II Tangent Space and Linearization

A. Tangent Space

we want to generalize the definition of derivative

to functions $f: M \to N$

but def$^{\underline{n}}$ from calculus uses the linear

structure on $\mathbb{R}^n$

add multiply

$$D_v f(p) = \lim_{h \to 0} \frac{f(p+hv) - f(p)}{h}$$

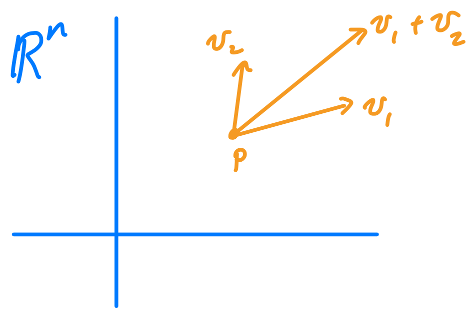
but M and N don't have such linear structure

and what is v for a manifold!

so we first try to understand a vector v at a point $p \in \mathbb{R}^n$

recall from calculus it was useful to consider the space

of vectors based at a point $p \in \mathbb{R}^n$



these for a vector space that we denote $\mathbb{R}^n_p$

we would like to generalize this to a manifold M

since we don't have a linear structure on M we need

another way to think about vectors

"judge a vector by its actions"

to this end we define

a derivation at $p \in \mathbb{R}^n$ is a linear map

$$D : C^\infty(\mathbb{R}^n, \mathbb{R}) \longrightarrow \mathbb{R}$$

such that

$$D(fg) = (Df)\, g(p) + f(p)(Dg)$$ product rule

lemma 1:

1) for any $v \in \mathbb{R}^n_p$ the directional derivative of f at p in the direction v

$$D_v f(p) = \lim_{h \to 0} \frac{f(p+hv) - f(p)}{h}$$
$$= \frac{d}{dt} f(p+tv) \Big|_{t=0}$$

is a derivation.

2) if $D : C^\infty(\mathbb{R}^n, \mathbb{R}) \longrightarrow \mathbb{R}$ is a derivation at p then there is a vector $v \in \mathbb{R}^n_p$ s.t. $D = D_v$

Proof: 1) D_v is clearly linear and satisfies the product rule from calculus

2) let $x^1, \dots, x^n$ be the coordinate functions

$$x^i : \mathbb{R}^n \to \mathbb{R} : (x^1, \dots, x^n) \mapsto x^i$$

set $v^i = D x^i \in \mathbb{R}$

given any function $f : \mathbb{R}^n \to \mathbb{R}$, then

$$f(x+p) - f(p) = \int_0^1 \frac{d}{dt} f(p+tx)\, dt$$

$$= \int_0^1 \frac{d}{dt} f(p^1 + t x^1, \dots, p^n + t x^n)\, dt$$

$$= \int_0^1 \sum_{i=1}^n x^i \frac{\partial f}{\partial x^i}(p + t x)\, dt$$

$$= \sum_{i=1}^n x^i \underbrace{\int_0^1 \frac{\partial f}{\partial x^i}(p + t x)\, dt}_{g_i(x)}$$

$$= \sum_{i=1}^n x^i g_i(x)$$

so $f(x+p) = f(p) + \sum_{i=1}^n x^i g_i(x)$

and $f(x) = f(p) + \sum_{i=1}^n (x^i - p^i)\, g_i(x-p)$

note: $Dc = cD1 = c\,D(1 \cdot 1)$

constant function

$$= c\,((D1)\cdot 1 + 1 \cdot (D1))$$

$$= c\,(2(D1)) = 2(Dc)$$

so $Dc = 0$

thus

$$Df = \cancelto{0}{D(f(p))} + D \sum_{i=1}^n (x^i - p^i)\, g_i(x-p)$$

$$= \sum_{i=1}^n (Dx^i - \cancelto{0}{Dp^i})\, g_i(0) + \cancelto{0}{(p^i - p^i)}\, D g_i(x-p)$$

$$= \sum_{i=1}^n v^i g_i(0)$$

if we set $v = \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix}$ then as above

$$D_v f(p) = \dots = \sum v^i g_i(0) = Df$$

so $D = D_v$

let $T_p\mathbb{R}^n = \{\text{set of derivations at } p\}$

<u>note</u>: $T_p\mathbb{R}^n$ is a vector space

$D_1, D_2 \in T_p\mathbb{R}^n$ then define $(D_1 + D_2)f = D_1 f + D_2 f$

and $(rD_1)f = r(D_1 f)$

<u>exercise</u>: these are derivations

<u>Th$^{\underline{m}}$ 2</u>:

the map

$$\begin{aligned} \mathbb{R}^n_p &\longrightarrow T_p\mathbb{R}^n \\ v &\longmapsto D_v \end{aligned}$$

is an isomorphism.

<u>Proof</u>: from calculus we know this is a linear map

$$D_{(v+aw)} f = D_v f + a D_w f$$

lemma 1 says the map is onto.

to see the map is injective suppose

$$D_v f = 0 \quad \text{for all } f \in C^\infty(\mathbb{R}^n)$$

let $v = \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix}$

consider the $i^{\underline{th}}$ coordinate function

$$x^i : \mathbb{R}^n \to \mathbb{R} : (x^1, \dots, x^n) \longmapsto x^i$$

then $\quad 0 = D_v x^i = \sum_{j=1}^{n} v^j \frac{\partial}{\partial x^j} x^i = v^j$

so $v = 0$ ▦

now given a manifold M and a point $p \in M$

a derivation at p is a linear map

$$D : C^\infty(M) \longrightarrow \mathbb{R}$$

such that

$$D(fg) = (Df)\, g(p) + f(p)(Dg)$$

set $T_p M = \{\text{derivations at } p\}$

as above $T_p M$ is a vector space

$T_p M$ is called the tangent space of M at p

You should think of elements of $T_p M$ as "directional derivatives" and hence "tangent vectors"

