IV Bundles

A. Fiber bundles

a **fiber bundle** (a.k.a. a **twisted product** or a **locally trivial fibration**) is a quadruple

(p, E, B, F) where:

- E, B, F are topological spaces
- $p: E \to B$ is a continuous surjection

such that

$\forall x \in B$, $\exists$ open set U about x and a homeomorphism

$$\phi_U : U \times F \longrightarrow p^{-1}(U)$$

such that

$$\begin{tikzcd} U \times F \arrow[rr, "\phi_U"] \arrow[dr, "p_1"'] & \arrow[d, phantom, "\circ"] & p^{-1}(U) \arrow[dl] \\ & U & \end{tikzcd}$$

commutes, where $p_1 : U \times F \to U$ is projection

F is called the **fiber**

E " " **total space**

B " " **base space**

p " " **projection**

usually write

$$\begin{array}{ccc} F & \to & E \\ & & \downarrow \\ & & B \end{array}$$

ϕ_U is called a local trivialization of E

if F, E, B, p are smooth manifolds/maps and the ϕ_U are all smooth, then we say it is a smooth fiber bundle

example:

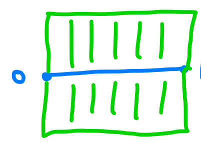
1) $E = B \times F$ and $p: E \to B$ is just projection

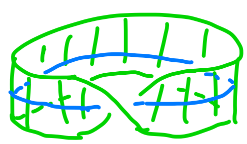
2) Möbius band

let $I = [0,1]$

$S = I \times \mathbb{R}$

$M = S/\sim$ where $(0,t) \sim (1,-t)$

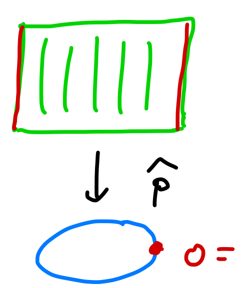




note: $\tilde{p} : S \to I : (t,r) \mapsto t$

induces a map

$\hat{p} : S \to S' = I/_{0\sim 1}$



and this induces a map

quotient space

$$\begin{tikzcd} S \arrow[r, "\hat{p}"] \arrow[d, "q"'] & S' \\ M \arrow[ur, "p"'] & \end{tikzcd}$$

$p : M \to S^1$ is a bundle with fiber $\mathbb{R}$

<u>indeed:</u>

for $U = (0,1) \subset [0,1]/_\sim = S^1$

we have

$$p^{-1}(U) = (0,1) \times \mathbb{R}$$

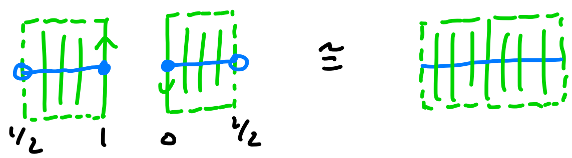
(no identifications!)

if $U' = [0,\tfrac{1}{2}) \cup (\tfrac{1}{2},1]/_\sim \subset [0,1]/_\sim = S^1$

then

$$p^{-1}(U) = \big([0,\tfrac{1}{2}) \times \mathbb{R}\big) \cup \big((\tfrac{1}{2},1] \times \mathbb{R}\big)/_\sim$$

where $(0,t) \sim (1,-t)$



<u>exercise:</u> a) $p^{-1}(U') \cong (-\tfrac{1}{2},\tfrac{1}{2}) \times \mathbb{R}$

b) so M a bundle, but show

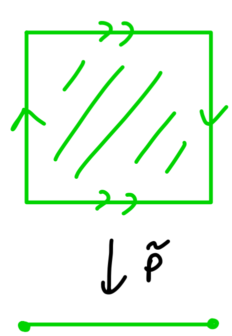
$$M \not\cong S^1 \times \mathbb{R}$$

(easier later!)

