Differential Topology

Differential topology is about studying the "most general" spaces on which you can "do calculus"

that is: differentiate functions

integrate functions (and other things)

solve differential equations ...

We will study manifolds which are roughly spaces that look locally like Euclidean space

In particular $\mathbb{R}^n$ is a manifold ($\mathbb{R}^n$ means n-dimensional Euclidean space)

So differential topology is in some sence the ultimate generalization of vector calc.

Why study manifolds?

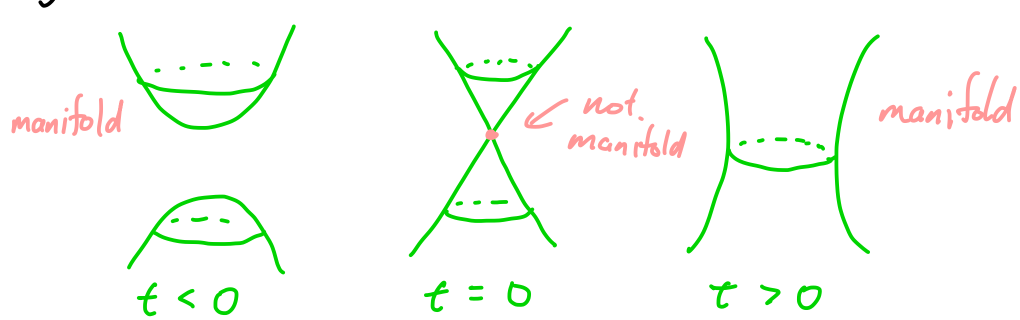
Manifolds are everywhere

they show up all over mathematics, the sciences, and engineering

for example

- Solutions to "most" equations

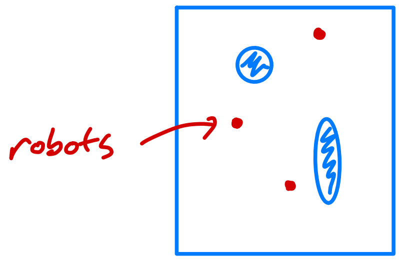
e.g. $x^2+y^2-z^2=t$



- Solutions to P.D.E.

- configuration spaces

  e.g. • robots moving on a factory floor

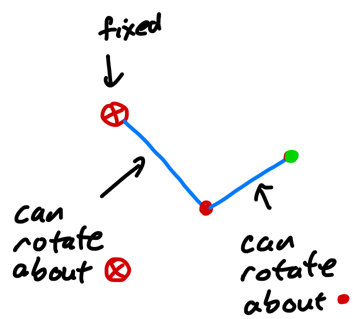


what are the possible positions, without collisions?

it is a 6-manifold

What is it?

• mechanical arms (linkages)



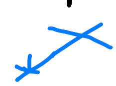
possible configurations are $S^1 \times S^1$

a 2-manifold

<u>Fact</u>: Any (nice) manifold can be realized as configuration space of some linkage

• Rigid bodies

e.g. positions of "airplane"



in $\mathbb{R}^3$

is a 6-manifold $\mathbb{R}^3 \times SO(3)$

- Models of the universe

  4-manifold (Relativity: space-time)

  is it $\mathbb{R}^4$?

  10-manifold (String theory ...)

I Manifolds

A Topological Manifolds

a topological space M is a manifold of dimension n (also known as an n-manifold) if

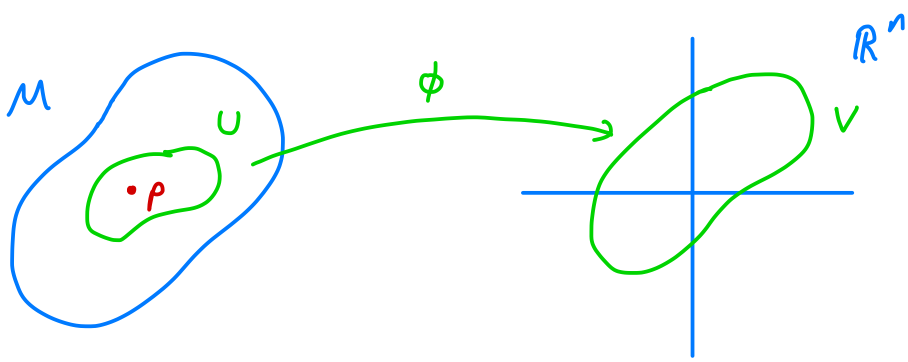
(1) M is Hausdorff and 2^{nd} countable and

(2) M is "locally Euclidean", that is for each point $p \in M$ there is

an open nbhd U of p in M,
an open set V in $\mathbb{R}^n$, and
a homeomorphism $\phi: U \to V$

Hausdorff: any 2 distinct pts can be separated by disjoint open sets

2^{nd} Countable: countable basis



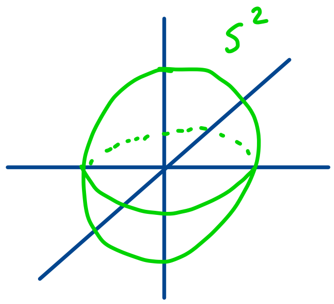
$\phi: U \to V$ is called a coordinate chart around p

Remark: Items 1) and M being a topological space can be replaced by M is a subset of $\mathbb{R}^k$, some k.
(we will prove this later)
this is psychologically satisfying, but make examples and proofs much harder

exercise: Show in item 2) one could say that V = open ball in $\mathbb{R}^n$ and $\phi(p) = 0$ and we would have an equivalent definition (similarly, could take $V = \mathbb{R}^n$)

examples:

1) any open subset of $\mathbb{R}^n$ is an n-manifold (e.g $\mathbb{R}^n$)
2) $S^n \subset \mathbb{R}^3$ unit sphere is a 2-manifold



let $U_{z+} = \{(x,y,z) \in S^2 \mid z > 0\}$

$V_{z+} = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 < 1\}$

$\phi : U_{z+} \to V_{z+} : (x,y,z) \mapsto (x,y)$

$\phi^{-1} : V_{z+} \to U_{z+} : (x,y) \mapsto (x,y,\sqrt{1-x^2-y^2})$

these are clearly continuous so ϕ a homeomorphism

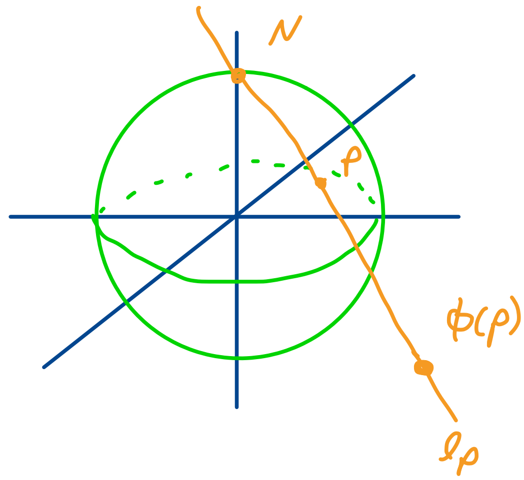
so $\phi : U_{z+} \to V_{z+}$ is a coordinate chart

similarly get $U_{x\pm} \to V_{x\pm}$, $U_{y\pm} \to V_{y\pm}$, $U_{z-} \to V_{z-}$

so S^2 is a 2-manifold (note: Hausdorff and 2nd countable since $\mathbb{R}^3$ is and subsets inherit these property)

can prove S^2 a 2-manifold in a different way

Stereographic Coordinates



let $N = (0,0,1) \in S^2$

$U = S^2 - \{N\}$

and $V = \mathbb{R}^2 = xy\text{-plane}$

given $p \in U$ let

ℓ_p = line through N and p

$\ell_p \cap xy$-plane in one point, call it $\phi(p)$

this defines a function $\phi : U \to V$

let's work out a formula for ϕ

given $p = (x,y,z)$ the line ℓ_p is parameterized by

$$r(t) = t(0,0,1) + (1-t)(x,y,z)$$

to see where $\ell_p \cap xy$-plane we need to find t

such that z-coordinate of $\phi(t)$ is 0

i.e.

$$t + (1-t)z = 0$$
$$(1-z)t = -z$$
$$t = \frac{z}{z-1} \qquad\qquad 1-t = \frac{-1}{z-1}$$

so $r\left(\frac{z}{z-1}\right) = \left(0,0,\frac{z}{z-1}\right) + \left(\frac{-x}{z-1}, \frac{-y}{z-1}, \frac{-z}{z-1}\right)$

$$= \frac{1}{z-1}(-x,-y,0)$$

and $\boxed{\phi(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right)}$

exercise: 1) compute (using a param. of the line through N and $(x,y,0)$)

that $\phi^{-1}(x,y) = \frac{1}{1+x^2+y^2}(2x, 2y, x^2+y^2-1)$

2) find similar coordinates with

$U_2 = S^2 - \{(0,0,-1)\}, \quad V_2 = \mathbb{R}^2$

3) find stereographic coordinates on $S^n \subset \mathbb{R}^{n+1}$

so S^n is an n-manifold

3) $\mathbb{R}P^n = \{\text{lines in } \mathbb{R}^{n+1}\}$ real projective space

note: 1) each line is determined by a non-zero vector

2) 2 vectors v_1, v_2 give same line

$\Leftrightarrow$

$\exists\, r \neq 0$ s.t. $v_1 = r v_2$

so $\mathbb{R}P^n = \mathbb{R}^{n+1} - \{(0,\dots,0)\} / \mathbb{R} - \{0\} = S^n/\sim$

quotient space so use quotient topology

where $p \sim -p$

denote the equivalence class by $[x^0 : \dots : x^n]$

(called homogeneous coords)

let $U_i = \{[x^0 : \dots : x^n] \in \mathbb{R}P^n \mid x^i \neq 0\}$

and $\phi : U_i \longrightarrow \mathbb{R}^n$

$[x^0 : \dots : x^n] \longmapsto (x^0/x^i, \dots, x^{i-1}/x^i, x^{i+1}/x^i, \dots, x^n/x^i)$

note $\phi^{-1} : \mathbb{R}^n \longrightarrow U_i$

$(x^1, \dots, x^n) \longmapsto [x^1 : \dots : x^{i-1} : 1 : x^i : \dots : x^n]$

exercise: ϕ_i is one-to-one and onto

ϕ_i is continuous

ϕ_i^{-1} is continuous

so ϕ_i is a coordinate chart

so $\mathbb{R}P^n$ is an n-manifold ($U_0, \dots, U_n$ cover $\mathbb{R}P^n$)

actually not! need

exercise: $\mathbb{R}P^n$ is Hausdorff and 2nd countable

Remark: this is the first example that is not obviously a subset of $\mathbb{R}^k$, some k.

the above exercise is some what hard but would be harder to get embedding

exercise: Show

$$\tilde{f}: S^2 \to \mathbb{R}^4 : (x, y, z) \mapsto (yz, xz, xy, x^2+2y^2+3z^2)$$

induces an embedding $f: \mathbb{R}P^2 \to \mathbb{R}^4$

exercise:

1) $\mathbb{C}P^n = \mathbb{C}^{n+1} - \{(0,\dots,0)\} / \mathbb{C} - \{0\}$ complex projective space