(and they are for $M = \mathbb{R}^n$)

now if we have a smooth map

$$f : M \to N$$

then we get a map

$$df_p : T_p M \to T_{f(p)} N$$

as follows

if $v \in T_p M$ and $g \in C^\infty(N)$

then $g \circ f \in C^\infty(M)$

so $v(g \circ f) \in \mathbb{R}$

thus we set

$$df_p(v)(g) = v(g \circ f)$$

note: 1) $df_p(v)(g_1 + g_2) = v((g_1+g_2) \circ f)$

$= v(g_1 \circ f + g_2 \circ f)$

linearity of v

$= v(g_1 \circ f) + v(g_2 \circ f)$

$= df_p(v)(g_1) + df_p(v)(g_2)$

similarly $df_p(v)(ag) = a\, df_p(v)(g)$

2) $df_p(v)(gh) = v((gh) \circ f) = v((g \circ f)(h \circ f))$

product rule for v

$= (v(g \circ f))(h \circ f(p)) + (g \circ f(p))(v(h \circ f))$

$= [df_p(v)(g)]\, h(f(p)) + g(f(p))\, [df_p(v)(h)]$

so $\boxed{df_p(v) \in T_{f(p)} N}$

call df_p the differential of f at p.

exercise: 1) $df_p : T_p M \to T_{f(p)} N$ is a linear map

2) $f : M \to N$, $g : N \to W$, then

$$d(g \circ f)_p = (dg_{f(p)}) \circ (df_p)$$ chain rule!

3) $id_M : M \to M$ then

$$d(id_M)_p = id_{T_pM}$$

4) $f: M \to N$ a diffeomorphism, then df_p an isomorphism

Thm 3:

$$\boxed{\begin{array}{l}\text{let } U \subseteq M \text{ be an open set, then} \\ \qquad i: U \to M \\ \text{the inclusion map, induces an isomorphism} \\ \qquad di_p : T_pU \to T_pM \\ \text{for all } p \in U\end{array}}$$

note: from the Theorem we can compute the dimension of T_pM. To see this let $\phi: U \to V$ be a coordinate chart for M about p

$$T_pM \underset{\text{Thm 3}}{\cong} T_pU \underset{\substack{\text{exercise}\\\text{above}}}{\cong} T_{\phi(p)}V \underset{\text{Thm 3}}{\cong} T_{\phi(p)}\mathbb{R}^n \underset{\text{Thm 2}}{\cong} \mathbb{R}^n_{\phi(p)} = \mathbb{R}^n$$

so

$$\boxed{\text{dimension of } T_pM = n}$$

to prove the theorem we need

lemma 4:

$$\boxed{\text{If } f, g \in C^\infty(M) \text{ and } f = g \text{ on an open set containing } p \text{ then for all } v \in T_pM \quad vf = vg}$$

Proof: let $h = f - g$

there is some open set $\mathcal{O}$ s.t. $p \in \mathcal{O}$ and $h = 0$ on $\mathcal{O}$

from the end of last section there is a "bump function" $\psi \in C^\infty(M)$ such that

$\psi_p = 1 -$ bump function from last section

1) $\psi = 0$ on a neighborhood of p

2) $\psi = 1$ outside $\mathcal{O}$

so $\psi h = h$ since $\psi = 1$ where $h \neq 0$

now
$$v(h) = v(\psi h) = (v\psi)\cancelto{0}{h(p)} + \cancelto{0}{\psi(p)}(vh)$$
$$= 0$$

so $vf - vg = v(f-g) = 0$ $\blacksquare$

Proof of Th^{m} 3: we know di_p is a linear map so we just need to show it is bijective

Injective: suppose $v \in T_pU$ and $di_p(v) = 0$ in T_pM

so $di_p(v) : C^\infty(M) \to \mathbb{R}$ is the zero map we want to show $v : C^\infty(U) \to \mathbb{R}$ is also zero map

given $f \in C^\infty(U)$ we need to extend it to M

to do this we use bump functions!

i.e. there are open sets $O_p \subset O_p' \subset U$

s.t. $p \in O_p$ and there is a

function $\psi : M \to \mathbb{R}$ s.t.

$\psi = 1$ on $\bar{O}_p$ and 0 outside O_p'

now set
$$\bar{f} = \begin{cases} \psi \cdot f & \text{on } U \\ 0 & \text{outside } U \end{cases}$$

clearly $\bar{f} \in C^\infty(M)$

by hypoth.

now
$$vf = v(\bar{f}|_U) = v(\underbrace{\bar{f} \circ i}) = di_p(v)(\bar{f}) = 0$$

lemma 4 since agree on O_p $\bar{f} \circ i = \bar{f}|_U$ def^n di_p

since $f \in C^\infty(U)$ was arbitrary, $v = 0$ ✓

<u>Surjective</u>: let $v \in T_p M$

for any $f \in C^\infty(U)$ let $\bar{f}$ be as above

and define $\bar{v} f = v \bar{f}$

it is clear $\bar{v} : C^\infty(U) \to \mathbb{R}$ is a derivation

if not clear check it!

