Math 6452 - Fall 2020
Homework 2

Work all these problems and talk to me if you have any questions on them, but carefully write up and turn in **only** *problems 4, 5, 7, 8, 9, 12.* **Due: September 11.**

1. If S^2 is the unit sphere in $\mathbb{R}^3$ and $N = (0,0,1)$ and $S = (0,0,-1)$ then we have the two stereographic coordinate maps $\pi_N : (S^2 - N) \to \mathbb{R}^2$ and $\pi_S : (S^2 - S) \to \mathbb{R}^2$. If p is a point in S^2 not equal to N or S then we can use the first to express a tangent vector in T_pS^2 in terms of the basis $\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}\}$ (where we are using Cartesian coordinates (x^1, x^2) on $\mathbb{R}^2$) as
$$v = v^1 \frac{\partial}{\partial x^1} + v^2 \frac{\partial}{\partial x^2}.$$
Similarly we can use the second to express the same vector in terms of the basis $\{\frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}\}$ (where we are using Cartesian coordinates (y^1, y^2) on $\mathbb{R}^2$) as
$$v = w^1 \frac{\partial}{\partial y^1} + w^2 \frac{\partial}{\partial y^2}.$$
Write the w^i in terms of the v^i (and the coordinate transform $\pi_S \circ \pi_N^{-1}$).
In particular if $\pi_N(p) = (1,0)$ and $v = \frac{\partial}{\partial x^1}$ then express v in the other coordinate system.
2. Let M and N be two smooth manifolds.
   (a) Show that for $(p,q) \in M \times N$ we have
   $$T_{(p,q)}(M \times N) = (T_pM) \times (T_qN).$$
   (b) If $\pi : M \times N \to M : (p,q) \mapsto p$ is the projection map then
   $$df_{(p,q)} : T_{(p,q)}(M \times N) \to T_pM$$
   is the projection map $(v,w) \mapsto v$.
   (c) Fix a point $q_0 \in N$ and let $f : M \to M \times N : p \mapsto (p, q_0)$ then show that
   $$df_p : T_pM \to T_{(p,q_0)}(M \times N)$$
   is given by $v \mapsto (v, 0)$.
3. Let $f : M \to N$ be a smooth map between smooth manifolds and define $F : M \to (M \times N) : p \mapsto (p, f(p))$. Show that $dF_p(v) = (v, df_p(v))$. (Here we are of course using Problem 3 (a) to write the tangent bundle of $M \times N$ as a product.)
4. If $f : M \to N$ is a submersion, then show f is an open map. (That is show that for any open set U in M the image $f(U)$ is open in N.)
5. If M is a compact smooth non-empty manifold and N is a connected smooth manifold, then show that any smooth submersion $f : M \to N$ is surjective. Is there a submersion from S^2 to any $\mathbb{R}^n$, with $n > 0$?
6. Let M be a compact smooth manifold and N a connected smooth manifold. If they both have the same dimension and are non-empty show that any embedding $f : M \to N$ is a diffeomorphism.

7. Show that $\mathbb{C}P^1$ is diffeomorphic to S^2.
Hint: Using stereographic coordinates on S^2 and our "standard" coordinates on $\mathbb{C}P^1$ we see both manifolds can be covered by 2 coordinate charts. Study the transition functions between these coordinate charts and see if you can define a map using the coordinate charts.
8. Define the map
$$f : \mathbb{C}P^n \to \mathbb{C}P^m$$
by
$$f([x^0 : \cdots : x^n]) = [x^0 : \cdots : x^n : 0 : \cdots : 0]$$
where $n \leq m$. Show that f is a smooth embedding. (Notice that this says that S^2 is submanifold of $\mathbb{C}P^2$, or any $\mathbb{C}P^n$ with $n > 0$ for that matter. Later we will see that this is a "non-trivial" S^2.)
9. With f as in the previous problem show that $\mathbb{C}P^{n+1} - f(\mathbb{C}P^n)$ is diffeomorphic to $\mathbb{C}^{n+1}$. (So for example $\mathbb{C}P^2$ is the union of $\mathbb{C}P^1 \cong S^2$ and $\mathbb{C}^2$. Thus we can think of $\mathbb{C}P^2$ is the compactification of $\mathbb{C}^2$ by an "S^2 at infinity".)
10. A smooth map $f : (\mathbb{C}^{n+1} - \{(0, \ldots, 0)\}) \to (\mathbb{C}^{k+1} - \{(0, \ldots, 0)\})$ is called homogeneous of degree k if $f(\lambda p) = \lambda^k f(p)$ for all $\lambda \neq 0$ and $p \in (\mathbb{C}^{n+1} - \{(0, \ldots, 0)\})$. Show that f induces a map
$$\tilde{f} : \mathbb{C}P^n \to \mathbb{C}P^k.$$
Show this map is smooth.
11. Define the map
$$f : \mathbb{C}P^n \times \mathbb{C}P^m \to \mathbb{C}P^{nm+n+m}$$
by
$$f([x^0 : \cdots : x^n], [y^0 : \cdots : y^m]) = [x^0y^0 : x^0y^1 : \cdots : x^0y^m : x^1y^0 : \cdots : x^ny^m].$$
Show f is a smooth map and that f it is an embedding. (Notice that this shows, for example, that $S^2 \times S^2$ is a submanifold of $\mathbb{C}P^3$).
Note: The last 4 problems could also have been carried out for real projective spaces.
12. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a homogeneous polynomial. This implies that there is some integer k such that
$$f(tx^1, \ldots, tx^n) = t^k f(x^1, \ldots, x^n)$$
for all $(x^1, \ldots, x^n)$. Prove that $f^{-1}(a)$, for $a \neq 0$, is an $(n-1)$-dimensional manifold. Moreover show that if a and b are both positive then $f^{-1}(a)$ and $f^{-1}(b)$ are diffeomorphic and similarly if a and b are both negative. Finally show that if a and b have different signs that $f^{-1}(a)$ and $f^{-1}(b)$ do not have to be diffeomorphic by considering $f(x, y, z) = x^2 + y^2 - z^2$.
Hint: It might be good to use the famous Euler identity for homogeneous functions
$$\sum_{i=1}^{n} x^i \frac{\partial f}{\partial x^i} = kf,$$
(you don't need to prove this identity, though feel free to if you like) to prove that 0 is the only critical value of f. To find the diffeomorphism consider the map $\mathbb{R}^n \to \mathbb{R}^n$ obtained by multiplication by an appropriate root of a/b.