III. Submanifolds

A. Immersions and Submanifolds

an immersion is a smooth map

$$f : M \to N$$

such that

$$df_p : T_pM \to T_{f(p)}N$$

is injective for all $p \in M$

a (smooth) embedding of M into N is a smooth map

$$f : M \to N$$

such that

1) f is an immersion and

2) f is a homeomorphism

$$M \to f(M)$$

where $f(M)$ is given the subspace topology

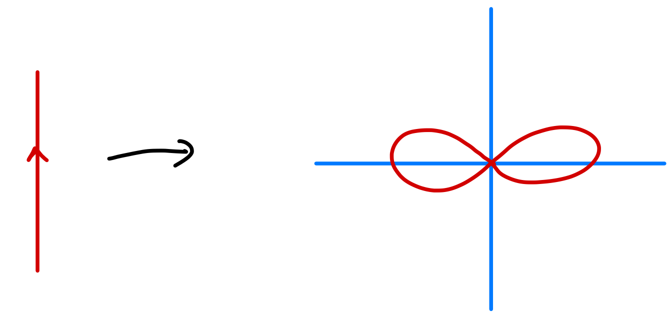
examples:

1) $\gamma : \mathbb{R} \to M$ is an immersion if

$$\gamma'(t) \neq 0 \quad \forall t$$

where $\gamma'(t) = d\gamma_t\left(\frac{\partial}{\partial t}\right) \in T_{\gamma(t)}M$

eg:



2) $f : M \to M \times N$
$p \mapsto (p, q_0)$

$df_p(v) = (v, 0) \in T_p M \times T_{q_0} N = T_{(p,q_0)}(M \times N)$

so f is an immersion

"clearly" f is a homeomorphism onto $M \times \{p_0\}$

so f is an embedding

3) consider

$$f : \mathbb{R} \to S^1 \times S^1$$
$$t \mapsto (e^{2\pi i \alpha t}, e^{2\pi i \beta t})$$

where $\alpha, \beta \in \mathbb{R}$ s.t. α/β irrational

clearly $df \neq 0$ so f is an immersion

also f is injective since

$$f(t_1) = f(t_2) \Rightarrow \alpha(t_1 - t_2) \in \mathbb{Z}$$
$$\beta(t_1 - t_2) \in \mathbb{Z}$$

so $t_1 = t_2 + \frac{n}{\alpha}$

and $\beta(t_2 + \frac{n}{\alpha} - t_2) = m$

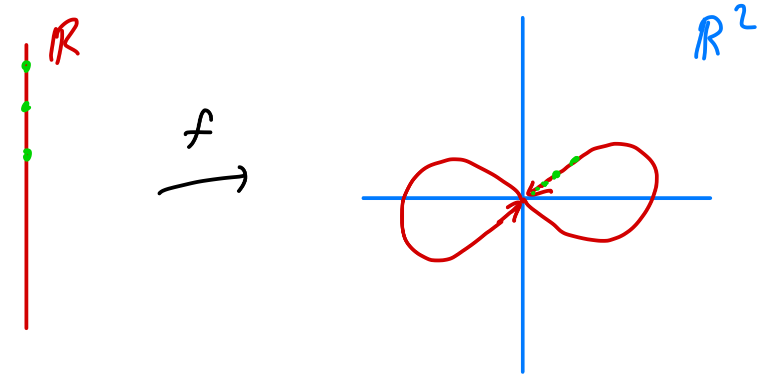
so $\beta/\alpha = m/n$ $\cancel{\emptyset}$

exercise: $\operatorname{im}(f)$ is dense in $S^1 \times S^1$

remark: we will see later an embedding of positive codim can't have dense image

So f is an injective immersion that is not an embedding

4) another injective immersion that is not an embedding



(look at sequence in $\mathbb{R}$ going to ∞)

exercise: if $f: M \to N$ is an injective immersion, then it is an embedding if any one of the following are true