and note for all $g \in C^\infty(M)$

$$di_p(\bar{v})g = \bar{v}(g \circ i)$$

$$= \bar{v}(g|_U) = v(\overline{g|_U})$$

$$= v\,g$$

lemma 4 since $g, g|_U$ agree on O_p

thus $di_p(\bar{v}) = v$ and di_p is surjective $\boxplus$

local coordinate representation of vectors

recall Th$^{\underline{m}}$ 2 says for any $D \in T_p\mathbb{R}^n$ there is a vector $v \in \mathbb{R}^n_p$ s.t. $D = D_v$

directional derivative

thus given coordinants $(x^1, \dots, x^n)$ on $\mathbb{R}^n$ we get a basis $\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\}$ for $\mathbb{R}^n_p$

and we can write $v = \sum_{i=1}^n v^i \frac{\partial}{\partial x^i}$

now given a coordinate chart $\phi: U \to V$ for a manifold M

for any $p \in U$ and $v \in T_p M$ we can consider $d\phi_p(v) \in T_{\phi(p)}\mathbb{R}^n \cong \mathbb{R}^n_{\phi(p)}$

so $d\phi_p(v) = \sum_{i=1}^n v^i \frac{\partial}{\partial x^i}$

for some $v^i \in \mathbb{R}$

and

$$v = \sum_{i=1}^{n} v^i \underbrace{d(\phi^{-1})_{\phi(p)}\left(\frac{\partial}{\partial x^i}\right)}$$

we will abuse notation and denote this $\frac{\partial}{\partial x^i}$

so $\frac{\partial}{\partial x^i}$ can mean a vector in $\mathbb{R}^n_{\phi(p)}$ or in T_pM depending on context

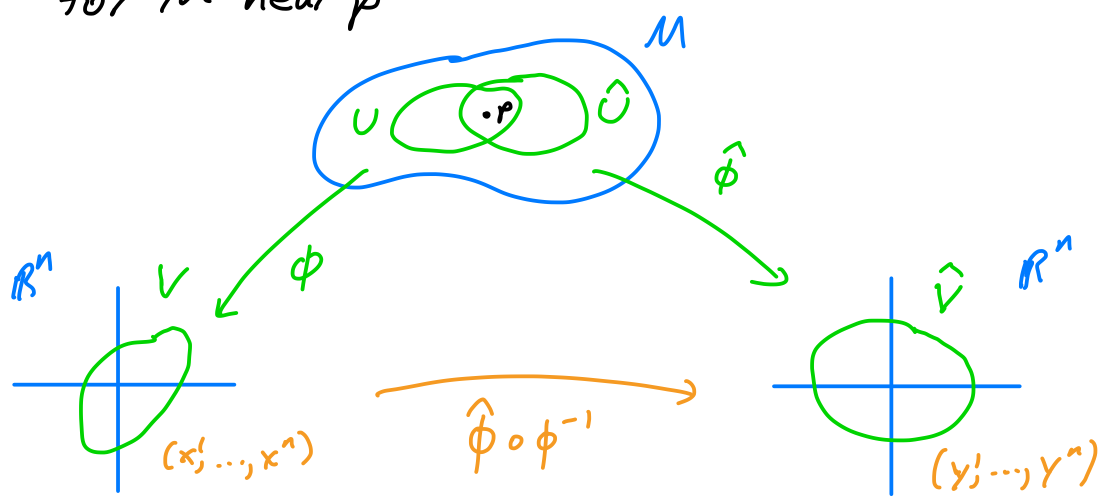
but recall ϕ identifies U and V so we really should not distinguish things in U and V (but recall we need ϕ for this)

if we want to emphasize $\frac{\partial}{\partial x^i}$ as a vector in T_pM we write "$\frac{\partial}{\partial x^i}$"

so $\boxed{\left\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right\} \text{ is a basis for } T_pM}$

using the chart $\phi : U \to V$

now suppose $\hat{\phi} : \hat{U} \to \hat{V}$ is another coordinate chart for M near p



a vector $v \in T_pM$ can be written

$$v = \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i} \quad \text{using } \phi$$

and

$$v = \sum_{i=1}^{n} \hat{v}^i \frac{\partial}{\partial y^i} \quad \text{using } \hat{\phi}$$

how do these representations relate?

note: $\hat{\phi} \circ \phi^{-1}(x^1, \dots, x^n) = (y^1(x^1, \dots, x^n), \dots, y^n(x^1, \dots, x^n))$

each $y^i : \phi(U \cap \hat{U}) \to \mathbb{R}$ is the $i^{\underline{th}}$ coordinate function

recall that to find the coefficients of a vector

$$w = \sum w^i \frac{\partial}{\partial y^i}$$

we just see how w "acts" on the coordinate functions

$$w(y_j) = \sum_{i=1}^{n} w^i \frac{\partial}{\partial y^i}(y^j) = w^j$$

so consider

vector

$$\left(d(\hat{\phi} \circ \phi^{-1})_{\phi(p)} \frac{\partial}{\partial x^i}\right)(y^j)$$

$$= \frac{\partial}{\partial x^i}\left(y^j \circ \hat{\phi} \circ \phi^{-1}\right)\Big|_{\phi(p)}$$

directional derivative

$$= D_{\frac{\partial}{\partial x^i}}\left(y^j(x^1, \dots, x^n)\right)\Big|_{\phi(p)}$$

$$= \frac{\partial y^j}{\partial x^i}(\phi(p))$$

so
$$d(\hat{\phi}\circ\phi^{-1})_{\phi(p)}\frac{\partial}{\partial x^i} = \sum_{j=1}^{n} \frac{\partial y^j}{\partial x^i}(\phi(p))\,\frac{\partial}{\partial y^j}$$

this shows how to "change bases" from $\frac{\partial}{\partial x^i}$ to $\frac{\partial}{\partial y^j}$

exercise: if $v = \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}$ (in ϕ coords)

then $v = \sum_{j=1}^{n} w^j \frac{\partial}{\partial y^j}$ (in $\hat{\phi}$ coords)

where
$$\boxed{w^j = \sum_{i=1}^{n} \frac{\partial y^j}{\partial x^i}\, v^i} \quad ⊛$$