3) <u>Klein bottle</u>

$$I \times I \xrightarrow{\tilde{p}} I$$

$$K = I \times I/_\sim \xrightarrow{p} S^1$$



<u>exercise:</u> as above show $S^1 \to K \xrightarrow{p} S^1$

4) $S^3 \xrightarrow{p} \mathbb{C}P^1$ recall $\mathbb{C}P^1 \cong S^2$

$(z^1, z^2) \mapsto [z^1 : z^2]$

<u>note</u>: if $(z^1)^2 + (z^2)^2 = 1$, then

$$p^{-1}([z^1:z^2]) = \left\{ (w^1, w^2) \;\middle|\; \begin{array}{l} w^1 = \lambda z^1 \\ w^2 = \lambda z^2 \\ \text{and } (w^1)^2 + (w^2)^2 = 1 \end{array} \right\}$$

$$= \{ (\lambda z^1, \lambda z^2) \mid |\lambda| = 1, \lambda \in \mathbb{C} \}$$

$$= S^1$$

<u>Claim</u>:

$$\begin{array}{ccc} S^1 & \longrightarrow & S^3 \\ & & \downarrow p \\ & & \mathbb{C}P^1 \end{array}$$

to see this consider

$$U_1 = \{ [z^1 : z^2] \in \mathbb{C}P^1 \mid z^1 \neq 0 \}$$

recall $\phi_1 : U_1 \longrightarrow V_1 \cong \mathbb{C}$

$$[z^1 : z^2] \longmapsto z^2 / z^1$$

and $\phi_1^{-1} : V_1 \longrightarrow U_1$

$$z \longmapsto [1 ; z]$$

now define

$$\phi_{U_1} : U_1 \times S^1 \longrightarrow p^{-1}(U_1)$$

$$([z^1 : z^2], \theta) \longmapsto \frac{e^{i\theta}}{\sqrt{1 + |z^2/z^1|^2}} (1, z_2/z_1)$$

exercise: a) check ϕ_{U_1} is a diffeomorphism and gives a local trivialization

Hint: $\phi_{U_1}^{-1}(w^1, w^2) = ([w^1 : w^2], \underbrace{\arg w^1}_{= \theta^1})$ $\in S^3$ where $w^1 = r^1 e^{i\theta^1}$

b) check the analogously defined ϕ_{U_2} is also a loc. triv.

so
$$\begin{tikzcd} S^1 \arrow[r] & S^3 \arrow[d] \\ & \mathbb{C}P^1 \end{tikzcd}$$
is a bundle

c) Show $S^3 \not\cong S^2 \times S^1$

(probably easier later!)

given two bundles
$$\begin{tikzcd} E \arrow[d, "p"] \\ B \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} E' \arrow[d, "p'"] \\ B' \end{tikzcd}$$

then a bundle map is a pair $(f, \bar{f})$ s.t.

$$\begin{tikzcd} E \arrow[r, "\bar{f}"] \arrow[d, "p"] & E' \arrow[d, "p'"] \\ B \arrow[r, "f"] & B' \end{tikzcd}$$
commutes

(bundle map is smooth if $f, \bar{f}$ are smooth and $\bar{f}$ on each fiber a diffeomorphism) if $\begin{tikzcd} E \arrow[d, "p"] \\ B \end{tikzcd}$ is a fiber bundle, then a <u>section</u> is a map $\sigma : B \to E$ such that $p \circ \sigma = id_B$

$$\begin{tikzcd} E \arrow[d, "p"'] \\ B \arrow[u, bend right, "\sigma"'] \end{tikzcd}$$

the set of sections is denoted $\Gamma(E)$

How to understand all bundles

given
$$\begin{tikzcd} F \arrow[r] & E \arrow[d, "p"] \\ & B \end{tikzcd}$$

take two local trivializations: $U_\alpha, U_\beta \subset B$ and

$$\begin{tikzcd} U_\alpha \times F \arrow[rr, "\phi_\alpha"] \arrow[dr, "p_1"'] & & p^{-1}(U_\alpha) \arrow[dl, "p"] \\ & U_\alpha & \end{tikzcd} \qquad \begin{tikzcd} U_\beta \times F \arrow[rr, "\phi_\beta"] \arrow[dr, "p_1"'] & & p^{-1}(U_\beta) \arrow[dl, "p"] \\ & U_\beta & \end{tikzcd}$$

if $U_\alpha \cap U_\beta \neq \emptyset$ then

$$\begin{tikzcd} (U_\alpha \cap U_\beta) \times F \arrow[r, "\phi_\alpha"] \arrow[dr, "p_1"'] & p^{-1}(U_\alpha \cap U_\beta) \arrow[r, "\phi_\beta^{-1}"] \arrow[d, "p"] & (U_\alpha \cap U_\beta) \times F \arrow[dl, "p_1"] \\ & U_\alpha \cap U_\beta & \end{tikzcd}$$