1) f an open map
2) f a closed map
3) M is compact

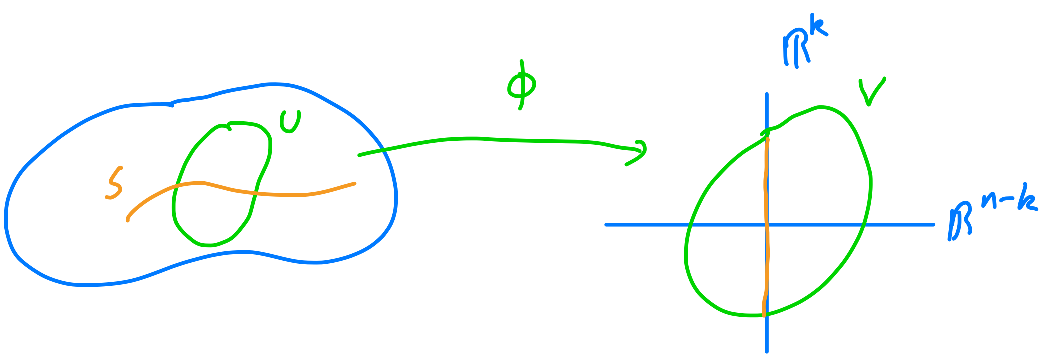
let M^n be a manifold

$S \subset M$ is a <u>submanifold of dimension k</u>

if for each point $p \in S$, $\exists$ a coord. chart

$\phi : U \to V$ about p for M s.t.

$$\phi(S \cap U) = V \cap \Big(\underbrace{\{0\} \times \mathbb{R}^k}_{\substack{\cap\vert \\ \mathbb{R}^{n-k} \times \mathbb{R}^k = \mathbb{R}^n}}\Big)$$



such a chart is called a <u>slice chart</u> for S

(if $\partial M \neq \emptyset$ then allow slice charts in $\mathbb{R}^n_{\geq 0}$ too)

$(n-k)$ is called the <u>codimension</u> of S in M

and M is called the <u>ambient manifold of S</u>

<u>exercise</u>:

1) Such an S is a k-dimensional manifold (possibly w/ boundary)

2) $(\partial M) \cap S = \partial S$ and ∂S is a submanifold of ∂M of the same codimension

3) $S \subset M$ inherits a smooth structure from M: if $\{\phi_\alpha : U_\alpha \to V_\alpha\}$ is part of a smooth atlas for M that contains slice charts for S and $S \subset \bigcup U_\alpha$ then

$$\{\phi_\alpha|_{S \cap U_\alpha} : (S \cap U_\alpha) \to \phi_\alpha(S \cap U_\alpha)\}$$

is a smooth atlas for S.

4) if S is a submanifold for M and $i : S \to M$ is the inclusion map then i is an <u>embedding</u>

5) if S is a submanifold of M and $f : M \to N$ is a smooth map then $f|_S : S \to N$ is a smooth map

<u>lemma 1:</u>

if $f : N \to M$ is a smooth embedding then $f(N)$ is a submanifold of M

<u>Remark</u>: so submanifolds are the same as the images of smooth embeddings

Proof: let $f : N \to M$ be an embedding

so $\operatorname{rank} df_p = \dim N \quad \forall p$

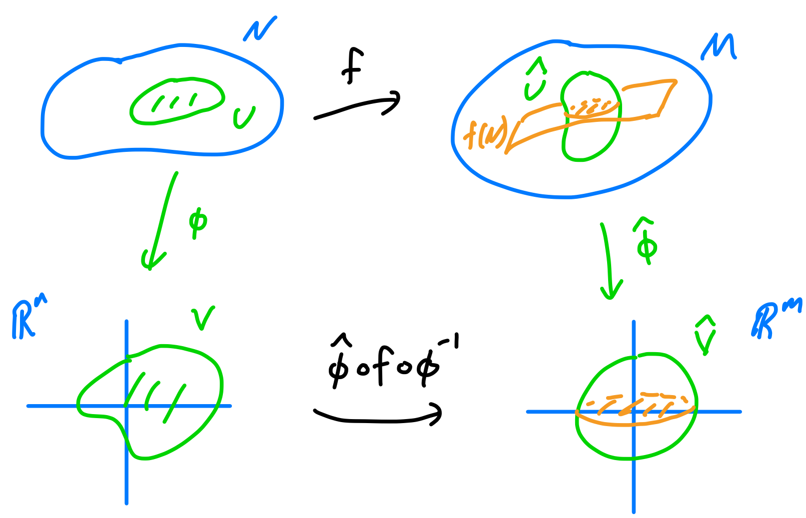
so df has maximal rank

now given $p \in f(N) \in M$ $\exists!$ $q \in N$ s.t. $f(q) = p$

$\therefore$ Th$^{\underline{m}}$ II.7 part 1 $\exists$ coord. charts

$$\phi : U \to V \quad \text{about } q$$
$$\hat{\phi} : \hat{U} \to \hat{V} \quad \text{about } p$$

