V Vector Fields and Flows

A. Vector fields

a vector field v on a manifold M is a smooth section of the tangent bundle TM

$$\begin{array}{c} TM \\ p\big\downarrow \;\big\uparrow v \\ M \end{array} \qquad p \circ v(x) = 0$$

i.e. a choice of vector at each point of M

note: in $\mathbb{R}^n$, $\quad T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$

so $\quad v : \mathbb{R}^n \longrightarrow \mathbb{R}^n \times \mathbb{R}^n$

$$(x^1, \dots, x^n) \longmapsto ((x^1, \dots, x^n), (v^1(x), \dots, v^n(x)))$$

$$x = (x^1, \dots, x^n)$$

or we can write

$$v(x) = \sum_{i=1}^{n} v^i(x) \frac{\partial}{\partial x^i}$$

so v is determined by a function

$$F_v(x) = (v^1(x), \dots, v^n(x))$$

$$F_v : \mathbb{R}^n \to \mathbb{R}^n$$

and given any function $F : \mathbb{R}^n \to \mathbb{R}^n$

we get a vector field

$$v_F : \mathbb{R}^n \longrightarrow T\mathbb{R}^n$$
$$x \longmapsto (x, F(x))$$

this is probably how you saw vector fields in ODE/Calc.

we denote the set of vector fields by

$$\mathfrak{X}(M) = \Gamma(TM)$$

exercise: show $\mathfrak{X}(M)$ is a vector space

no given a vector field $v \in \mathfrak{X}(M)$ and a function

$$f : M \to \mathbb{R}$$

then

$$v \cdot f : M \to \mathbb{R}$$
$$x \mapsto v(x) \cdot f$$

derivation at x

is again a function on M check!

so $v \in \mathfrak{X}(M)$ gives a linear map

$$v : C^\infty(M) \to C^\infty(M)$$

and $v \cdot (fg) = (v \cdot f) g + f (v \cdot g)$

we say v is a derivation on M

exercise: vector fields are in one-to-one correspondance with derivations on M

note: $df(v) : M \to \mathbb{R}$

$$x \mapsto df_x(v(x))$$

$df_x(v(x)) \in T_{f(x)}\mathbb{R} \cong \mathbb{R}$

canonically by Thm II.2

and the local coord. computation of df from Section II says

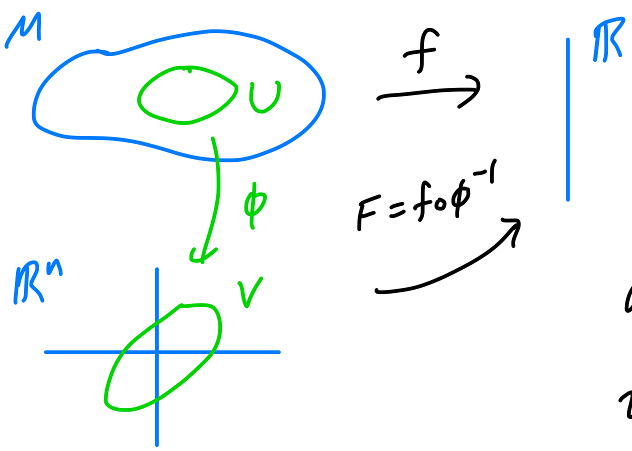
$$df(v)(x) = v \cdot f(x)$$

<u>In local coordinants:</u>

$f : M \to \mathbb{R}$

$v \in \mathfrak{X}(M)$

$\phi : U \to V$ a local chart



now $F(x^1, \ldots, x^n) = f \circ \phi^{-1}(x^1, \ldots, x^n)$

$$v_{loc} = d\phi(v) \in \Gamma(TV)$$
$$\quad \overset{\shortparallel}{\sum_{i=1}^{n} v^i(x^1, \ldots, x^n) \frac{\partial}{\partial x^i}}$$

recall $dF_x\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial F}{\partial x^i}(x)$

so
$$v \cdot f = df(v) = dF(v_{loc}) = \sum_{i=1}^{n} v^i(x) \frac{\partial F}{\partial x^i}(x)$$