$U_\alpha \cap U_\beta$ F

so $(\phi_\beta^{-1} \circ \phi_\alpha)(p, x) = (p, g_{\beta\alpha}(p)(x))$

where $g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathrm{Homeo}(F)$

is continuous

space of homeomorphisms

$F \to F$

in the compact-open topology

U open, K compact in F

$W(U,K) = \{ h : F \to F \mid F(K) \subset U \}$

$\{W(U,K)\}_{\substack{\text{all } U,K \\ \text{in } F}}$ basis for topology

If F a metric space then this is the topology of uniform convergence.

if the bundle is smooth use $\mathrm{Diff}(F)$ diffeom.

instead of $\mathrm{Hom}(F)$

the $g_{\alpha\beta}$ is called a transition map

(or a clutching function)

exercise: if $U_\alpha, U_\beta, U_\gamma$ are as above then their trasition maps satisfy

$$\circledast \begin{cases} g_{\gamma\beta} \circ g_{\beta\alpha} = g_{\gamma\alpha} & \text{and} \\ g_{\alpha\alpha} = id_F \end{cases}$$

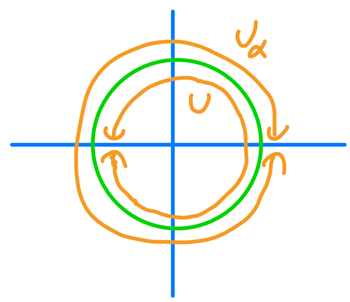
so given $\begin{array}{c} F \to E \\ \downarrow \\ B \end{array}$ we get a cover $\{U_\alpha\}_{\alpha \in A}$ of B

by local trivializations and functions $\{g_{\alpha\beta}\}$ satisfying $\otimes$

example:

1) Möbius Band:

$$U_\alpha = S^1 - \{(1,0)\}$$
$$U_\beta = S^1 - \{(-1,0)\}$$



$$g_{\beta\alpha} : \underset{\substack{\shortparallel \\ (0,1/2) \cup (1/2,1)}}{U_\alpha \cap U_\beta} \to \mathrm{Homeo}(\mathbb{R})$$

exercise: $g_{\beta\alpha}(t) = \begin{cases} id_{\mathbb{R}} & t \in (0, 1/2) \\ -id_{\mathbb{R}} & t \in (1/2, 1) \end{cases}$

2) $\begin{array}{c} S^1 \to S^3 \\ \quad\ \downarrow \\ \quad\ \mathbb{C}P^1 \; (= S^2) \end{array}$

$$\mathbb{C}P^1 = U_1 \cup U_2$$

$$U_1 = \{[z^1 : z^2] \mid z_1 \neq 0\}$$
$$U_2 = \{[z^1 : z^2] \mid z_2 \neq 0\}$$

exercise:

$$(U_1 \cap U_2) \times S^1 \longrightarrow (U_1 \cap U_2) \times S^1$$

$$([z^1 : z^2], \theta) \longmapsto ([z^1 : z^2], \theta + \underbrace{\arg z^2/z^1}_{\text{note: well-defined}})$$

Th$^{\underline{m}}$ 1:

if $\{U_\alpha\}_{\alpha \in A}$ is an open cover of B and you have

$$g_{\beta\alpha} : U_\alpha \cap U_\beta \longrightarrow \mathrm{Hom}(F)$$

satisfying $\circledast$

then

disjoint union

$$E = \left(\coprod_{\alpha \in A} U_\alpha \times F \right) \Big/ \sim$$

where $(n, x) \in U_\alpha \times F \sim (m, y) \in U_\beta \times F$

if $n = m$ and $g_{\beta\alpha}(n)(x) = y$

is a fiber bundle over B with fiber F

(with the obvious projection)

and if B, F are smooth manifolds and

$$g_{\beta\alpha} : U_\alpha \cap U_\beta \longrightarrow \mathrm{Diff}(F)$$

then E is a smooth manifold and

p is a smooth map

Proof:

note:
$$\begin{tikzcd} U_\alpha \times F \arrow[r, hook] \arrow[rr, bend right, "i"'] & \coprod_{\beta \in A} U_\beta \times F \arrow[r, "q"] & E \end{tikzcd}$$

