IX Tensors

A. Linear Algebra

let V, W be vector spaces

let $F(V \times W)$ be the vector space generated by $V \times W$

let $R(V \times W)$ be the subspace of $F(V \times W)$ generated by

$$(v_1 + v_2, w) - (v_1, w) - (v_2, w)$$
$$(v, w_1 + w_2) - (v, w_1) - (v, w_2)$$
$$(av, w) - a(v, w)$$
$$(v, aw) - a(v, w)$$

the tensor product of V and W is

$$V \otimes W = F(V \times W) / R(V \times W)$$

the coset of (v, w) is denoted $v \otimes w$

clearly:
$$v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2$$
$$(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w$$
$$(av) \otimes w = a(v \otimes w) = v \otimes (aw)$$

Universal Property:

let $\phi : V \times W \to V \otimes W$ be the composition of

$$V \times W \xrightarrow{i} F(V \times w) \xrightarrow{\rho} V \otimes W$$

if U is any vector space and

$$\ell : V \times W \to U$$

is a bilinear map, then $\exists!\ \tilde{\ell} : V \otimes W \to U$

such that

$$\tilde{\ell} \circ \phi = \ell \qquad \begin{tikzcd} V \times W \arrow[r, "\ell"] \arrow[d, "\phi"'] & U \\ V \otimes W \arrow[ur, dashed, "\tilde{\ell}"'] \arrow[u, phantom, shift left=4ex, "\circ"] & \end{tikzcd}$$

exercise:

1) Show this

2) if X and $\Phi : V \times W \to X$ satisfy the universal property, then show $X \cong V \otimes W$

3) $\left\{ \begin{matrix} \text{bilinear maps} \\ V \times W \to U \end{matrix} \right\} \xleftrightarrow[\text{one-to-one correspondence}]{} \left\{ \begin{matrix} \text{linear maps} \\ V \otimes W \to U \end{matrix} \right\}$

4) $V \otimes W \cong W \otimes V$

canonically

5) $V \otimes (W \otimes U) \cong (V \otimes W) \otimes U$

6) $\phi : V \to V'$
$\psi : W \to W'$ linear maps

then $\exists!$ linear map

$$\phi \otimes \psi : V \otimes W \to V' \otimes W'$$

s.t. $\phi \otimes \psi (v \otimes w) = \phi(v) \otimes \psi(w)$

and if $\phi' : V' \to V''$, $\psi' : W' \to W''$

then $(\phi' \otimes \psi') \circ (\phi \otimes \psi) = (\phi' \circ \phi) \otimes (\psi' \circ \psi)$

now suppose $e_1, \dots, e_n$ is a basis for V and

$f_1, \dots, f_m$ is a basis for W

then $e_i \otimes f_j \in V \otimes W$

<u>note</u>: $\alpha \in V \otimes W$, then

equivalence class

$$\alpha = \left[\sum_{i=1}^{\ell} a_i \, (v_i, w_i) \right]$$

$v_i \in V$, $w_i \in W$

and $v_i = \sum v_i^j e_j$

$w_i = \sum w_i^k f_k$

so $\alpha = \left[\sum_{i=1}^{\ell} a_i (v_i, w_i) \right] = \sum_{i=1}^{\ell} a_i \, v_i \otimes w_i$

$$= \sum_{i=1}^{\ell} a_i \left(\sum v_i^j e_j \right) \otimes \left(\sum w_i^k f_k \right)$$

$$= \sum_{i,j,k} a_i v_i^j w_i^k \, e_j \otimes f_k$$

thus $\{e_i \otimes f_j\}_{\substack{i=1\dots n \\ j=1\dots m}}$ span $V \otimes W$

if $\sum a^{ij} e_i \otimes f_j = 0$ then let

$e^i : V \to \mathbb{R}$ be dual to e_i

$f^j : W \to \mathbb{R}$ " " f_j

and note

$$0 = e^k \otimes f^\ell \left(\sum a^{ij} e_i \otimes f_j\right) = a_{k\ell} \quad \forall k, \ell$$

so $\{e_i \otimes f_j\}$ linear independent

thus we have shown

lemma 1:

$$\boxed{\{e_i \otimes f_j\}_{\substack{i=1,\dots,n \\ j=1,\dots,m}} \text{ is a basis for } V \otimes W \text{ and } \dim V \otimes W = (\dim V)(\dim W)}$$

lemma 2:

$$\boxed{V^* \otimes W^* \cong (V \otimes W)^* \cong Bilin(V \times W, \mathbb{R})}$$

Proof:

$$V^* \times W^* \xrightarrow{\tilde{\varphi}} (V \otimes W)^*$$

$$(a, b) \longmapsto \varphi_{a,b} : V \otimes W \to \mathbb{R}$$

$$v \otimes w \longmapsto a(v)\, b(w)$$

check well-defined!