(in vector notation $\begin{bmatrix} w^1 \\ \vdots \\ w^n \end{bmatrix} = \left[\frac{\partial y^j}{\partial x^i}\right] \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix}$)

Jacobian of $\hat{\phi}\circ\phi^{-1}$

(this illustrates the use of supperscripts
Einstein summation convention: sum over repeated upper and lower indicies
ie. ⊛ can be written $w^j = \frac{\partial y^j}{\partial x^i} v^i$)

"Physics definition" of tangent space

a tangent vector at $p \in M$ is an assignment of n numbers to each coordinate chart containing p that transform according to ⊛ (mention story of Karen and Dennis)

exercise: Show this is equivalent to our definition of tangent vectors now let's see what df_p is in local coordinates

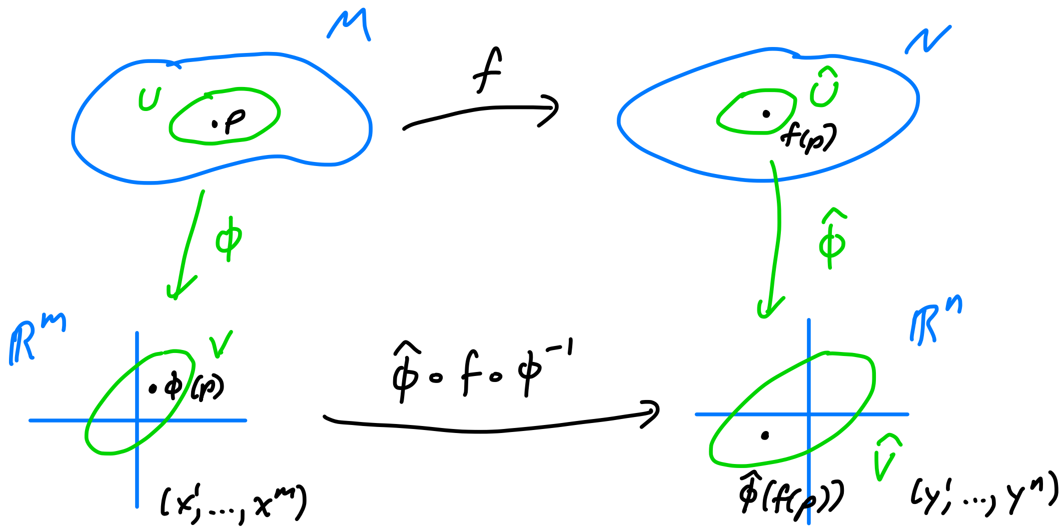
given

$$f : M \to N$$

a smooth function

let $\phi : U \to V$ be a coord. chart for M about p

and $\hat{\phi} : \hat{U} \to \hat{V}$ be a coord. chart for N about $f(p)$



if $v \in T_pM$ then

$$df_p(v) = d\left(\hat{\phi}^{-1}\circ(\hat{\phi}\circ f\circ\phi^{-1})\circ\phi\right)_p v$$

chain rule

$$= \left[d\hat{\phi}^{-1}_{\hat{\phi}(f(p))}\circ d(\hat{\phi}\circ f\circ\phi^{-1})_{\phi(p)}\circ d\phi_p\right](v)$$

$$= d\hat{\phi}^{-1}_{\hat{\phi}(f(p))}\circ d(\hat{\phi}\circ f\circ\phi^{-1})_{\phi(p)}\left(\sum_{i=1}^{m} v^i\frac{\partial}{\partial x^i}\right)$$

as above given a function $g : \mathbb{R}^m \to \mathbb{R}^n$

$$dg_p : T_p\mathbb{R}^m \to T_{g(p)}\mathbb{R}^n$$

is given by the total derivative of g

from vector calc / analysis

$$dg_p = \underbrace{Dg}_{\text{total derivative}}(p) = \underbrace{\left(\frac{\partial y^j}{\partial x^i}\right)}_{\text{Jacobian}}$$

where $g(x^1,\dots,x^n) = (y^1(x^1,\dots,x^m),\dots,y^n(x^1,\dots,x^m))$

so

$$\begin{aligned}
df_p(v) &= d\hat{\phi}^{-1}_{\hat{\phi}(f(p))}\left(D(\hat{\phi}\circ f\circ\phi^{-1})(\phi(p))\right)\left(\sum_{i=1}^{m} v^i \frac{\partial}{\partial x^i}\right)\\
&= d\hat{\phi}^{-1}_{\hat{\phi}(f(p))}\left(\frac{\partial y^j}{\partial x^i}(\phi(p))\right)\left(\sum_{i=1}^{m} v^i \frac{\partial}{\partial x^i}\right)\\
&= d\hat{\phi}^{-1}_{\hat{\phi}(f(p))}\sum_{j=1}^{n}\left(\sum_{i=1}^{m}\left(\frac{\partial y^j}{\partial x^i} v^i\right)\frac{\partial}{\partial y^j}\right)\\
&= \sum_{j=1}^{n}\left(\sum_{i=1}^{m}\frac{\partial y^j}{\partial x^i} v^i\right)\text{“}\frac{\partial}{\partial y^j}\text{”}
\end{aligned}$$

recall $\text{“}\frac{\partial}{\partial y^j}\text{”} = d\hat{\phi}^{-1}_{\hat{\phi}(f(p))}\left(\frac{\partial}{\partial y^j}\right)$

and

$$df_p\left(\sum_{i=1}^{m} v^i\,\text{“}\frac{\partial}{\partial x^i}\text{”}\right) = \sum_j\left(\sum_i v^i\frac{\partial y^j}{\partial x^i}(\phi(p))\right)\text{“}\frac{\partial}{\partial y^j}\text{”}$$

that is

$$\boxed{df_p \underset{\text{in local coordinates}}{=} \left(\frac{\partial y^j}{\partial x^i}(\phi(p))\right)}$$

so df_p is a generalization of the total derivative from calculus to manifolds!