now given another vector $w = \sum w^i(x) \frac{\partial}{\partial x^i}$

then note
$$w \cdot (v \cdot f) = w \cdot \sum_i v^i(x) \frac{\partial F}{\partial x^i}(x)$$
$$= \sum_{i,j} w^j \left(\frac{\partial v^i}{\partial x^j} \frac{\partial F}{\partial x^i} + v^i \frac{\partial^2 F}{\partial x^i \partial x^j} \right)$$

and $\quad w\cdot(v\cdot f) - v\cdot(w\cdot f) = \sum \left(w^j \frac{\partial v^i}{\partial x^j} - v^j \frac{\partial w^i}{\partial x^j}\right) \frac{\partial F}{\partial x^i}$

so if we set $\displaystyle X = \sum_{i,j=1}^{n} \left(w^j \frac{\partial v^i}{\partial x^j} - v^j \frac{\partial w^i}{\partial x^j}\right) \frac{\partial}{\partial x^i}$

then X is a vector field in these local coords.

<u>exercise:</u> X is independent of local coords.

so X is well-defined on M and only depends on v, w we denote it $[v,w]$ and call it the <u>Lie bracket</u>

$$[\cdot,\cdot] : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

<u>Properties:</u> $f,g \in C^\infty(M)$, $v,w,u \in \mathfrak{X}(M)$, $a,b \in \mathbb{R}$

skew-symmetric 1) $[v,w] = -[w,v]$

2) $[fv, gw] = fg[v,w] + f(v\cdot g)w - g(w\cdot f)v$

linear 3) $[av+bu, w] = a[v,w] + b[u,w]$

Jacobi identity 4) $[[v,u],w] + [[u,w],v] + [[w,v],u] = 0$

recall a vector space V with a bilinear pairing $[\cdot,\cdot] : V \times V \to \mathbb{R}$ satisfying 1) and 4) is called a <u>Lie algebra</u>

so $(\mathfrak{X}(M), [\cdot,\cdot])$ is a Lie algebra example: in local coordinates in $\mathbb{R}^n$

$$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] f = \frac{\partial}{\partial x^i}\left(\frac{\partial}{\partial x^j} f\right) - \frac{\partial}{\partial x^j}\left(\frac{\partial}{\partial x^i} f\right) = 0$$

(if f is smooth)

so $\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0$

from this and the properties above we can compute any Lie bracket

e.g. $v = \frac{\partial}{\partial y}$ $\quad u = y\frac{\partial}{\partial x} + \frac{\partial}{\partial z}$

$$\begin{aligned}
[v,u] &= \left[\frac{\partial}{\partial y}, y\frac{\partial}{\partial x} + \frac{\partial}{\partial z}\right] \\
&= \left[\frac{\partial}{\partial y}, y\frac{\partial}{\partial x}\right] + \cancel{\left[\frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right]}^{\,0} \\
&= y\cancel{\left[\frac{\partial}{\partial y}, \frac{\partial}{\partial x}\right]}^{\,0} + \frac{\partial y}{\partial y}\frac{\partial}{\partial x} + \cancel{\frac{\partial 1}{\partial x}}^{\,0}\frac{\partial}{\partial y} \\
&= \frac{\partial}{\partial x}
\end{aligned}$$

B Flows

Th$^{\underline{m}}$ 1:

let M be a manifold without boundary and $v \in \mathfrak{X}(M)$ a vector field

then there exist positive continuous functions

$$\varepsilon, \delta : M \to (0, \infty]$$

and a unique smooth function

$$\Phi : W_{\varepsilon,\delta} \longrightarrow M$$
$$W_{\varepsilon,\delta} = \{ (p,t) \in M \times \mathbb{R} \mid -\varepsilon(p) < t < \delta(p) \}$$

such that

① $\Phi(p,0) = p$ and

② $d\Phi_{(p,t)}\left(\frac{\partial}{\partial t}\right) = v(\Phi(p,t))$

the map satisfies

$$\Phi(\Phi(p,s),t) = \Phi(p,s+t)$$

if $-\varepsilon(p) < s+t < \delta(p)$

if v has compact support (e.g. if M cpt), then $\varepsilon, \delta = \infty$

i.e. $\Phi : M \times \mathbb{R} \to M$

Remarks:

1) if we set $\gamma^p(t) = \Phi(p,t)$ then

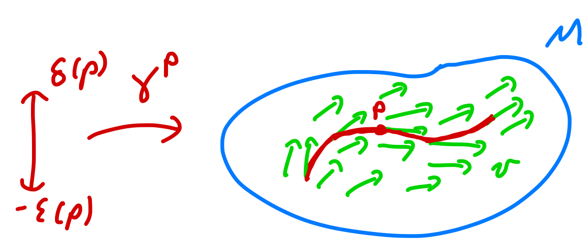
$$\gamma^p : (-\varepsilon(p), \delta(p)) \longrightarrow M$$

is a curve such that

$\gamma^p(0) = p$ and

$(\gamma^p)'(t) = (d\gamma^p)_{\gamma(t)}\left(\frac{\partial}{\partial t}\right) = v(\gamma(t))$

γ^p is called a flow line or integral curve of v through p



2) any map $\Phi : W_{\varepsilon,\delta} \to M$ satisfying

- $\Phi(\Phi(p,s),t) = \Phi(p,s+t)$
- $\Phi(p,0) = p$

is called a flow on M or a dynamical system on M

given a flow Φ set $v(p) = d\Phi_{(p,0)}\left(\frac{\partial}{\partial t}\right)$

exercise: $v : M \to TM$ is a smooth section

this gives a vector field associated to Φ called the velocity field of Φ

note: Th$^{\underline{m}}$ says associated to v is a flow Φ' and uniqueness $\Rightarrow \Phi = \Phi'$

3) fix $t \in (-\min \varepsilon, \min \delta)$ and set

$$\phi^t : M \to M : p \mapsto \Phi(p,t)$$

this is a smooth map and if $t, -t \in (-\min \varepsilon, \min \delta)$

then
$$\begin{aligned} \phi^t \circ \phi^{-t}(p) &= \Phi(\Phi(p,-t), t) \\ &= \Phi(p,0) = p \end{aligned}$$

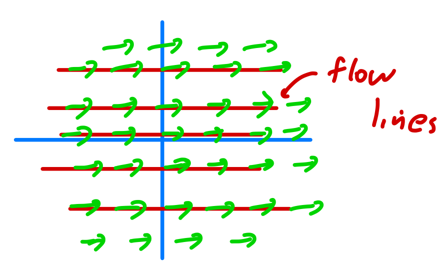
so $\phi^t \circ \phi^{-t} = id_M = \phi^{-t} \circ \phi^t$

similarly

thus ϕ^t is a diffeomorphism!

$$\boxed{\text{this is a very good way to build diffeomorphisms}}$$

examples: 1) $\Phi : \mathbb{R}^2 \times \mathbb{R} \longrightarrow \mathbb{R}^2$
$((x,y),t) \longmapsto (x+t, y)$

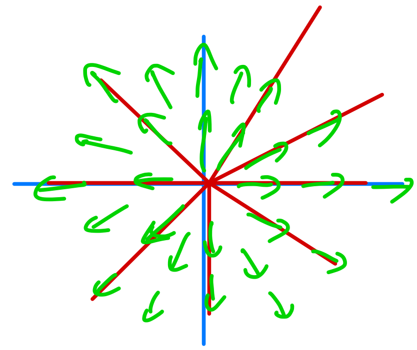


$$d\Phi_{((x,y),t)}\left(\frac{\partial}{\partial t}\right) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

so $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is the <u>velocity field</u> of the <u>flow</u> Φ

2) $\Phi : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$

$((x,y),t) \longmapsto (e^t x, e^t y)$



$$d\Phi_{((x,y),t)}\left(\frac{\partial}{\partial t}\right) = \begin{bmatrix} e^t & 0 & e^t x \\ 0 & e^t & e^t y \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} e^t x \\ e^t y \end{bmatrix}$$

so $v = \begin{bmatrix} xe^t \\ ye^t \end{bmatrix}$ is the velocity field of the flow Φ

For the proof of the theorem we need

Fundamental Theorem of ODE:

$U \subset \mathbb{R}^n$ open set

$F : U \to \mathbb{R}^n$ a smooth function

for $a \in \mathbb{R}$ and $x_0 \in U$

<u>Existence</u>: $\exists$ an open set J_0 containing a and an open set $U_0 \subset U$ containing x_0 such that