s.t. $\hat{\phi} \circ f \circ \phi^{-1}(x^1, \dots, x^n) = (x^1, \dots, x^n, 0, \dots, 0)$



so $f(N) \cap \hat{U} = f(U)$

$$\begin{aligned}
\hat{\phi}(f(N) \cap \hat{U}) &= \hat{\phi}(f(U)) \\
&= \hat{\phi} \circ f \circ \phi^{-1}(\phi(U)) \\
&= \hat{\phi} \circ f \circ \phi^{-1}(V) \\
&= \hat{V} \cap (\mathbb{R}^n \times \{0\})
\end{aligned}$$

so $\hat{\phi} : \hat{U} \to \hat{V}$ is a slice chart ▦

note: $f : N \to f(N)$ a homeomorphism implies $f(N) \cap \hat{U}$ is $\mathbb{R}^n \times \{0\} \cap \hat{V}$ (after possibly shrinking $\hat{U}$)

B. Submersions:

a submersion is a smooth map

$$f : M \to N$$

such that $df_p : T_pM \to T_{f(p)}N$ is surjective $\forall p \in M$

call $p \in M$ a regular point if df_p is surjective

" " a critical point if df_p is not " "

$q \in N$ is a regular value if df_p is surjective $\forall p \in f^{-1}(q)$

and a critical point if not

note: f a submersion $\Leftrightarrow$ all $p \in N$ regular values

lemma 2:

$$\boxed{\begin{array}{l}\text{given a smooth map } f : M \to N \\ \text{if } q \in N \text{ is a regular value, then} \\ \qquad S = f^{-1}(q) \\ \text{is a submanifold of } M \text{ of dimension } \dim M - \dim N \\ \qquad (\text{i.e. codim} = \dim N) \\ \text{moreover } T_pS = \ker df_p\end{array}}$$

