Math 6452 - Fall 2020
Homework 1

Work all these problems and talk to me if you have any questions on them, but carefully write up and turn in **only** *problems 2, 4, 6, 7, 8, 9.* **Due: In class on August 28.**

1. Lee's book, Problem 1-7
2. Lee's book, Problem 1-9
3. Lee's book, Problem 1-10
4. Let M be a smooth manifold with or without boundary. Show that $C^\infty(M)$ is a commutative ring where multiplication and addition are defined point-wise. (Recall $C^\infty(M)$ is the set of smooth maps from M to $\mathbb{R}$.)
5. Let M, X, and Y be smooth manifolds without boundary. Let $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ be the projection maps. Show that a function $f : M \to X \times Y$ is smooth if and only if $\pi_X \circ f : M \to X$ and $\pi_Y \circ f : M \to Y$ are smooth.
6. Prove the following maps are smooth.
   (a) $p_n : S^1 \to S^1$, where S^1 is the unit circle in $\mathbb{C}$ and p_n is he map $z \mapsto z^n$ restricted to S^1.
   (b) $f : S^3 \to S^2$ given by $f(w, z) = (z\overline{w} + w\overline{z}, iw\overline{z} - iz\overline{w}, z\overline{z} - w\overline{w})$ where S^3 is the unit sphere in $\mathbb{C}^2$ and S^2 is the unit sphere in $\mathbb{R}^3$.
7. Let $f : (\mathbb{R}^{n+1} - \{0\}) \to (\mathbb{R}^{k+1} - \{0\})$ be a smooth homogeneous function of degree $d \in \mathbb{Z}$. This means that $f(cx) = c^d f(x)$ for all $c \in \mathbb{R}$ and $x \in (\mathbb{R}^{n+1} - \{0\})$. Show that the map $\widetilde{f} : \mathbb{R}P^n \to \mathbb{R}P^k$ defined by $\widetilde{(}[x]) = [f(x)]$ is a well-defined, smooth map.
8. Let M and N be smooth manifolds.
   (a) Show that a continuous function $f : M \to N$ defines a linear map $f^* : C(N) \to C(M)$ by $h \in C(N)$ maps to $h \circ f$. (Here $C(M)$ is the set of continuous functions from M to $\mathbb{R}$.)
   (b) Show that $f : M \to N$ is smooth if and only if $f^*(C^\infty(N)) \subset C^\infty(M)$.
   (c) If $f : M \to N$ is a homeomorphism, then show it is a diffeomorphism if and only if f^* restricts to an isomorphism from $C^\infty(N)$ to $C^\infty(M)$.
9. Recall the Grassmann of k-dimensional subspaces of $\mathbb{R}^n$ is denoted $G(k, n)$ (see Lee pages 22-24). Using the standard inner product on $\mathbb{R}^n$ and denoting the orthogonal complement of a subspace V of $\mathbb{R}^n$ by $V^\perp$, we can define a map $f : G(k, n) \to G(n-k, n)$ by $f(V) = V^\perp$ for every V in $G(k, n)$. Show that f is a diffeomorphism.