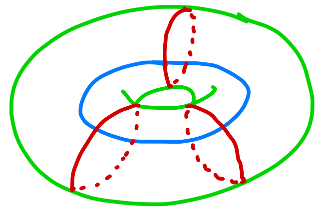
is a $2n$-manifold

2) Show $\mathbb{C}P^1$ is homeomorphic to S^2

4) Products of an n-manifold and an m-manifold is an $(n+m)$-manifold

exercise: Prove this

so, for example, $T^2 = S^1 \times S^1$ is a 2-manifold



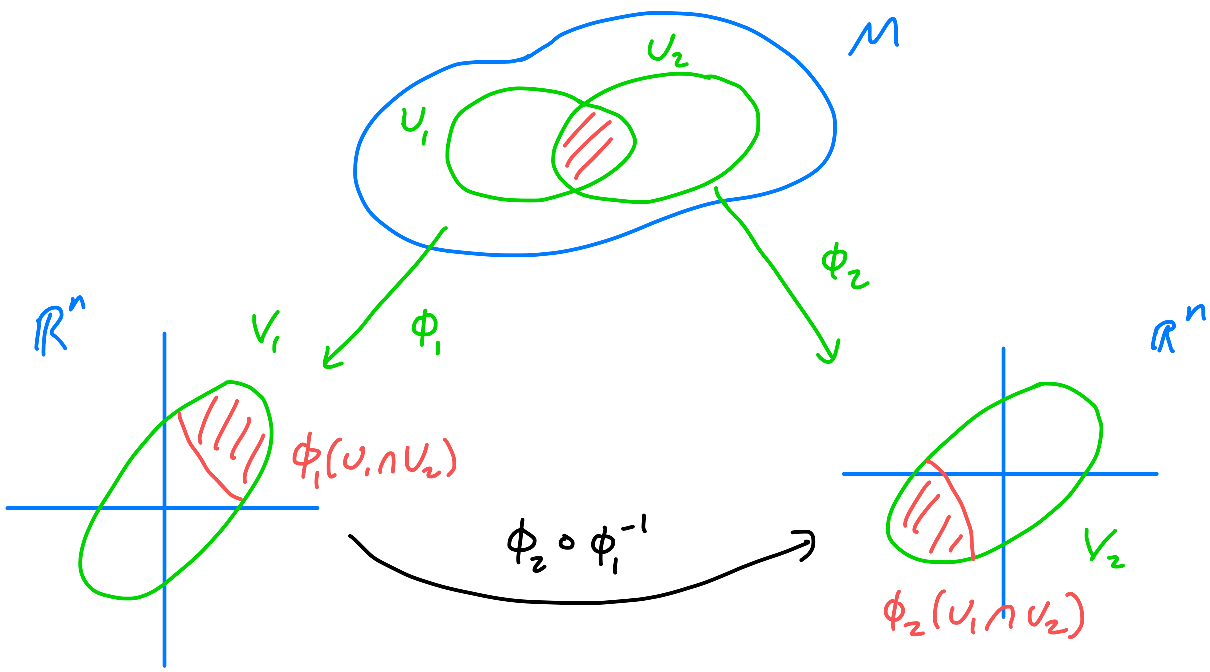
B. Smooth Manifolds

topological manifolds are interesting but to "do calculus" we need more structure

we begin with:

given an n-manifold M and two coordinate charts

$$\phi_1 : U_1 \to V_1 \quad \text{and} \quad \phi_2 : U_2 \to V_2$$



we say they are smoothly compatible if

called coordinate transformation or transition map

$$\begin{cases} \phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \longrightarrow \phi_2(U_1 \cap U_2) & \text{and} \\ \phi_1 \circ \phi_2^{-1} : \phi_2(U_1 \cap U_2) \longrightarrow \phi_1(U_1 \cap U_2) \end{cases}$$

have continuous partial derivatives of all orders at all points a smooth atlas for M is a collection of coordinate charts $\mathcal{A} = \{\phi_\alpha : U_\alpha \to V_\alpha\}_{\alpha \in A}$ such that

1) $\{U_\alpha\}_{\alpha \in A}$ covers M (ie. $M = \bigcup_{\alpha \in A} U_\alpha$) and
2) all charts are smoothly compatible

Remark: could similarly define

C^k-atlas (for $\phi_2 \circ \phi_1^{-1}$ k-times continuously diff.)

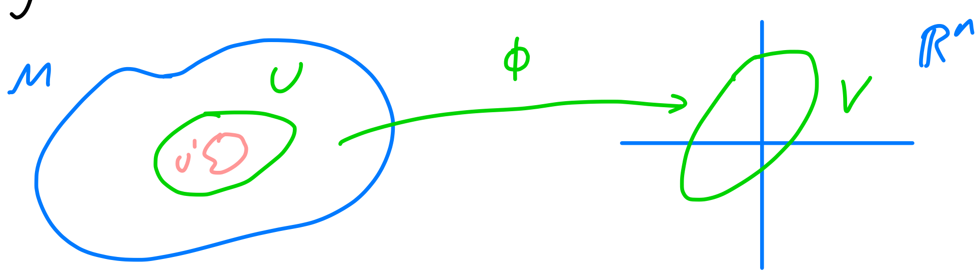
C^ω-atlas (for $\phi_2 \circ \phi_1^{-1}$ real analytic)

complex atlas (for $V_\alpha \in \mathbb{C}^n$ and $\phi_2 \circ \phi_1^{-1}$ complex analytic)

C^0-atlas any topological manifold has this!

we want to say that a smooth atlas gives a smooth structure on M, but this leads to problems

e.g. given $\phi : U \to V$ in an atlas $\mathcal{A}$ for M



pick any open $U' \subset U$ then let

$$\mathcal{A}' = \mathcal{A} \cup \{\phi|_{U'} : U' \to \phi(U')\}$$

this is also a smooth atlas for M different from $\mathcal{A}$!

so we would get infinitely many smooth structures on M if we just used a smooth atlas to define them

lemma 1:

M a manifold

1) every smooth atlas for M is contained in a unique maximal smooth atlas

2) two smooth atlases for M determine the same maximal atlas
$\iff$
their union is a smooth atlas