Alternate definition of tangent vectors

a curve in M is a smooth map

$$\gamma : (a,b) \to M$$

(we can always reparameterize γ so that $0 \in (a,b)$ and $\gamma(0)$ is any preassigned point in image of γ)

let $\mathcal{P}_p = \{ \gamma : (a,b) \to M \mid \gamma(0) = p \}$

set of curves through p

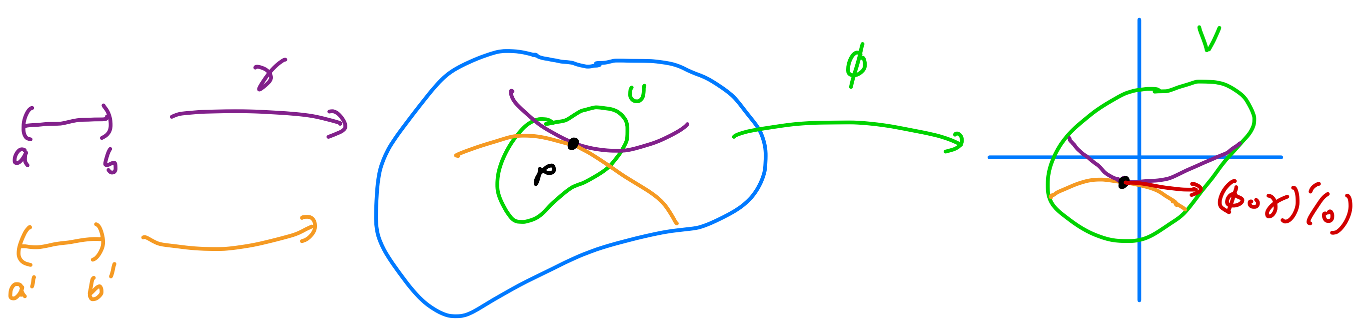
If $\gamma, \delta \in \mathcal{P}_p$ we say $\gamma \sim_p \delta$ if $\exists$ a coord. chart

$\phi : U \to V$ about p

such that

$$\frac{d}{dt} \underbrace{(\phi \circ \gamma)}_{} \Big|_{t=0} = \frac{d}{dt} \underbrace{(\phi \circ \delta)}_{} \Big|_{t=0}$$

curves in $\mathbb{R}^n$



exercise: 1) this is true in one coord chart about p

$\Leftrightarrow$

true in all coord charts about p

2) this is an equivalence relation

now define $\tilde{T}_p M = P_p/\sim_p$

exercise: use a coordinate chart to show $\tilde{T}_p M$ is a vector space (and vector space structure is independent of the coord. chart).

Th$^{\underline{m}}$ 5:

$$\boxed{\begin{array}{l} \text{The map} \\ \qquad \Phi : \tilde{T}_p M \longrightarrow T_p M \\ \qquad\quad [\gamma] \longmapsto D_\gamma \\ \text{is an isomorphism, where} \\ \qquad D_\gamma : C^\infty(M) \longrightarrow \mathbb{R} \\ \qquad\qquad f \longmapsto \frac{d}{dt} f \circ \gamma |_{t=0} \end{array}}$$

Proof:

one can easily check D_γ is a derivation and (using the exercise above) Φ is linear so we just need to check it is bijective

injective:

suppose $D_\gamma = D_\delta$

given a coordinate chart $\phi: U \to V$ about p (write $\phi(p) = (x^1(p), \dots, x^n(p))$)

then $D_\gamma(x^i) = D_\delta(x^i)$

$$\frac{d}{dt} x^i \circ \gamma \Big|_{t=0} \qquad\qquad \frac{d}{dt} x^i \circ \delta \Big|_{t=0}$$

so $$\frac{d}{dt}(\phi \circ \gamma)\Big|_{t=0} = \begin{bmatrix} \frac{d}{dt}(x^1 \circ \gamma)\big|_{t=0} \\ \vdots \\ \frac{d}{dt}(x^n \circ \gamma)\big|_{t=0} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{d}{dt}(x^1 \circ \delta)\big|_{t=0} \\ \vdots \\ \frac{d}{dt}(x^n \circ \delta)\big|_{t=0} \end{bmatrix} = \frac{d}{dt}(\phi \circ \delta)\Big|_{t=0}$$

and $\gamma \sim \delta$ ✓

<u>surjective:</u>

given $v \in T_pM$

let $\phi : U \to V$ be a coordinate chart about p

set $\tilde{\gamma}(t) = \phi(p) + \underbrace{(d\phi_p(v))}_{\in T_{\phi(p)}\mathbb{R}^n = \mathbb{R}^n_{\phi(p)} = \mathbb{R}^n} t$

and $\gamma(t) = \phi^{-1} \circ \tilde{\gamma}(t)$

<u>note:</u> if $g : \underset{t}{\mathbb{R}} \to \underset{x}{\mathbb{R}}$, then

$$dg_0\left(\frac{\partial}{\partial t}\right) \in T_{g(0)}\mathbb{R} \cong \mathbb{R}$$

$v \longmapsto v(x)$

unit tangent vector at any pt of $\mathbb{R}$

v acting on coord. funct.