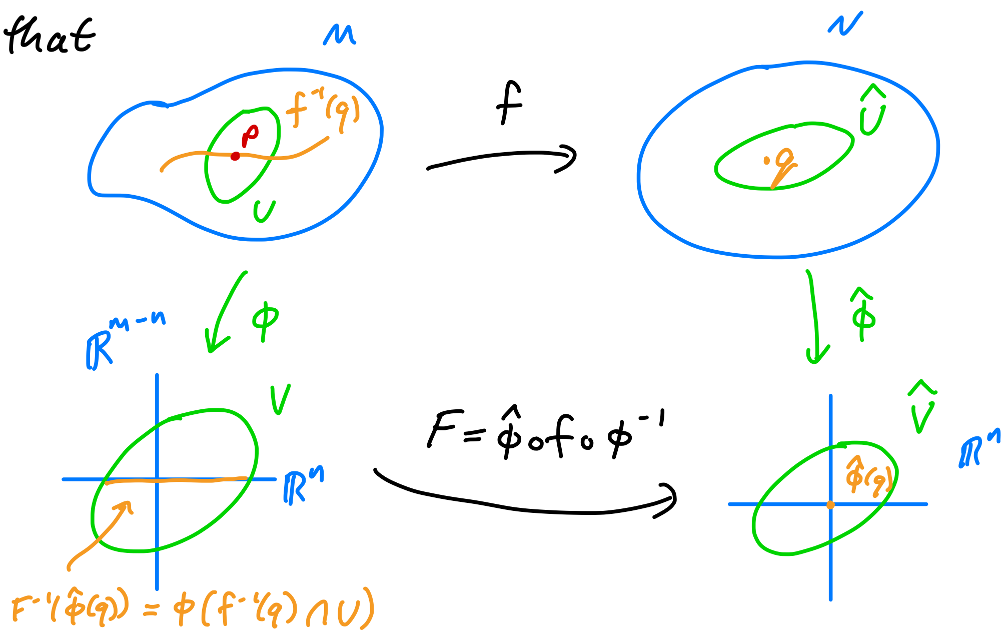
Proof: given a point $p \in f^{-1}(q)$

Thm II.7 part ② gives coord. charts

$\phi : U \to V$ for p in M

$\hat{\phi} : \hat{U} \to \hat{V}$ for $q = f(p)$ in N

such that



$$\hat{\phi} \circ f \circ \phi^{-1}(x^1, \ldots, x^m) = (x^1, \ldots, x^n)$$

so ϕ is a slice chart for $f^{-1}(q)$

$\therefore$ $f^{-1}(q)$ an $m-n$ manifold

<u>note</u>: if $v \in T_p(f^{-1}(q))$, then $\exists$ a path $\gamma : (-\varepsilon, \varepsilon) \to f^{-1}(q)$

s.t. $v = [\gamma]$

now $df_p(v) = \underbrace{[f \circ \gamma]}_{\text{constant path}} = 0 \in T_q N$

so $T_p(f^{-1}(q)) \subset \ker df_p$

but both are dimension $m-n$ so they are equal

<u>examples</u>:

1) $f : \mathbb{R}^{n+1} \to \mathbb{R}$

$(x^1, \ldots, x^{n+1}) \mapsto \sum_{i=1}^{n+1} (x^i)^2$

$df_x = [2x^1 \; \cdots \; 2x^{n+1}]$ $1 \times (n+1)$ matrix

so if $(x^1, \ldots, x^{n+1}) \neq 0$, then $\operatorname{rank} df_x = 1$

so any $a>0$ is a regular value

and $f^{-1}(a) = S^n$ a manifold (sphere) of dim n

$$T_x S^n = \ker df_x = \{v \in \mathbb{R}^{n+1} \text{ s.t. } [(x^1, \dots, x^n)]v = 0\}$$

2) recall $M(n;\mathbb{R}) = n \times n$ matricies $\cong \mathbb{R}^{n^2}$

$GL(n,\mathbb{R})$ = invertable $n \times n$ matrix

earlier we saw this is an open subset of $M(n;\mathbb{R})$

so also a smooth n^2-manifold

now set

orthogonal group

$$O(n) = \{A \in GL(n,\mathbb{R}) : A A^T = I\}$$

transpose

= {linear maps that preserve standard inner product on $\mathbb{R}^n$ }

special orthogonal group

$$SO(n) = \{A \in O(n) : \det A = 1\}$$

<u>exercise</u>:

a) $\det : M(n;\mathbb{R}) \to \mathbb{R}$ is smooth

b) if $Sym(n) = \{A \in M(n;\mathbb{R}) : A = A^T\} \cong \mathbb{R}^{\frac{n(n+1)}{2}}$

then show
$$f : M(n;\mathbb{R}) \to Sym(n)$$
$$A \longmapsto A A^T$$
is smooth

<u>note</u>: $O(n) = f^{-1}(I)$ $n \times n$ idetity matrix

c) Show I is a regular value of f

so $O(n)$ is a manifold of dim

$$n^2 - \frac{(n+1)n}{2} = \frac{n(n-1)}{2}$$

d) show $df_I(B) = B + B^T$

more generally: $df_A(B) = B^T A + A^T B$

hint: if $B \in T_A M(n;\mathbb{R})$ then $\gamma(t) = A + tB$ represents B

so $T_I O(n) = \ker df_I = \{A \in M(n;\mathbb{R}) \text{ s.t. } A = -A^T\}$

skew-symmetric matricies

e) $\det : O(n) \to \{\pm 1\}$ is onto

so $SO(n) = \det^{-1}(1)$ is an open subset of $O(n)$

(connected component of I)

3) let $M(n;\mathbb{C})$ = $n \times n$ matricies w/ entries in $\mathbb{C}$

$\cong \mathbb{C}^{n^2}$

$GL(n,\mathbb{C})$ = invertable elts of $M(n;\mathbb{C})$

unitary group $U(n) = \{A \in GL(n,\mathbb{C}) : A\bar{A}^T = I\}$

special unitary group $SU(n) = \{A \in U(n) : \det A = 1\}$

exercise:

a) Show $U(n)$ is a manifold of diminsion

$$2(n^2) - \underbrace{\left(2\frac{(n-1)n}{2} + n\right)}_{A = \bar{A}^T} = n^2$$

b) $T_I U(n) = \{A \in GL(n,\mathbb{C}) : A = -\bar{A}^T\}$

skew Hermetian

c) $SU(n)$ is a manifold of dimension n^2-1

d) $T_I\, SU(n) = \{A \in GL(n;\mathbb{C}) : A = -\bar{A}^T,\ \operatorname{tr} A = 0\}$ ← trace