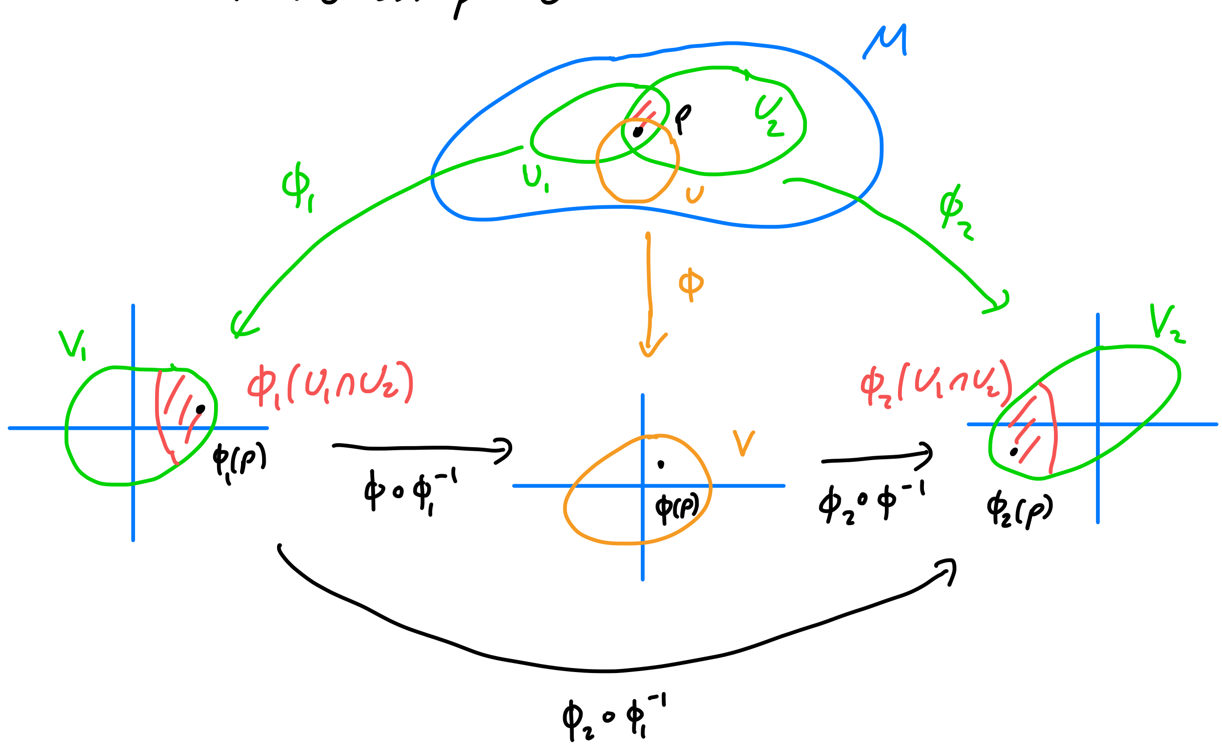
Proof: 1) given an smooth atlas $\mathcal{A}$ for M

let $\bar{\mathcal{A}} = \{$ all coordinate charts for M smoothly compatible with all charts in $\mathcal{A}\}$

Claim: $\bar{\mathcal{A}}$ is an atlas for M

Indeed, if $\phi_1 : U_1 \to V_1$ and $\phi_2 : U_2 \to V_2$ are in $\bar{\mathcal{A}}$

for any $p \in U_1 \cap U_2$ there is a chart $\phi : U \to V$ in $\mathcal{A}$ s.t. $p \in U$



so $$\phi_2 \circ \phi_1^{-1} = \underbrace{(\phi_2 \circ \phi^{-1})}_{\substack{\text{smooth at} \\ \phi(p) \text{ by} \\ \text{hypothesis}}} \circ \underbrace{(\phi \circ \phi_1^{-1})}_{\substack{\text{smooth at} \\ \phi_1(p) \text{ by} \\ \text{hypothesis}}}$$

so $\phi_2 \circ \phi_1^{-1}$ is smooth at $\phi_1(p)$ by the chain rule

since p was any point in $U_1 \cap U_2$ we see

$\phi_2 \circ \phi_1^{-1}$ is smooth at all points of $\phi_1(U_1 \cap U_2)$ ✓

proof of 2 is an exercise

We define a <u>smooth manifold</u> to be a manifold M together with a maximal smooth atlas

We also say a maximal smooth atlas for M is a <u>smooth structure</u> on M

We can easily describe smooth structures by giving a smooth atlas.

<u>examples</u>:

1) $M = \mathbb{R}^n$
$$\mathcal{A} = \{ id : \mathbb{R}^n \to \mathbb{R}^n \}$$
gives a smooth structure on $\mathbb{R}^n$

2) S^2 can be given a smooth structure with stereographic coordinates.

exercise: show the transition function between the two charts is

$$\phi : \mathbb{R}^2 - \{(0,0)\} \to \mathbb{R}^2 - \{(0,0)\}$$
$$(x,y) \longmapsto \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$$

and similarly S^n is a smooth manifold via stereographic coordinates

3) $\mathbb{RP}^n$ has coordinate charts

$$\phi_i : U_i \longrightarrow V_i$$

$\{[x^0 : \dots : x^n] \mid x^i \neq 0\}$

$\mathbb{R}^n$

means leave out

$$\phi_i([x^0 : \dots : x^n]) = \left(x^0/x^i, \dots, \widehat{x^i/x^i}, \dots x^n/x^i\right)$$

$$\phi_i^{-1}(x^1, \dots, x^n) = [x^1 : \dots : 1 : \dots : x^n]$$

i^{th} slot

so $\phi_i \circ \phi_j^{-1}(x^1, \dots, x^n) = \left(x^1/x^i, \dots, x^n/x^i\right)$

and $\phi_j(U_i \cap U_j) = \{(x^1, \dots x^n) \mid x^i \neq 0\}$

$\therefore$ $\phi_i \circ \phi_j^{-1}$ is smooth

and $\mathbb{RP}^n$ has a smooth structure

4) If N is an open subset of a smooth manifold M it gets a smooth structure

i.e. given $\mathcal{A}$ for M let

$$\mathcal{A}_N = \{\phi|_{U \cap N} : U \cap N \to \phi(U \cap N) \text{ for all } \phi : U \to V \text{ in } \mathcal{A}\}$$