so we think of $dg_0\left(\frac{\partial}{\partial t}\right) \in \mathbb{R}$ by

$$dg_0\left(\frac{\partial}{\partial t}\right) = \frac{d}{dt}(x \circ g)\Big|_{t=0} = \frac{d}{dt} g \Big|_{t=0}$$

now

$$D_\gamma f = \frac{d}{dt}(f \circ \gamma)\Big|_{t=0} \overset{\text{from above}}{=} d(f \circ \gamma)_0 \left(\frac{\partial}{\partial t}\right)$$

$$= d(f \circ \phi^{-1} \circ \tilde{\gamma})_0 \left(\frac{\partial}{\partial t}\right)$$

$$= df_p \circ d\phi^{-1}_{\phi(p)} \left(\frac{d}{dt} \tilde{\gamma}\Big|_{t=0}\right)$$

$$= df_p \circ d\phi^{-1}_{\phi(p)} \left(d\phi_p(v)\right)$$

$$= df_p(v) \in T_{f(p)} \mathbb{R}$$

same as above

$$= v \cdot f$$

so Φ onto

now if $f : M \to N$ is a smooth function, then

$$df_p : \tilde{T}_p M \to \tilde{T}_{f(p)} N$$
$$[\gamma] \mapsto [f \circ \gamma]$$

is simple! (this can make computations easier)

<u>exercise</u>: Show this agrees with

$$df_p : T_p M \to T_{f(p)} N$$

using Φ.

3 ways to think of T_pM

① derivations on $C^\infty(M)$

② paths through p

③ coord. charts

there are <u>many</u> other ways

B. Linearization and local behavior of functions

Key Idea:
- given $f: M \to N$ smooth the "linearization" $df_p : T_pM \to T_{f(p)}N$ should tell you about f near p
- "usually" f and df_p "look the same" near p

Thm 6 (linearization Thm):

given $f: M^m \to N^n$ a smooth map
if $df_p : T_pM \to T_{f(p)}N$ has maximal rank
(i.e. $\operatorname{rank} df_p = \min\{m, n\}$)
then there are coord. charts

$\phi : U \to V$ about p

$\hat{\phi} : \hat{U} \to \hat{V}$ about $f(p)$

such that

① if $m \leq n$, then

$$\hat{\phi} \circ f \circ \phi^{-1}(x^1, \dots, x^m) = (x^1, \dots, x^m, 0, \dots, 0)$$

(so f looks like standard inclusion $\mathbb{R}^m \to \mathbb{R}^n$)

② if $m \geq n$, then

$$\hat{\phi} \circ f \circ \phi^{-1}(x^1, \dots, x^m) = (x^1, \dots, x^n)$$

(so f looks like standard projection $\mathbb{R}^m \to \mathbb{R}^n$)

This theorem follows from

Th$^{\underline{m}}$ 7 (Rank Th$^{\underline{m}}$):

if $f: M^m \to N^n$ is a smooth map and

$df_p: T_pM \to T_{f(p)}N$ has constant rank k

for all points p in some open set $\mathcal{O}$

then for all $p \in \mathcal{O}$, $\exists$ coordinate charts

$$\phi: U \to V \text{ about } p \text{ and}$$

$$\hat{\phi}: \hat{U} \to \hat{V} \text{ about } f(p)$$

such that

$$\hat{\phi} \circ f \circ \phi^{-1}(x^1, \dots, x^m) = (x^1, \dots, x^k, 0, \dots, 0)$$

(standard projection $\mathbb{R}^m \to \mathbb{R}^k$ followed by stanandard inclusion $\mathbb{R}^k \to \mathbb{R}^n$)

Proof of Th$^{\underline{m}}$ 6 given Th$^{\underline{m}}$ 7:

we just need to see that if df_p has maximal rank at p then there is a neighborhood $\mathcal{O}$ of p s.t. df_x has constant rank for $x \in \mathcal{O}$

This follows from

lemma 8:

$$\boxed{\begin{array}{l}\text{let } \Psi: S \to Mat(n,m;\mathbb{R}) \text{ be a continuous function from some space } S \text{ to } n\times m \text{ matrices} \\ \text{Then } \quad \operatorname{rank}\Psi : S \longrightarrow \mathbb{R} : p \longmapsto \operatorname{rank}(\Psi(p)) \\ \text{is lower semi-continuous} \\ (\text{i.e. if } \operatorname{rank}\Psi(p) = k \text{ then } \exists \text{ nbhd } \mathcal{O} \text{ of } p \\ \quad \text{s.t. } \operatorname{rank}\Psi(x) \geq k \ \ \forall x \in \mathcal{O})\end{array}}$$

now since df_p written in local coordinates is an $n\times m$ matrix that depends continuously on p

lemma 8 and Th$^{\underline{m}}$ 7 $\Rightarrow$ Th$^{\underline{m}}$ 6 ∎

Proof of lemma 8:

suppose $\Psi(p)$ has rank k

so $\Psi(p)$ is an $n\times m$ matrix and there is a $k\times k$-submatrix with rank k

let

$$\Phi : Mat(n,m;\mathbb{R}) \to \mathbb{R}$$
$$A \longmapsto \det(\text{this } k\times k\text{-submatrix})$$

exercise: show Φ is continuous

so $\Phi\circ\Psi : S \to \mathbb{R}$ is continuous and $\Phi\circ\Psi(p) \neq 0$

thus $\exists$ an open set $\mathcal{O} \subset S$ around p st.

$$\Phi \circ \Psi(x) \neq 0 \quad \forall x \in \mathcal{O}$$

$\therefore x \in \mathcal{O} \Rightarrow \operatorname{rank} \Psi(x)$ is at least k ▦

to prove the rank theorem we need

Inverse function theorem:

> if $f: M \to N$ is a smooth function and
>
> $$df_p : T_p M \to T_{f(p)} N$$
>
> is an isomorphism
>
> then f is a local diffeomorphism near p
>
> (i.e. $\exists$ an open set $U \subset M$ st. $f|_U : U \to f(U)$ is a diffeomorphism)
>
> moreover $(df^{-1})_{f(p)} = (df_p)^{-1}$

exercise: prove this using the inverse function theorem from calculus

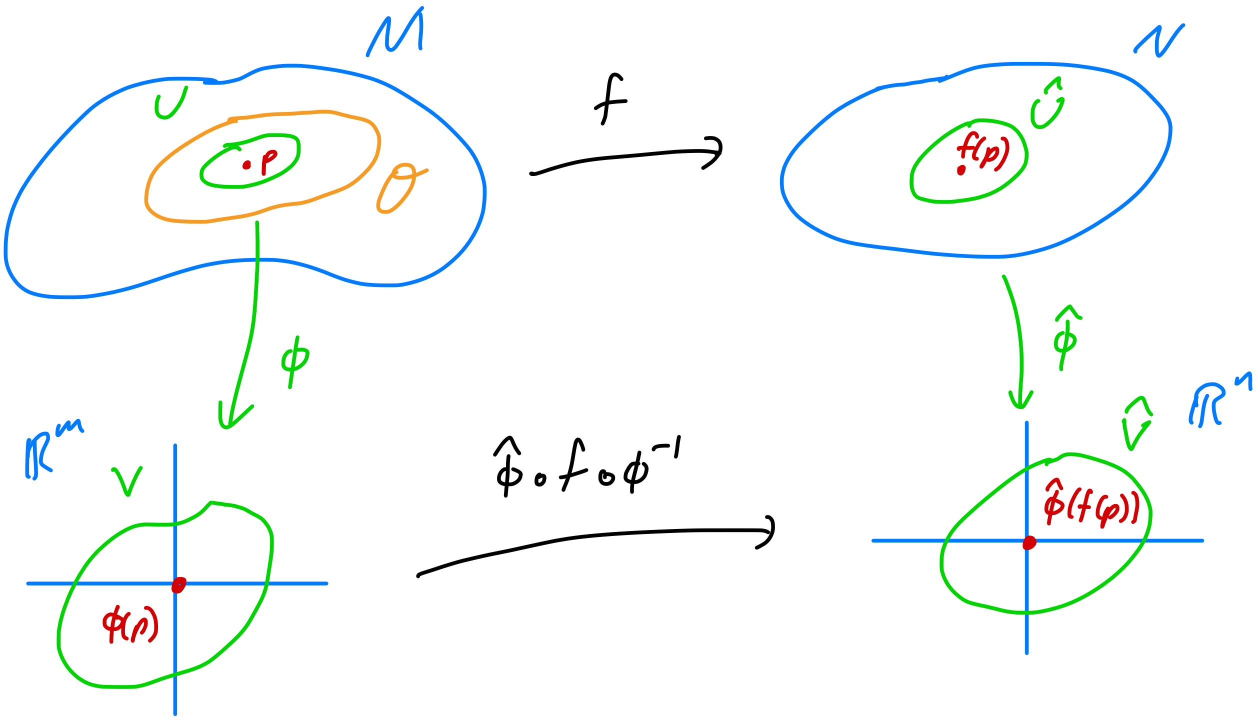
Proof of Th$^{\underline{m}}$ 7 (Rank Th$^{\underline{m}}$):

assume $\operatorname{rank} df_x = k \quad \forall x \in \mathcal{O}$

for $p \in \mathcal{O}$ let $\phi : U \to V$ be chart about p $(U \subset \mathcal{O})$

$\hat{\phi} : \hat{U} \to \hat{V}$ be chart about $f(p)$

(we may assume w.l.o.g. that $\phi(p) = 0 = \hat{\phi}(f(p))$)



set $F = \hat{\phi}\circ f\circ \phi^{-1}$

note $F(0) = 0$

write $F(x^1, \dots, x^m) = (F^1(x), \dots, F^n(x))$

where $x = (x^1, \dots, x^m)$

we can reorder F^j's and x^i's (i.e. rechoose coord charts)

s.t. $\begin{pmatrix} \frac{\partial F^1}{\partial x^1} & \cdots & \frac{\partial F^1}{\partial x^k} \\ \vdots & \ddots & \vdots \\ \frac{\partial F^k}{\partial x^1} & \cdots & \frac{\partial F^k}{\partial x^k} \end{pmatrix}$ is rank k

now write the coordinates $(x,y) = (x^1, \dots, x^k, y^1, \dots, y^{m-k})$

and let $Q(x,y) = (F^1, \dots, F^k)$ and

$R(x,y) = (F^{k+1}, \dots, F^n)$

so $F(x,y) = (Q(x,y), R(x,y))$

and $\left(\frac{\partial Q^i}{\partial x^j}\right)$ is rank k (so invertable)