Hint: Show $d(\det)_A(B) = \det A \operatorname{tr}(A^{-1}B)$

for $A \in GL(n;\mathbb{C})$ and $B \in M(n,\mathbb{C})$

Remark: The above are examples of Lie groups.

A Lie group is a smooth manifold G equipped with a group structure such that multiplication

$$G \times G \to G : (g,h) \mapsto g\cdot h$$

and inversion

$$G \to G : g \mapsto g^{-1}$$

are smooth maps

exercise: Show all examples in 2) and 3) are Lie groups

Remark: If G is a Lie group then $T_I G$ is a Lie algebra. (I ← identity in G) A Lie algebra is a vector space V and a binary operation

$$[\cdot,\cdot] : V \times V \to V$$

called the Lie bracket that satisfies

1) bilinearity $[av+bw, u] = a[v,u] + b[w,u]$
$[u, av+bw] = a[u,v] + b[u,w]$

2) anticommutative $[v,u] = -[u,v]$

3) Jacobi identity

$$[u,[v,w]] + [w,[u,v]] + [v,[w,u]] = 0$$

later we will see how to give $T_I G$ such a structure

we see regular values are useful, but how common are they?

a subset $A \subset \mathbb{R}^n$ is said to have measure zero if for any $\delta > 0$, A can be covered by a countable collection of open cubes/balls, the sum of whose volumes is $< \delta$

$A \subset M$ has measure zero if $\forall$ coordinate charts $\phi : U \to V$, $\phi(A)$ has measure zero in $\mathbb{R}^n$

exercise:

1) a countable union of sets of measure zero has measure zero

2) if $A \subset M$ has measure zero then $M - A$ is dense in M
(if not, A contains an open set...)

3) $\mathbb{Q} \subset \mathbb{R}$ has measure zero

Sard's $Th^{\underline{m}}$:

$$\boxed{\begin{array}{l}\text{if } f: M \to N \text{ is a smooth map, then the set} \\ \quad\text{of critical values has measure zero in } N \\ \quad\text{(so regular values are dense)}\end{array}}$$

we will not prove this since the proof has a very different "flavor" to the rest of the class (see book if interested)

Remarks:

1) Continuous and Smooth maps are <u>very</u> different

recall $\exists$ a continuous <u>surjection</u>

$$f: [0,1] \to [0,1] \times [0,1]$$

<u>space filling curve</u>

but if f is smooth, $\exists$ regular value $q \in [0,1] \times [0,1]$

and if $p \in f^{-1}(q)$ then

$$df_p : \underbrace{T_p[0,1]}_{= \mathbb{R}} \to \underbrace{T_q([0,1]\times[0,1])}_{= \mathbb{R}^2}$$

is <u>onto</u> $\cancel{\emptyset}$

2) image of an immersion $f: N^n \to M^m$ with $n \lneq m$

has measure zero (since $\mathrm{im}(f) \subset \{\text{critical values}\}$)

C. Whitney's Embedding Th^{m}

we now prove

Th^{m} 3 (Whitney's Embedding Th^{m}):

$\boxed{\text{every compact manifold } M \text{ embeds in } \mathbb{R}^k \text{ for } k \text{ sufficiently large}}$

Remarks:

1) don't need compact
2) later we will see, we can take $k = 2n+1$, $n = \dim M$
3) can actually take $k = 2n$ and can immerse M in $\mathbb{R}^{2n-1}$ (these results much harder)
4) given a manifold M it is interesting to see minimal k such that $M \subseteq \mathbb{R}^k$

   e.g. $S^n \subset \mathbb{R}^{n+1}$

   any orientable surface $\subset \mathbb{R}^3$

   if $n \neq 2^k$ then M^n embeds in $\mathbb{R}^{2n-1}$

   if $n = 2^k$ then $\mathbb{R}P^n$ does not embed in $\mathbb{R}^{2n-1}$

Proof: for each $p \in M$ let $\phi_p : U_p \to V_p$ be a coord chart about p

recall $\exists$ bump function $f_p : M \to \mathbb{R}$

such that $\exists$ open sets $p \in \mathcal{O}_p \subset \mathcal{O}' \subset U_p$

and $f_p = \begin{cases} 1 & \text{on } \overline{\mathcal{O}_p} \\ 0 & \text{outside } \mathcal{O}_p' \end{cases}$

note: $\{\mathcal{O}_p\}_{p \in M}$ a cover of M

M compact so $\exists$ a finite subcover $\{\mathcal{O}_{p_1} \dots \mathcal{O}_{p_\ell}\}$

note: $f_{p_i}(x)\,\phi_{p_i}(x) : M \to \mathbb{R}^n$

$$x \longmapsto \begin{cases} f_{p_i}(x)\,\phi_{p_i}(x) & x \in \mathcal{O}_{p_i} \\ 0 & x \notin \mathcal{O}'_{p_i} \end{cases}$$

set $\Psi : M \to \mathbb{R}^{n\ell + n}$

$$x \longmapsto \left(f_{p_1}(x)\phi_{p_1}(x), \dots, f_{p_\ell}(x)\phi_{p_\ell}(x), f_{p_1}(x), \dots, f_{p_\ell}(x)\right)$$

Claim: Ψ is injective

if $\Psi(x) = \Psi(y)$, then $\exists\, i$ s.t. $f_{p_i}(x) = 1$

so $f_{p_i}(y) = 1$ too and $x, y \in \mathcal{O}_{p_i}$

also

$$\phi_{p_i}(x) = f_{p_i}(x)\,\phi_{p_i}(x) = f_{p_i}(y)\,\phi_{p_i}(y) = \phi_{p_i}(y)$$

but $\phi_{p_i} : U_{p_i} \to V_{p_i}$ a diffeomorphism

$\therefore$ injective and we see $x = y$.

Claim: Ψ an immersion

given $p \in M$, $\exists\, i$ s.t. $p \in \mathcal{O}_{p_i}$

thus $d(f_{p_i}\phi_{p_i})_p = d(\phi_{p_i})_p$ $\leftarrow$ rank n (invertable)

$\therefore d\Phi_p$ which contains $(d\phi_{p_i})_p$ has a rank n factor $\therefore$ its rank $\geq n$

but $\dim M = n$ so rand $d\Phi_p = n$

i.e. $d\Phi_p$ is injective

<u>note</u>: $\Phi : M \to \Phi(M)$ is a continuous map from a compact space to a Hausdorff space

this implies Φ a homeomorphism

$\therefore \Phi$ an embedding ▦

Th$^{\underline{m}}$ 4 (Strong(er) Whitney Embedding Th$^{\underline{m}}$):

every compact manifold M of dimension n embeds in $\mathbb{R}^{2n+1}$

Proof:

from Th$^{\underline{m}}$ 3 we know $M \subseteq \mathbb{R}^N$ as a smooth submfd

given a vector $v \in \mathbb{R}^N$ with $v \neq 0$

let $v^\perp = \{w \in \mathbb{R}^N : w \cdot v = 0\} \cong \mathbb{R}^{N-1}$

and

$$\pi_v : \mathbb{R}^N \to \mathbb{R}^{N-1} = v^\perp$$
$$w \longmapsto w - \left(\frac{v \cdot w}{w \cdot w}\right) v$$

orthogonal projection onto $v^\perp$

so $\pi_v|_M : M \to \mathbb{R}^{N-1}$ is a smooth map

we will show that if $N > 2n+1$ then we

can find v (a dense set of v!) s.t. $\pi_v|_M$ is an embedding

(this clearly finishes the proof)