5) $Mat(n,m;\mathbb{R}) = \{n\times m \text{ matricies w/ real coefficients}\}$

<u>note</u>: $Mat(n,m;\mathbb{R}) = \mathbb{R}^{n\times m}$ so it is a smooth manifold

6) $GL(n,\mathbb{R}) = \{\text{invertable } n\times n \text{ matricies}\}$

recall $GL(n,\mathbb{R}) = \det^{-1}(\mathbb{R}-\{0\})$

<u>note</u>: $\det : Mat(n,n;\mathbb{R}) \to \mathbb{R}$ is continuous (smooth)

e.g. $\det : Mat(2,2;\mathbb{R}) \longrightarrow \mathbb{R}$

$$\mathbb{R}^4 \ni (a,b,c,d) \longmapsto \underset{\text{polynomial}}{ad-bc}$$

$$(a,b,c,d) \leftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

by induction

$$\det M = \sum_{i=1}^{n} a_{1i} (-1)^i \underbrace{\det M_{1i}}_{\text{polynomial by induction}}$$

delete 1^{st} row i^{th} column of M

is smooth

so $GL(n,\mathbb{R})$ is an open subset of $Mat(n,n;\mathbb{R})$

$\therefore$ a smooth manifold (of dimension n^2)

see book(s) for more examples, but first

<u>worrisome example:</u>

consider $M = \mathbb{R}$ and $\mathcal{B} = \{ f : \mathbb{R} \to \mathbb{R},\ x \mapsto x^3 \}$

<u>note</u>: $\mathcal{B}$ is not compatible with $\mathcal{A} = \{ id : \mathbb{R} \to \mathbb{R} \}$

since $x \mapsto x^{1/3}$ is not differentiable at 0

so $\mathcal{A}$ and $\mathcal{B}$ are different smooth structures on $\mathbb{R}$!

but note $g: \mathbb{R} \to \mathbb{R} : x \to x^3$ takes $id \in \mathcal{A}$ to $\underset{\underset{id \circ g}{\shortparallel}}{f} \in \mathcal{B}$

exercise: let $\bar{\mathcal{A}}, \bar{\mathcal{B}}$ be the maximal atlases associated to $\mathcal{A}$ and $\mathcal{B}$, then

$$\{\phi : U \to V\} \in \bar{\mathcal{A}}$$
$$\Longleftrightarrow$$
$$\{\phi \circ g : g^{-1}(U) \to V\} \in \bar{\mathcal{B}}$$

so the smooth structures on $\mathbb{R}$ are really "isomorphic" (that is related by a bijection) the term we use for this is "diffeomorphic"

two smooth manifolds $(M, \mathcal{A})$ and $(N, \mathcal{B})$ are diffeomorphic if there is a homeomorphism $f: M \to N$ such that

$$\phi \in \mathcal{B} \quad \text{iff} \quad \phi \circ f \in \mathcal{A}$$

wouldn't work if didn't take maximal atlas!

we will come back to this later but first

Interesting Facts (way beyond this course)

1) if M is a topological n-manifold, then

a) $n \leq 3$, M has a smooth structure

($n=1$ "easy"
$n=2$ Rådo
$n=3$ Moise)

b) $n \geq 4$, some M do not have any smooth structure!

2) if M is an n-manifold (and compact) and M has a smooth structure, then

a) $n \leq 3$, smooth structure is unique (upto diffeomorphism)

b) $n \geq 5$, there are finitely many smooth structures on M

e.g. S^7 has 28 smooth structures! (Milnor)

c) $n = 4$, M frequently has infinitely many smooth structures!

3) $\mathbb{R}^n$ has a unique smooth structure (upto diffeo.)

$$\Longleftrightarrow$$

$$n \neq 4$$

(Stallings)

$\mathbb{R}^4$ has uncountably many smooth structures (Donaldson + Freedman)

4) Classification

1-manifolds: S^1 compact

$\mathbb{R}^1$ non compact

2-manifolds:

compact: S^2 

T^2 

$\vdots$

Σ_g 

$\mathbb{R}P^2$ 

$N_n = N_{n-1} \# \mathbb{R}P^n$ ← connect sum (later)

n-manifolds, $n \geq 5$:

no classification, but if you fix the "fundamental group" then the classification reduces to hard problems in algebraic topology

3-manifolds:

hard, no classification, but we know a <u>lot</u>

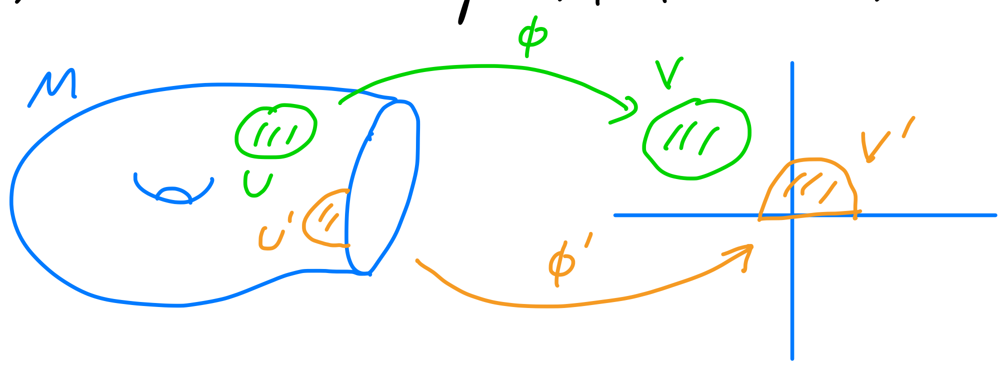
4-manifolds:

no idea!