$\forall p \in U_0 \ \exists \ \gamma_p : J_0 \to U$ such that

$$\boxed{\begin{array}{ll} \gamma_p'(t) = F(\gamma_p(t)) & \forall\, t \in J_0 \\ \gamma_p(a) = p & \end{array}} \quad \circledast$$

Uniqueness: if $\gamma, \tilde{\gamma}$ both satisfy $\circledast$ and

$$\gamma(a) = \tilde{\gamma}(a)$$

then $\gamma(t) = \tilde{\gamma}(t)$ on common domain

Smoothness: if J_0, U_0 as a above then the map

$$\begin{aligned} \Gamma : U_0 \times J_0 &\longrightarrow U \\ (p, t) &\longmapsto \gamma_p(t) \end{aligned}$$

is smooth

the proof of this is standard in an analysis course or advanced ODE course

Proof of Th^{m} 1: given $p \in M$, let $\phi : U \to V$ be a coordinate chart about p

$d\phi(v)$ is a vector field on $V \subset \mathbb{R}^n$ so can be represented as a function $F : V \to \mathbb{R}^n$ so

$$d\phi(v)(x) = (x, F(x)) \in V \times \mathbb{R}^n = TV$$

by ODE theorem, $\exists$ an interval J_0 (containing 0) and $V_0 \subset V$ and function

$$\begin{aligned} \tilde{\Gamma} : V_0 \times J_0 &\longrightarrow V \\ (x, t) &\longmapsto \tilde{\gamma}_x(t) \end{aligned}$$

where $\tilde{\gamma}_x$ satisfies $\tilde{\gamma}_x(0) = p$ and

$$\tilde{\gamma}_x'(t) = F(\tilde{\gamma}_x(t))$$

so we get $\Gamma : U_0 \times J_0 \to U$ where $U_0 = \phi^{-1}(V_0)$

$$(x,t) \longmapsto \gamma_x(t)$$

where $\gamma_x(t) = \phi^{-1} \circ \gamma_{\phi(x)}(t)$

<u>note</u>: Γ is Φ near p and Γ is smooth by ODE thm

<u>exercise</u>: Show you can do the same for other points in M and by uniqueness in ODE thm they fit together to give Φ on all of M and we can assume at each p, Φ is defined on a maximal subinterval

<u>exercise</u>: check $\Phi(\Phi(p,t),s) = \Phi(p,t+s)$

the result about compact support follows from

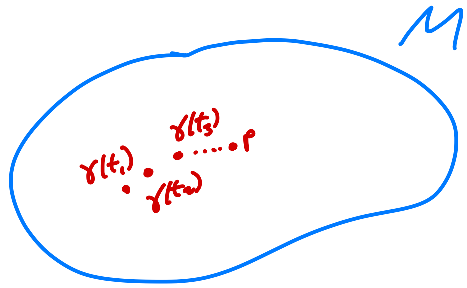
<u>lemma 2</u>:

> let $\gamma : (a,b) \to M$ be a flow line of v through p (i.e. $\gamma(0) = p$)
> assume (a,b) is the maximal interval on which γ can be defined
> if $a \neq -\infty$ or $b \neq \infty$ then $\gamma((a,b))$ is <u>not</u> contained in any compact subset of M

<u>Proof</u>: suppose $b < \infty$ and $\operatorname{im}\gamma \subset$ compact set in M

take $t_i \to b$ an increasing sequence in (a, b)

by compactness $\gamma(t_i) \to p$



from ODE th$^{\underline{m}}$ $\exists\, U_0$ a nbhd of p and $J_0 = (-\varepsilon, \varepsilon)$

