Math 6452 - Fall 2020
Homework 3

Work all these problems and talk to me if you have any questions on them, but carefully write up and turn in **only** *problems 3, 5, 6, 9, 11, 12.* **Due: September 25.**

1. Prove that TS^1 is diffeomorphic to $S^1 \times \mathbb{R}$.
2. If S^k is the unit sphere in $\mathbb{R}^{k+1}$ show that S^k has a non-zero vector field if k is odd. Hint: For $k = 1$ you can use the vector field $v(x^1, x^2) = (-x^2, x^1)$. Here we are thinking of $S^1 \subset \mathbb{R}^2$ and $T_xS^1 \subset T_x\mathbb{R}^2 = \mathbb{R}^2$.
3. A rank k vector bundle $p : E \to M$ is called trivial if $E \cong M \times \mathbb{R}^k$. (Here "rank k" means the fiber of the bundle is a k dimensional vector space). Show that E is trivial if and only if there are k sections $\sigma_1, \ldots, \sigma_k$ of E such that at each point $x \in M$ the vectors $\sigma_1(x), \ldots, \sigma_k(x)$ form a basis for $p^{-1}(x)$.
4. Suppose E and $\widehat{E}$ are two rank k vector bundles over M. Suppose that $\{U_\alpha\}_{\alpha\in A}$ is an open cover of M such that both E and $\widehat{E}$ have local trivializations over the U_α and their transition functions are $\tau_{\alpha\beta} : U\alpha \cap U_\beta \to GL(k, \mathbb{R})$ and $\widehat{\tau}_{\alpha\beta} : U\alpha \cap U_\beta \to GL(k, \mathbb{R})$, respectively. Show that there is a smooth bundle isomorphism

$$\begin{array}{ccc} E & \xrightarrow{\ f\ } & \widehat{E} \\ \Big\downarrow{\scriptstyle p} & & \Big\downarrow{\scriptstyle \widehat{p}} \\ M & \xrightarrow{Id_M} & M \end{array}$$

if and only if there are smooth maps $\sigma_\alpha : U_\alpha \to GL(k, \mathbb{R})$ for all $\alpha \in A$ such that

$$\widehat{\tau}_{\alpha\beta}(x) = \sigma_\alpha^{-1}(x)\tau_{\alpha\beta}(x)\sigma_\beta(x)$$

for all $x \in U_\alpha \cap U_\beta$.
5. Stereographic coordinates provide a local trivialization of the tangent bundle of S^2. Compute the transition functions for the tangent bundle to S^2 using the trivializations determined by stereographic coordinates.
6. Let v be a vector field on M and $f : M \to \mathbb{R}$ a positive function. If $\gamma : \mathbb{R} \to M$ is a flow line of v show there is a function g with positive derivative such that the reparameterization $\gamma \circ g$ of γ is a flow line of fv.
7. Let A be an $n \times n$ symmetric real matrix and $b \in \mathbb{R}$ a nonzero real number. Show that

$$M = \{x \in \mathbb{R}^n : x^tAx = b\}$$

is a manifold of dimension $n - 1$.
8. Let

$$H(m, n) = \{(z, w) \in \mathbb{C}P^m \times \mathbb{C}P^n : \sum_{i=0}^{m} z^i w^i = 0\}$$

is a manifold of dimension $2(m + n - 1)$, where $m \leq n$ and $z = [z^1 : \cdots : z^m]$ and $w = [w^1 : \cdots : w^n]$ are homogeneous coordinates.

9. Given a submanifold N of M we say a smooth map $f : W \to M$ is transverse to N if for every $p \in f^{-1}(N)$ we have $T_{f(p)}M$ being spanned by vectors in $T_{f(p)}N$ and the image of df_p. Note: a point $p \in M$ is a regular value of f if and only if f is transverse to p. So this notion of transversallity generalized the notion of a regular value.
   Show that if $f : W \to M$ is transverse to the submanifold N of M then $f^{-1}(N)$ is a submanifold of W whose codimension is the same as the codimension of N in M.
10. If S_1 and S_2 are two submanifolds of M then we say they are transverse if for all $p \in S_1 \cap S_2$ we have T_pM spanned by T_pS_1 and T_pS_2. Note if $I_i : S_i \to M$ is the inclusion map then S_1 is transverse to S_2 if and only if I_1 is transverse to S_2 if and only if I_2 is transverse to S_1. If S_1 and S_2 are transverse submanifolds in M show that $S_1 \cap S_2$ is a submanifold of dimension $dim(S_1) + dim(S_2) - dim(M)$ (said another way the codimension of $S_1 \cap S_2$ is the sum of the codimensions of S_1 and S_2).
11. Let $f : M \to \mathbb{R}^k$ be a smooth map and $N \subset \mathbb{R}^k$ a submanifold. Show that for any $\epsilon > 0$ there is a vector v with $\|v\| < \epsilon$ such that the map $M \to \mathbb{R}^k : x \mapsto f(x) + v$ is transverse to N.
    Hint: consider the map $M \times N \to \mathbb{R}^k : (x, y) \mapsto y - f(x)$.
12. Given a function $f : M \to \mathbb{R}$ and a vector field v on M, show that $\mathcal{L}_v f = 0$ if and only if f is constant on the flow lines of v. (Here $\mathcal{L}_v f$ means the Lie derivative of f along v.)