C. Manifolds with boundary

an n-manifold with boundary is a Hausdorff, 2^{nd} countable space M such that for every point $p \in M$ there is

1) a neighborhood U of p in M
2) an open set $V \subset \mathbb{R}^n_{\geq 0} = \{(x^1, \dots, x^n) \mid x^n \geq 0\}$
3) and a homeomorphism $\phi : U \to V$



interior of M

let $\operatorname{int} M = \{p \in M$ with a chart $\phi : U \to V$ s.t. $\phi(p)$ has positive x^n-coordinate$\}$

$\partial M = \{p \in M$ with a chart $\phi : U \to V$ s.t. $\phi(p)$ has x^n-coordinate $0\}$

boundary of M

exercises:
1) ∂M is an $(n-1)$-manifold
2) $\operatorname{int} M$ is an n-manifold
3) $\partial(\partial M) = 0$
4) $\partial(\operatorname{int} M) = 0$

5) $\operatorname{int}(\partial M) = \partial M$

6) $(\operatorname{int} M) \cap \partial M = \emptyset$

to discuss smooth manifolds with boundary we need:

if $A \subset \mathbb{R}^n$ is an arbitrary subset and $f : A \to \mathbb{R}^k$ is a function

then f is smooth if for each $x \in A$ $\exists$ an open neighborhood U of x in $\mathbb{R}^n$ and a smooth function $F : U \to \mathbb{R}^k$ such that

$$F|_{U \cap A} = f|_{U \cap A}$$

note: if $x \in$ open set $\subset A$, then this is the ordinary def$^{\underline{n}}$ of smooth function

now a smooth structure on a manifold with boundary M is a maximal atlas of smoothly compatible coordinate charts.