st. $\tilde{\Gamma} : U_0 \times J_0 \to M$

$(x, t) \longmapsto \gamma_x(t)$ is defined

now $\exists$ some t_i st. $t_i > b - \varepsilon$ set

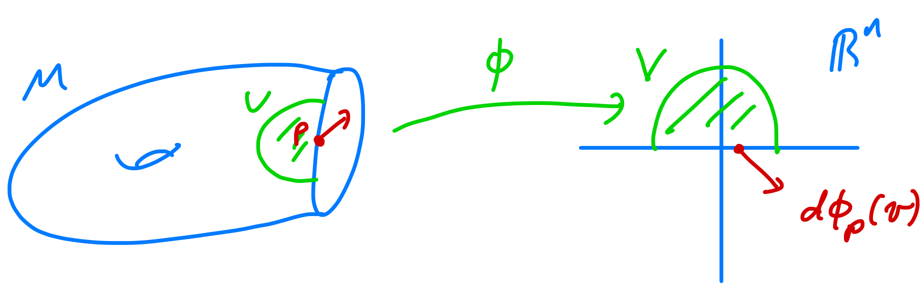
$$\tilde{\gamma} : (a, t_i + \varepsilon) \to M$$

$$t \longmapsto \begin{cases} \gamma(t) & a < t < b \\ \tilde{\Gamma}(\gamma(t_i), t - t_i)) & t_i - \varepsilon < t < \underbrace{t_i + \varepsilon}_{> b} \end{cases}$$

you can check that $\tilde{\gamma}$ is well defined an extention of γ to a larger subinterval

exercise:

1) if M is a manifold with boundary we say $v \in T_x M$ with $x \in \partial M$, points out of M if there is a local coordinate chart $\phi : U \to V$ such that $d\phi_x(v)$ has negative x^n-coordinate



Show this is independent of coordinates

2) if v is a vector field on a manifold with boundary and $v(x)$ never points out of M then show there exists $\delta : M \to (0,\infty)$ such that

$$\Phi : W_{0,\delta} \to M \text{ is as in th}^{m}\ 1$$

and if v has compact support then

$$\Phi : M \times [0,\infty) \to M$$

Great application!

Th^{m} 3:

every open neighborhood of ∂M is a compact mfd M contains a collar neighborhood

that is a smooth map

$$\psi : (\partial M \times [0,\varepsilon)) \to M$$

s.t. $\psi|_{\partial M \times \{0\}} : \partial M \times \{0\} \to \partial M$ is a diffeomorphism

and $\psi : (\partial M \times [0,\varepsilon)) \to im(\psi)$ is a diffeomorphism

Proof:

Claim: $\exists$ a vector field v such that v points into M and $v(x) \neq 0$ for all $x \in \partial M$

given v as in claim we get

$$\Phi : M \times [0,\infty) \to M$$

set

$$\psi : \partial M \times [0,\delta) \to M$$
$$(p,t) \longmapsto \Phi(p,t)$$

note: 1) $\psi(p,0)=p$ so $\psi|_{\partial M\times\{0\}}$ is a diffeomorphism (just inclusion)

2) $d\psi_{(p,0)}\left(\frac{\partial}{\partial t}\right) = v(p)$ so $d\psi_{(0,p)}$ an isomorphism $T_{(p,0)}(\partial M\times[0,\delta))$ to $T_p M$

$\in T_0[0,\delta)$ $\subset T_{(p,0)}\partial M\times[0,\delta)$ $\neq 0$ in T_pM

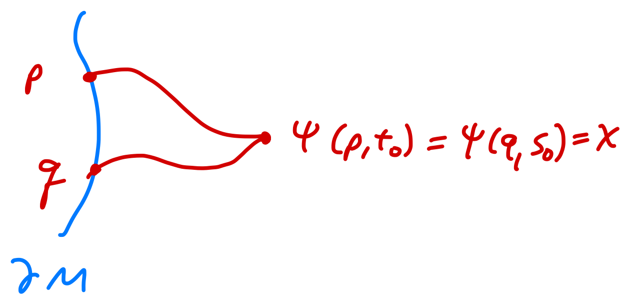
thus $d\psi_{(p,t)}$ an isomorphism for all t near 0

$\therefore$ ψ a local diffeomorphism at (p,t) for all $(p,t)\in\partial M\times[0,\delta')$ for some $\delta'>0$.