is clearly injective (no identifications)

and onto $p^{-1}(U_\alpha)$

also $i^{-1}: p^{-1}(U_\alpha) \to U_\alpha \times F$ can be defined

as follows

$$\tilde{h}: q^{-1}(p^{-1}(U_\alpha)) \to U_\alpha \times F$$

$$(n,x) \longmapsto \begin{cases} (n,x) & (n,x) \in U_\alpha \times F \\ (n, g_{\alpha\beta}(n)(x)) & (n,x) \in U_\beta \times F \end{cases}$$

$$\begin{tikzcd} q^{-1}(p^{-1}(U_\alpha)) \arrow[d, "q"] \arrow[dr, "\tilde{h}"] & \\ p^{-1}(U_\alpha) \arrow[r, dashed, "h"'] & U_\alpha \times F \end{tikzcd}$$

induced map

easy to see $h = i^{-1}$

exercise: 1) show h is continuous
(need to play with compact-open topol.)

2) Show E is smooth manifold if B, F are
and $g_{\alpha\beta}: U_\alpha \cap U_\beta \to \mathrm{Diff}(F)$

B. Vector Bundles

a vector bundle is a fiber bundle $\begin{array}{c} E \\ \downarrow \\ B \end{array}$ s.t. $p^{-1}(x)$ is a vector space $\forall x \in B$ and there are local trivializations

$$\phi : U \times \mathbb{R}^n \longrightarrow p^{-1}(U)$$

that cover B s.t.

$$\phi\big|_{\{x\}\times\mathbb{R}^n} : \mathbb{R}^n \longrightarrow p^{-1}(x)$$

is a linear isomorphism $\forall x \in U$

lemma 2:

if $\begin{array}{ccc} F & \rightarrow & E \\ & & \downarrow p \\ & & B \end{array}$ is a fiber bundle,

then it is a vector bundle

$\iff$

1) $F = \mathbb{R}^n$ (some n) and

2) $\exists$ a cover $\{U_\alpha\}$ of B s.t.

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow GL(n, \mathbb{R})$$

for all α, β

Proof: ($\Rightarrow$) 1) clear

2) $g_{\alpha\beta}(x) \in \mathrm{Homeo}(\mathbb{R}^n)$ and a linear map

$\therefore$ an elt of $GL(n,\mathbb{R})$

($\Leftarrow$) give each fiber the structure of a vector space using local trivializations and the condition on $g_{\alpha\beta}(x)$ says this structure does not depend on the trivialization

now given a manifold M^n

set $$TM = \coprod_{x \in M} T_x M$$

and $p : TM \to M$ the obvious projection.

$$p(v) = x \iff v \in T_x M$$

we want to make TM a vector bundle

first we need a topology on TM:

let $\{U_\alpha\}_{\alpha \in A}$ be a cover of M by coordinate charts $\phi_\alpha : U_\alpha \to V_\alpha \subseteq \mathbb{R}^n$

<u>note</u>: $$d\phi_\alpha : TU_\alpha \to TV_\alpha = \underbrace{V_\alpha \times \mathbb{R}^n}_{\text{gets a topology from}} \subseteq \mathbb{R}^{2n}$$

$$v \longmapsto (d\phi_\alpha)_{p(v)}(v)$$

one-to-one correspondence

now set $\mathcal{U} = \{ d\varphi_\alpha^{-1}(W) \mid W \text{ open set in } V_\alpha \times \mathbb{R},\ \alpha \in A \}$

Thm 3:

1) $\mathcal{U}$ is a basis for a topology on TM and with this topology TM is a $2n$-dimensional manifold.

2) $\{ d\varphi_\alpha : TU_\alpha \to TV_\alpha \}_{\alpha \in A}$ give a smooth atlas on TM

3) with the above smooth structure

$$p : TM \to M$$

is a smooth map making TM a vector bundle

4) given $f : M \to N$ a smooth map

then
$$\begin{tikzcd} TM \arrow[r, "df"] \arrow[d, "p"'] & TN \arrow[d, "p"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

is a smooth bundle map

Proof: exercise!

$$\begin{tikzcd} TM \arrow[d, "p"] \\ M \end{tikzcd}$$
is called the tangent bundle of M