first note that $\pi_v = \pi_w$ if $v\mathbb{R} = w\mathbb{R}$

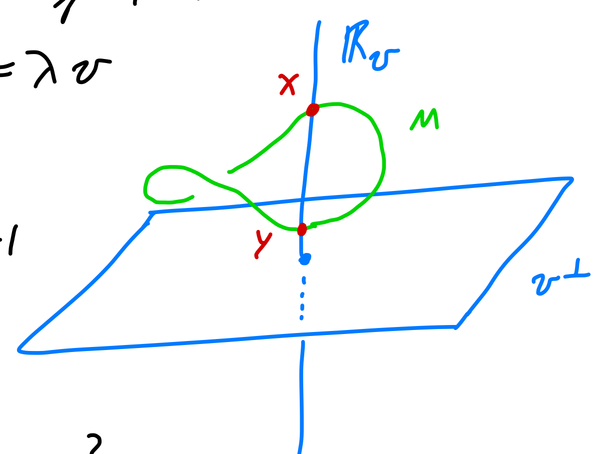
i.e. $[v] = [w] \in \mathbb{RP}^{N-1}$

so we show there is a dense set of points in $\mathbb{RP}^{N-1}$ s.t. corresponding $\pi_v|_M$ is an embedding

<u>$\pi_v|_M$ Injective:</u>

$\pi_v(x) = \pi_v(y) \Leftrightarrow (x-y)$ in span of v

$\Leftrightarrow (x-y) = \lambda v$



so consider

$$g : [(M\times M)\setminus\Delta] \to \mathbb{RP}^{N-1}$$
$$(x,y) \longmapsto [x-y]$$

where $\Delta = \{(x,x) : x\in M\}$

<u>note</u>: g is smooth, since a composition of

$$(M\times M)-\Delta \to \mathbb{R}^N - \{0\}$$
$$(x,y) \longmapsto x-y$$

and

$\mathbb{R}^N - \{0\} \to \mathbb{RP}^{N-1}$ projection

if $p\in\mathbb{RP}^{N-1}$ is a regular value then

$g^{-1}(p)$ is a submanifold of

dimension $= 2n - (N-1)$

$< 2n - (2n-1) + 1 = 0$

so $g^{-1}(p) = \emptyset$

i.e. there is no $x, y \in M$, $x \neq y$ s.t. $[x-y] = p$

$\therefore$ if $[v] = p$, then $\pi_v|_M : M \to \mathbb{R}^{N-1}$

is injective

by Sard's Th$^{\underline{m}}$ there is a dense set of

p's so that $p = [v]$ then $\pi_v|_M$ is inj ✓

$\pi_v|_M$ Immersion:

note: $\pi_v : \mathbb{R}^N \to \mathbb{R}^{N-1}$ is a linear map

exercise: $d(\pi_v)_p : T_p\mathbb{R}^N \to T_{\pi_v(p)}\mathbb{R}^{N-1}$ is π_v

(more generally: the derivative of a linear map is the linear map)

so π_v is an immersion of M at p

$\iff \forall w \neq 0$ in T_pM, $\pi_v(w) \neq 0$

$\iff w \notin \operatorname{span} v \ \ \forall w \neq 0$ in T_pM

$\iff [v] \neq [w] \ \ \forall w \neq 0$ in T_pM

let $TM = \bigcup_{p \in M} T_pM$

this is called the tangent space of M

we will study this later, but it is a $2n$-dimensional manifold and we have the smooth map

$$h : (TM - Z) \longrightarrow \mathbb{R}P^{N-1}$$
$$w \longmapsto [w]$$

where $Z = \{ w \in T_p M : p \in M, w = 0 \}$

<u>note</u>: $w \in TM - Z \Rightarrow w \in T\mathbb{R}^N - Z$

$\Rightarrow w \in T_p \mathbb{R}^N - \{0\}$ some p

$\Rightarrow w \in \mathbb{R}^N - \{0\}$

$\Rightarrow [w] \in \mathbb{R}P^{N-1}$ well-def!

if $p \in \mathbb{R}P^{N-1}$ is a regular value,

then as above $h^{-1}(p) = \emptyset$

so there are no vectors in $TM - Z$ s.t. $[w] = p$

thus if $v \in p$ we have $[v] \neq [w]$ $\forall w \in TM - Z$

and there for $\pi_v|_M$ is an immersion

the set of critical values of g & h are both of measure zero, so there is a dense set of $p \in \mathbb{R}P^{N-1}$ that are regular values for g & h

this gives a p s.t. $v \in p$ has $\pi_v|_M$ is injective and an immersion

$\therefore$ since M is compact $\pi_v|_M$ is an embedding