$\tilde{\varphi}$ is bilinear so by universal property

$\exists!$ linear $\varphi : V^* \otimes W^* \to (V \times W)^*$

easy to check φ is one-to-one and onto

(dim's are equal so just check onto

$\{e_i \otimes f_j\}$ is a basis for $V \otimes W$

let $\{g^{ij}\}$ be the dual basis

now $e^i \otimes f^j \mapsto g^{ij}$)

second isomorphism similar (exercise)

Remark: we could use this to define $V \otimes W$

$$V \otimes W = V^{**} \otimes W^{**} \cong (V^* \otimes W^*)^*$$

$$\cong \mathrm{Bilin}(V^* \times W^*, \mathbb{R})$$

(recall $V \cong V^{**}$ canonically

$v \mapsto (\phi_v : V^* \to \mathbb{R} : a \mapsto a(v))$)

exercise: $V^* \otimes W \cong \mathrm{Hom}(V, W)$

canonically

Hint: given $(a, w) \in V^* \otimes W$

define $\phi_{a,w} : V \to W$

$v \mapsto a(v)w$

to get a map

$$V^* \otimes W \to \mathrm{Hom}(V, W)$$

notation: $T_q^p(V) = \underbrace{V \otimes \dots \otimes V}_{q\text{-times}} \otimes \underbrace{V^* \otimes \dots \otimes V^*}_{p\text{-times}}$

$$\cong \mathrm{Bilin}(V^* \times \dots \times V^* \times V \times \dots \times V, \mathbb{R})$$

$T_0^0(V) = \mathbb{R}$

set $T_*(V) = \sum_k T^0_k(V)$

this is called the tensor algebra of V

(i.e. vector space with multiplication)

note: $a \in T^0_k(V)$, $b \in T^0_\ell(V)$ then

$a \otimes b \in T^0_{k+\ell}(V)$

note: $T^k(V) = T^k_0(V) = V^* \otimes \dots \otimes V^* = T_k(V^*)$

$= \mathrm{MultiLin}(V \times \dots \times V \to \mathbb{R})$

B. Tensor Fields

define $T^k M = \coprod_{p \in M} T^k(T_p M)$

$T_k M = \coprod_{p \in M} T_k(T_p M)$

$T^\ell_k M = \coprod_{p \in M} T^\ell_k(T_p M)$

exercise: these are all vector bundles over M

the fiber of $T^\ell_k M$ has dim $n^{k+\ell}$

where $\dim M = n$

note: $T^1 M = T^* M$

$T^0 M = M \times \mathbb{R} = T_0 M$

$T_1 M = TM$

In local coordinates $\sigma \in \Gamma(T^k M)$ is

$$\sigma = \sum_{i_1 \ldots i_k = 1}^{n} \sigma_{i_1 \ldots i_k} \, dx^{i_1} \otimes \ldots \otimes dx^{i_k}$$

and similarly for $\Gamma(T_k M)$ and $\Gamma(T_k^l M)$

called tensor fields

if $\sigma \in \Gamma(T^k N)$ and $f: M \to N$

then $f^*: T^*N \to T^*M$ is a linear map

so get $f^*: T^k N \to T^k M$

and $f^*\sigma \in \Gamma(T^k M)$

exercise: if f as above and $g: N \to \mathbb{R}$, then

1) $f^*(g\sigma) = (f^*g)(f^*\sigma) = (g \circ f) f^*\sigma$

2) $f^*(\sigma \otimes \tau) = (f^*\sigma) \otimes (f^*\tau)$

3) $f^*: \Gamma(T^k N) \to \Gamma(T^k M)$ is linear

4) $h: N \to W$ then

$$(h \circ f)^* = f^* \circ h^*$$

5) $(id_N)^* = id_{\Gamma(T^k N)}$

now if $f : \mathbb{R}^n \to \mathbb{R}^m$

$$(x^1, \dots x^n) \mapsto (y^1(x^1 \dots x^n), \dots, y^m(x^1 \dots x^n))$$

and $\sigma = \sum \sigma^{i_1 \dots i_k} dy^{i_1} \otimes \dots \otimes dy^{i_k} \in \Gamma(T^k \mathbb{R}^m)$

then $f^* \sigma = \sum (\sigma^{i_1 \dots i_k} \circ f) \, d(y^{i_1} \circ f) \otimes \dots \otimes d(y^{i_k} \circ f)$

example: $\sigma = xy \, dx \otimes dy$

$$f : \mathbb{R}^2 \to \mathbb{R}^2 : (x, y) \mapsto (x \cos y, x \sin y)$$

then

$$
\begin{aligned}
f^* \sigma &= x^2 \cos y \sin y \, d(x \cos y) \otimes d(x \sin y) \\
&= x^2 \cos y \sin y [\cos y \, dx - x \sin y \, dy] \\
&\qquad \otimes [\sin y \, dx + x \cos y \, dy] \\
&= x^2 \cos^2 y \sin^2 y (dx \otimes dx - x^2 dy \otimes dy) \\
&\qquad + x^3 \cos^3 y \sin y \, dx \otimes dy \\
&\qquad - x^3 \cos y \sin^3 y \, dy \otimes dx
\end{aligned}
$$