<u>let's "straighten out" the domain:</u>

consider $g: V \to \mathbb{R}^m : (x,y) \to (Q(x,y), y)$

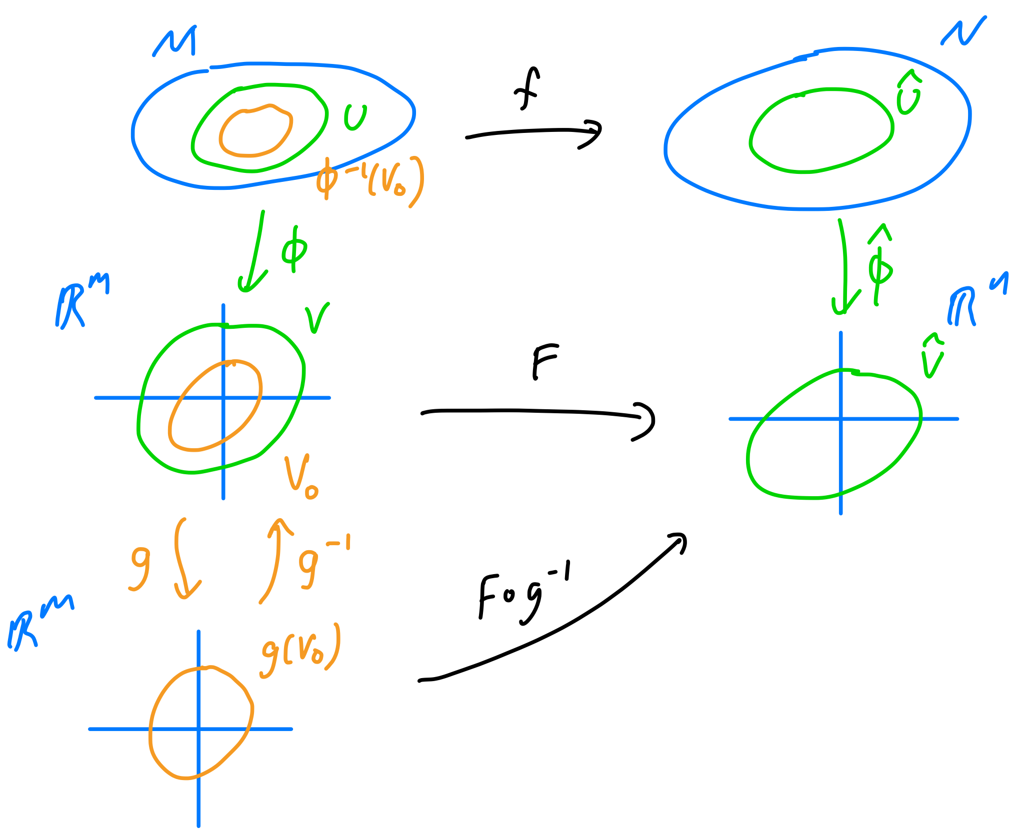
so
$$dg_{(0,0)} = \begin{pmatrix} \frac{\partial Q^i}{\partial x^j} & \frac{\partial Q^i}{\partial y^j} \\ 0 & I_{m-k} \end{pmatrix}$$ is invertable!

$\therefore$ Inverse function theorem $\Rightarrow \exists$ a smaller set

$V_0 \subset V$ s.t. $g|_{V_0} : V_0 \to g(V_0)$

is a diffeomorphism

note $g \circ \phi : \phi^{-1}(V_0) \to g(V_0)$ is a <u>new</u> coordinate chart about p



write $g^{-1}(x,y) = (A(x,y), B(x,y))$

first k coords

so

$$(x,y) = g \circ g^{-1}(x,y) = g(A(x,y), B(x,y))$$

$$= \big(Q(A(x,y), B(x,y)),\ B(x,y)\big)$$

$\therefore\ B(x,y) = y$ for all (x,y)

so $g^{-1}(x,y) = (A(x,y), y)$

and $Q(A(x,y), y) = x$

$$\therefore\ F \circ g^{-1}(x,y) = \big(\underbrace{Q(A(x,y),y)}_{=\,x},\ \underbrace{R(A(x,y),y)}_{=\,\tilde{R}(x,y)}\big)$$

and

$$d(F\circ g^{-1})_{(x,y)} = \begin{pmatrix} I_k & 0 \\ \frac{\partial \tilde{R}^i}{\partial x^j} & \frac{\partial \tilde{R}^i}{\partial y^j} \end{pmatrix}$$

$$= dF_{g^{-1}(x,y)} \circ \underbrace{dg^{-1}_{(x,y)}}_{\text{invertable}}$$

so $d(F\circ g^{-1})$ has same rank as dF which is k

thus $\frac{\partial \tilde{R}^i}{\partial y^j} = 0$ for all (x,y) (here is where constant rank comes in)

so $\tilde{R}^i$ is only a function of x, not y!

so set $S(x) = R(x,0) = R(x,y)$

note: $F \circ g^{-1}(x,y) = (x, S(x))$

<u>now we "straighten out" range:</u>

set $\hat{g} : \hat{V} \longrightarrow \mathbb{R}^n$

$$\underbrace{((v^1, \dots, v^k)}_{u}, \underbrace{(v^1, \dots, v^{n-k})}_{v}) \mapsto (u, v - S(u))$$

$$d\hat{g}_0 = \begin{pmatrix} I_k & 0 \\ \frac{\partial S^i}{\partial x^j} & I_{n-n} \end{pmatrix}$$ has rank n so is invertable

thus IFT $\Rightarrow \exists\, \hat{V}_0 \subset \hat{V}$ such that

$\hat{g} : \hat{V}_0 \longrightarrow \hat{g}(\hat{V})$ is diffeomorphism

note: $\hat{g} \circ F \circ g^{-1}(x,y) = \hat{g}(x, S(x))$

$= (x, S(x) - S(x)) = (x, 0)$

so setting $\psi : \phi^{-1}(V_0) \longrightarrow g(V_0) : q \longmapsto g \circ \phi(q)$

$\hat{\psi} : \hat{\phi}^{-1}(\hat{V}_0) \longrightarrow \hat{g}(\hat{V}_0) : q \longmapsto \hat{g} \circ \hat{\phi}(q)$

we get $\hat{\psi} \circ f \circ \psi^{-1}(x,y) = \hat{g} \circ \underbrace{(\hat{\phi} \circ f \circ \phi^{-1})}_{F} \circ g^{-1}(x,y) = (x,0)$