D. Smooth Maps

let M and N be smooth manifolds

$f : M \to N$ a map

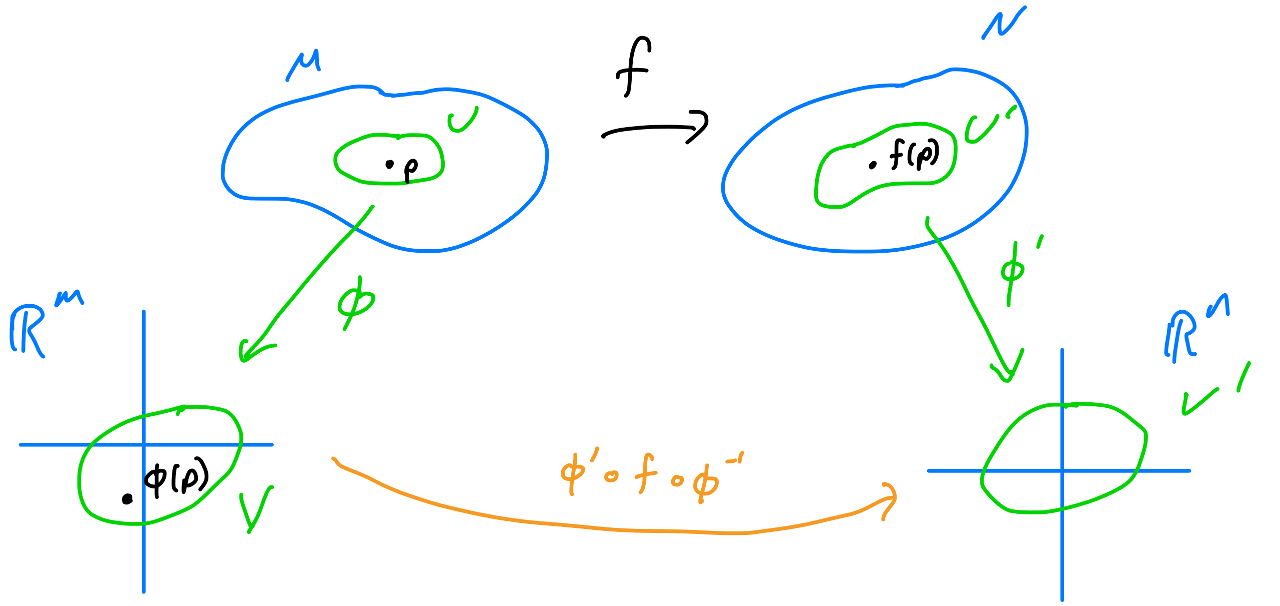
f is smooth at $p \in M$ if there is

1) a coordinate chart $\phi : U \to V$ about p in M

2) a coordinate chart $\phi' : U' \to V'$ about $f(p)$ in N

such that

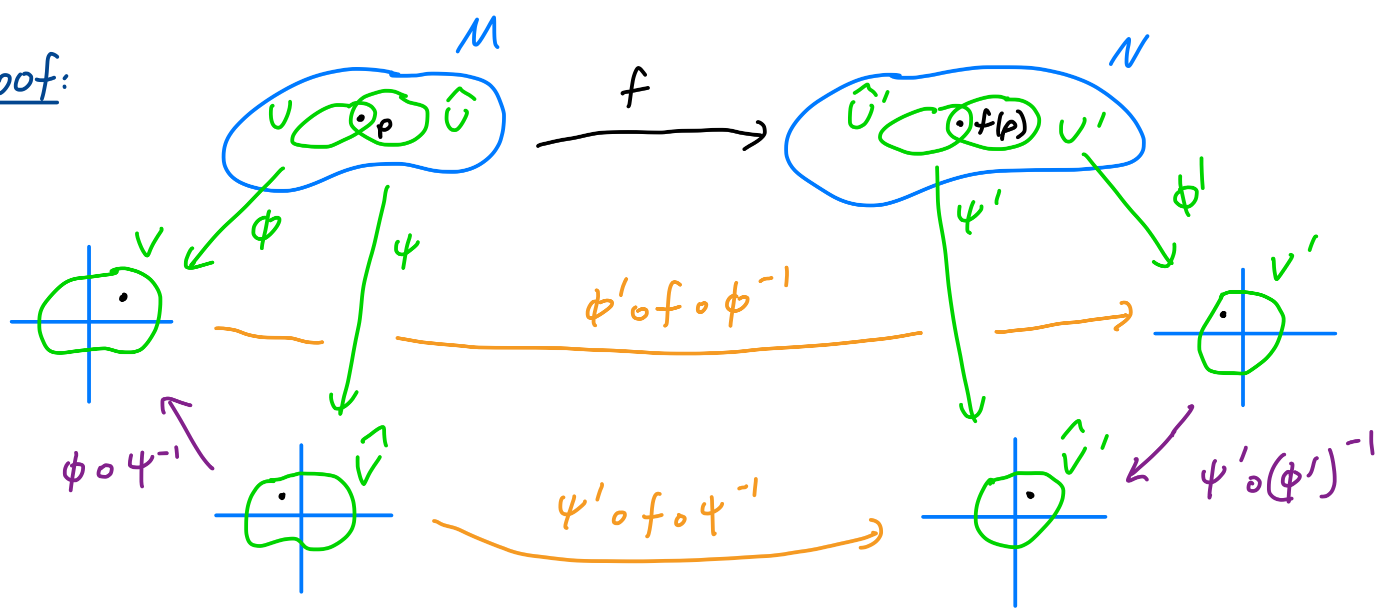
$$\phi' \circ f \circ \phi^{-1} : \underset{\mathbb{R}^m}{\underset{\cap\!\shortmid}{V}} \to \underset{\mathbb{R}^n}{\underset{\cap\!\shortmid}{V'}} \quad \text{is smooth at } \phi(p)$$



lemma 2:

> if f is smooth at p then for *any* coord. chart
> $\psi : \hat{U} \to \hat{V}$ about p and $\psi' : \hat{U}' \to \hat{V}'$
> about $f(p)$ we have
> $$\psi' \circ f \circ \psi^{-1}$$
> is smooth at $\psi(p)$

Proof:



$$\psi'\circ f\circ\psi^{-1} = \underbrace{(\psi'\circ(\phi')^{-1})}_{\substack{\text{smooth at} \\ \text{all points} \\ \text{since charts} \\ \text{compatible}}}\circ\underbrace{(\phi'\circ f\circ\phi^{-1})}_{\substack{\text{smooth at} \\ \phi(p)=\phi(\psi^{-1}(\psi(p))) \\ \text{by hypoth.}}}\circ\underbrace{(\phi\circ\psi^{-1})}_{\substack{\text{smooth at} \\ \text{all points} \\ \text{since charts} \\ \text{compatible}}}$$

so $\psi'\circ f\circ\psi^{-1}$ smooth at $\psi(p)$ by the chain rule

$f: M\to N$ is <u>smooth on an open set</u> $U\subset M$ if it is smooth at all points $p\in U$

it is <u>smooth</u> if it is smooth at all $p\in M$

<u>exercise:</u> $f: M\to N$ is smooth on M

$\iff$

for any atlas $\mathcal{A}$ for M and $\mathcal{B}$ for N
$\phi\circ f\circ\psi^{-1}$ is smooth (where defined)
for $\phi\in\mathcal{B}$ and $\psi\in\mathcal{A}$

<u>examples:</u> 1) since an atlas for $\mathbb{R}^k$ is $\{id:\mathbb{R}^k\to\mathbb{R}^k\}$

a function

$$f:\mathbb{R}^n\to\mathbb{R}^m$$

is smooth $\iff$ it is smooth in the calculus sense!

2) $f: M\to\mathbb{R}^k$ is smooth

$\iff$

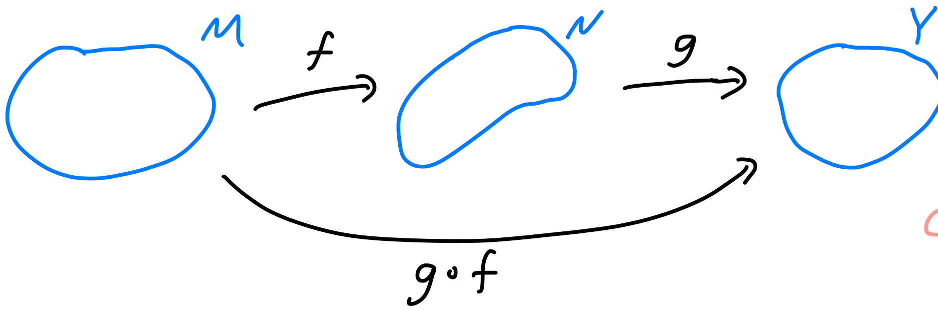
for every chart $\phi: U\to V$ of M,

$f\circ\phi^{-1}: V\to\mathbb{R}^k$ is smooth.

exercises:

1) $f: M \to N$ a smooth map, then f is continuous

2) compositions of smooth maps are smooth



chain rule

notation: $C^\infty(M,N) = \{\text{set of smooth maps } M \to N\}$

$C^\infty(M) = C^\infty(M, \mathbb{R})$ ← note this is a vector space

a map $f: M \to N$ is a diffeomorphism if it is a homeomorphism and both f and f^{-1} are smooth

example: $\mathbb{R}$ with its "standard" smooth structure

$$f: \mathbb{R} \to \mathbb{R} : x \mapsto x^3$$

is ① a homeomorphism and
② smooth

but f^{-1} is not smooth so f not a diffeomorphism

Remark: as mentioned above, diffeomorphism is the fundamental equivalence in the study of smooth manifolds

exercise: Show this definition of diffeomorphism agrees with the one in Section B.

examples of smooth maps:

1) let $i : S^2 \to \mathbb{R}^3 : (x,y,z) \mapsto (x,y,z)$ be the inclusion map

Recall we have $\phi : (S^2 - \{N\}) \to \mathbb{R}^2$

$$(x,y,z) \longmapsto \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$$

and

$$\phi^{-1} : \mathbb{R}^2 \to (S^2 - \{N\})$$

$$(x,y) \mapsto \frac{1}{1+x^2+y^2}(2x, 2y, x^2+y^2-1)$$

so $\quad i \circ \phi^{-1} : \mathbb{R}^2 \to \mathbb{R}^3$

$$(x,y) \longmapsto \left(\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, \frac{x^2+y^2-1}{1+x^2+y^2}\right)$$

is smooth on $\mathbb{R}^2$

similarly for the other coordinate chart

so i is smooth

(similarly for $S^n \to \mathbb{R}^{n+1}$)

2) $\pi : (\mathbb{R}^{n+1} - \{(0,\dots,0)\}) \to \mathbb{R}P^n$

$$(x^0, \dots, x^n) \longmapsto [x^0 : \dots : x^n]$$

local charts $\quad U_i = \{[x^0 : \dots : x^n] \mid x^i \neq 0\}$

$\quad V_i = \mathbb{R}^n$

$\quad \phi_i([x^0 : \dots : x^n]) = (x^0/x^i, \dots, \widehat{x^i/x^i}, \dots, x^n/x^i)$

leave out

so

$$\phi_i \circ \pi : (\mathbb{R}^{n+1} - \{(0,\dots,0)\}) \to \mathbb{R}^n$$

is smooth ~~at~~ on $\pi^{-1}(U_i) = \{(x^0, \dots, x^n) \mid x^i \neq 0\}$

so π smooth at all points of $\mathbb{R}^{n+1} - \{(0,\dots,0)\}$

$\therefore$ π is smooth.

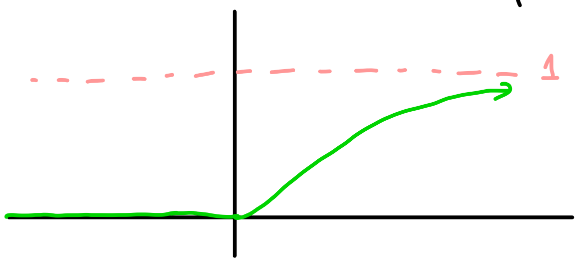
3) $p : S^n \to \mathbb{R}P^n$

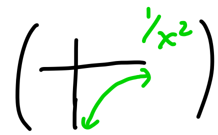
where $p = \pi \circ i$ is smooth by composition rule

(but could also check using charts)

4) Bump functions

consider $f : \mathbb{R} \to \mathbb{R} : x \longmapsto \begin{cases} 0 & x \le 0 \\ e^{-1/x^2} & x > 0 \end{cases}$

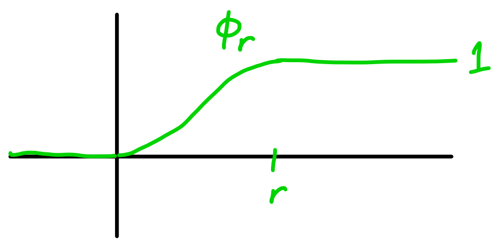




exercise: show f is smooth

set $\phi_r(x) = \dfrac{f(x)}{f(x) + f(r-x)}$ $\quad r > 0$

note: $f(x) + f(r-x) > 0$ so $\phi_r(x)$ is well-defined

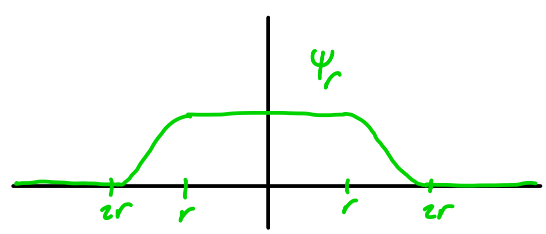


set $\psi_r(x) = 1 - \phi_r(|x| - r)$

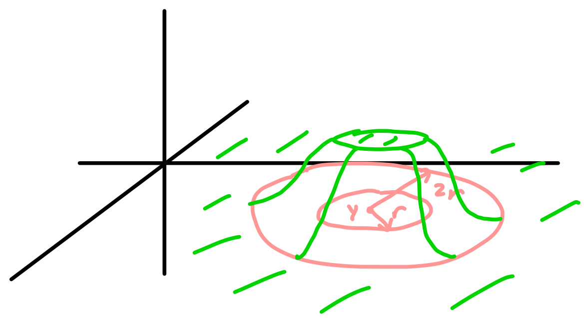
note: $\psi_r(x) = 1 \iff |x| \le r$

$\psi_r(x) = 0 \iff |x| \ge 2r$

$0 \le \psi_r(x) \le 1$



set $\Psi_{y,r} : \mathbb{R}^m \to \mathbb{R} : x \mapsto \psi_r(\|x-y\|)$



now given $p \in M$ let $\phi : U \to V$ be a coordinate chart about p and say $y = \phi(p)$

there is some $r_0 > 0$ s.t. $B_{2r_0}(y) \subset V$

set $f_p : M \to \mathbb{R} : x \longmapsto \begin{cases} \Psi_{y,r_0} \circ \phi(x) & x \in U \\ 0 & x \notin U \end{cases}$

note: 1) f is smooth

2) given any open set $\mathcal{O}$ containing p we could have arranged that $\exists$ open sets $\mathcal{O}_p$ and $\mathcal{O}_p'$ s.t

$p \in \mathcal{O}_p \subset \mathcal{O}_p' \subset \mathcal{O}$ and

$f_p(x) = 1 \Leftrightarrow x \in \mathcal{O}_p$ and $f_p(x) = 0 \Leftrightarrow x \notin \mathcal{O}_p'$

f_p is a bump function at p

so any manifold has lots of non-constant smooth functions, that is $C^\infty(M)$ is big!