is smooth if and only if there is a smooth function $g : N \to \mathbb{R}$ such that $g \circ f : M \to \mathbb{R}$ is smooth.

   (d) The image if an injective immersion is a submanifold.

(e) Regular values of a smooth function $f : M \to N$ are dense in N.

(f) The subset $\{(x, y) \in \mathbb{R}^2 : xy = 0\}$ is a submanifold of $\mathbb{R}^2$.

(g) There is a smooth surjection from $[0, 1]$ to $[0, 1] \times [0, 1]$.

(h) If M and N are manifolds of the same dimension then a smooth map $f : M \to N$ is a submersion if and only if it is an immersion.