exercise: an injective local diffeomorphism is a diffeomorphism

so we now show ψ is injective

suppose $\psi(p,t_0)=\psi(q,s_0)$



so we have flow line $\gamma_p:[0,\varepsilon)\to M$

$\gamma_p'(t)=v(\gamma_p(t))$

$\gamma_q:[0,\varepsilon)\to M$

$\gamma_q'(t)=v(\gamma_q(t))$

and $\gamma_p(t_0)=\gamma_q(s_0)=x$

$\therefore$ γ_p, γ_q are flow lines through x

by uniqueness in ODE th^{m}

$$\gamma_p(t-t_0)=\gamma_q(t-s_0)$$

if $t_0<s_0$ then $\gamma_q(t_0-s_0)$ is not defined but $\gamma_p(t_0-t_0)$ is $\emptyset$

similarly for $t_0>s_0$ $\therefore$ $t_0=s_0$

$\therefore \gamma_p = \gamma_q$ and $p = q$

Now for Claim:

for each $p \in \partial M$ let $\phi_p : U_p \to V_p$ be a coordinate chart about p

let $r_p > 0$ be such that $B_{\phi(p)}(r_p) \cap \mathbb{R}_{\geq 0} \subset V_p$

ball of radius r_p about $\phi(p)$

set $\mathcal{O}_p = \phi_p^{-1}\left(B_{\phi(p)}(r_p/2)\right)$ and

$\mathcal{O}_p' = \phi_p^{-1}\left(B_{\phi(p)}(r_p)\right)$

and $f_p : M \to \mathbb{R}$ a bump function st.

$f_p = 1$ on $\overline{\mathcal{O}_p}$,

$f_p = 0$ outside $\mathcal{O}_p'$, and

$0 \leq f_p \leq 1$

$\{\mathcal{O}_p\}_{p \in \partial M}$ a cover of ∂M

take a finite subcover $\{\mathcal{O}_{p_1}, \dots, \mathcal{O}_{p_k}\}$

let $\tilde{v}_{p_i} = d\phi_{p_i}^{-1}\left(\frac{\partial}{\partial x^n}\right)$

set $$v_{p_i}(x) = \begin{cases} f_{p_i} \tilde{v}_{p_i} & x \in \mathcal{O}_{p_i}' \\ 0 & x \notin \mathcal{O}_{p_i}' \end{cases}$$

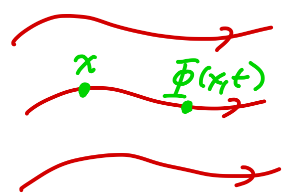
and $$v(x) = \sum_{i=1}^{k} v_{p_i}(x)$$

C. Lie derivatives

given a vector field v on a manifold M

let $\Phi : W_{\varepsilon,\delta} \to M$ be its flow

and $\phi^t : M \to M$ the associated diffeomorphisms for t small



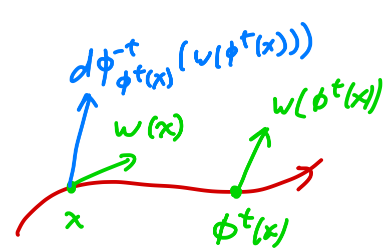
suppose $f : M \to \mathbb{R}$ is a function then define the Lie derivative of f to be

$$\mathcal{L}_v f(x) = \lim_{t \to 0} \frac{f \circ \Phi(x,t) - f(x)}{t}$$

$$= \frac{d}{dt}\left(f \circ \Phi(x,t)\right)\Big|_{t=0}$$

note: this is just f along a flow line so $\mathcal{L}_v f$ is the rate of change of f along the flow line

if w is another vector field then define the Lie derivative of w along v to be



$$\mathcal{L}_v w(x) = \lim_{t \to 0} \frac{d\phi^{-t}_{\phi^t(x)}(w(\phi^t(x))) - w(x)}{t}$$

$$= \frac{d}{dt}\left(d\phi^{-t}_{\phi^t(x)}(w(\phi^t(x)))\right)\Big|_{t=0}$$

all vectors in $T_x M$

Th$^{\underline{m}}$ 4:

$$\boxed{\begin{array}{l} 1)\ \mathcal{L}_v f = v\cdot f = df(v) \\ 2)\ \mathcal{L}_v w = [v,w] \end{array}}$$

Proof:

1) for a fixed x let

$$\gamma_x : (-\varepsilon,\varepsilon) \to M$$
$$t \mapsto \Phi(x,t)$$

so $\gamma_x'(0) = v(x)$ and $\gamma_x(0) = x$

so γ_x represents the vector $v(x)$ in $T_x M$

thus $df(v) = \frac{d}{dt}(f\circ\gamma_x)\big|_{t=0} = \frac{d}{dt}(f\circ\Phi(x,t))\big|_{t=0} = \mathcal{L}_v f(x)$

alt. def$^{\underline{n}}$ of df
in terms of paths

2) $\mathcal{L}_v w$ is clearly a vector field so to see $\mathcal{L}_v w = [v,w]$

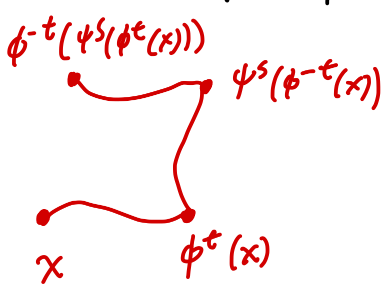
we just need to see

$$(\mathcal{L}_v w)\cdot f = [v,w]\cdot f \qquad \forall f \in C^\infty(M)$$

to do this let $\Psi : W_{\varepsilon',\delta'} \to M$ be the flow of w

and $\psi^t : M \to M$ the associated diffeomorphisms

for t,s near $(0,0)$ in $\mathbb{R}^2$ set



$$H(t,s) = f\left(\phi^{-t}\left(\psi^s\left(\phi^t(x)\right)\right)\right)$$

<u>note</u>: $\frac{\partial H}{\partial s}\Big|_{(t,0)} = \frac{\partial}{\partial s}\left[(f\circ\phi^{-t})\circ\Psi(s,\phi^t(x))\right]\Big|_{s=0}$

$= \mathcal{L}_w(f\circ\phi^{-t})(\phi^t(x))$

↑ by definition

$= \left(w\cdot(f\circ\phi^{-t})\right)(\phi^t(x))$

↑ by 1)

so $(\mathcal{L}_v w)\cdot f(x) = \left[\frac{d}{dt}\left(d\phi^{-t}_{\phi^t(x)}\, w(\phi^t(x))\right)\Big|_{t=0}\right]\cdot f$

$= \frac{d}{dt}\, d(f\circ\phi^{-t})(w(\phi^t(x)))\Big|_{t=0}$

↑ def$^{\underline{n}}$ of pushing vector forward

$= \frac{d}{dt}\left(w\cdot(f\circ\phi^{-t})\right)(\phi^t(x))\Big|_{t=0}$

$= \frac{\partial^2 H}{\partial t\partial s}\Big|_{(0,0)}$

now consider

$$K(t,s,u) = f\circ\phi^u\circ\psi^s\circ\phi^t(x)$$

so $\quad H(t,s) = K(t,s,-t)$

and

$\frac{\partial^2 H}{\partial t\partial s}\Big|_{(0,0)} = \frac{\partial^2}{\partial t\partial s}\left(K(t,s,-t)\right)\Big|_{(0,0)}$

$= \frac{\partial}{\partial t}\left(\frac{\partial K}{\partial s}(t,s,-t)\right)\Big|_{(0,0)}$

$= \left(\frac{\partial^2 K}{\partial u\partial s} - \frac{\partial^2 K}{\partial s\partial t}\right)\Big|_{(0,0)}$

now $\frac{\partial K}{\partial s}\Big|_{(t,0,0)} = (\mathcal{L}_w f)(\phi^t(x)) \underset{\text{by 1)}}{=} (w\cdot f)(\phi^t(x))$

and $$\left.\frac{\partial^2 K}{\partial u \partial s}\right|_{(0,0,0)} = \mathcal{L}_v (w \cdot f)(x) \underset{\text{by 1)}}{=} v \cdot (w \cdot f)(x)$$

similarly $$\frac{\partial^2 K}{\partial s \partial t} = w \cdot (v \cdot f)(x)$$

$$\therefore (\mathcal{L}_v w) \cdot f = \left.\frac{\partial^2 H}{\partial s \partial t}\right|_{(0,0)} = v \cdot (w \cdot f) - w \cdot (v \cdot f)$$

$$= [v, w] \cdot f$$