VI Approximation and Stability

A. Approximation of continuous functions

differential topology is about <u>smooth</u> functions but
many times we only have continuous information
so we begin with

<u>Th$^{\underline{m}}$1:</u>

> Suppose M is a smooth manifold and
> $f: M \to \mathbb{R}^k$ is a continuous function
> given any continuous positive function
> $\delta: M \to \mathbb{R}$
> there is a smooth function $\hat{f}: M \to \mathbb{R}^k$
> such that
> $$\|f(x) - \hat{f}(x)\| < \delta(x) \quad \forall x \in M$$
> if f is already smooth on a closed subset $A \subset M$
> then we can take $\hat{f} = f$ on A

"can jiggle a continuous function to make it smooth"

"smooth functions are dense in continuous functions"

for the proof we need the following

given an open cover $\{U_\alpha\}_{\alpha \in A}$ of a manifold M, then
a <u>partition of unity</u> subordinate to $\{U_\alpha\}$ is

a family of functions $\{\psi_\alpha : M \to \mathbb{R}\}_{\alpha \in A}$ such that

1. $0 \le \psi_\alpha(x) \le 1 \quad \forall \alpha \in A$ and $x \in M$
2. support of $\psi_\alpha \subset U_\alpha \quad \forall \alpha \in A$
3. $\forall x, \exists$ a nbhd V_x s.t. $\psi_\alpha|_{V_x} \equiv 0$ for all but finitely many α
4. $\sum_{\alpha \in A} \psi_\alpha(x) = 1 \quad \forall x \in M$

lemma 2:

$$\boxed{\text{any open cover of a smooth manifold admits a partition of unity subordinate to it}}$$

Proof: "Manifolds are paracompact" need Hausdorff and second countable for this see text book

i.e. given $\{U_\alpha\}_{\alpha \in A}$ an open cover, can find an open cover $\{V_\beta\}_{\beta \in B}$ by sets homeomorphic to open balls in coordinate charts s.t. $\forall \beta \in B, \exists \alpha \in A$ s.t. $V_\beta \subset U_\alpha$ and $\{V_\beta\}$ are "locally finite" (that is satisfy ③ above)

now for each $\beta \in B$ take a bump function

$$f_\beta : M \to \mathbb{R}$$

s.t. $f_\beta > 0$ on V_β and $= 0$ on $M - V_\beta$

choose a function $i : B \to A$ s.t. $V_\beta \subset U_{i(\beta)}$

and set
$$\tilde{\varphi}_\alpha(x) = \sum_{\substack{\beta\in B \\ i(\beta)=\alpha}} f_\beta(x)$$

$$\psi(x) = \sum_{\alpha\in A} \tilde{\varphi}_\alpha(x)$$

and
$$\varphi_\alpha(x) = \tilde{\varphi}_\alpha(x) / \psi(x)$$

Cor 3:

Suppose M a smooth manifold
A a closed subset and
$f: A \to \mathbb{R}^k$ a smooth function

for any open set U containing A, $\exists$ a smooth function $\tilde{f}: M \to \mathbb{R}^k$
such that $\tilde{f} = f$ on A and support $\tilde{f} \subset U$

Proof: by def$^{\underline{n}}$ of "smooth on A" we know
for each $a \in A$, $\exists$ an open set W_a and a smooth function $f_a: W_a \to \mathbb{R}^k$ s.t. $f_a = f$ on $W_a \cap A$
and we can assume $W_a \subset U$ (by intersection)

$\{W_a\}_{a\in A} \cup \{M \setminus A\}$ is an open cover of M

so there is a partition of unity $\{\psi_a\}_{a\in A} \cup \{\psi_0\}$

subordinate to it

set $\tilde{f}_a : M \to \mathbb{R}^k : x \longmapsto \begin{cases} \psi_a(x) f_a(x) & x \in W_a \\ 0 & x \notin W_a \end{cases}$

these are smooth on all of M

and since $W_a \subset U$, the functions are 0 outside U

set $\tilde{f}(x) = \sum_{a\in A} \tilde{f}_a(x)$ ← note: finite sum since ψ_a are locally finite

clearly support $\tilde{f} \subset U$

and

$x \in A$, $\tilde{f}(x) = \sum \psi_a(x) f_a(x)$

$= f_a(x) \sum \psi_\alpha(x) = f_a(x)$

Proof of Th^m 1:

by Cor 3 $\exists$ a smooth function $\tilde{f} : M \to \mathbb{R}^k$

that agrees with f on A

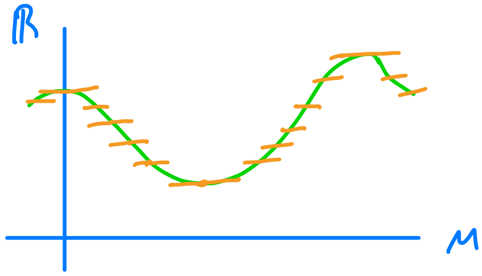
let $U_0 = \{x \in M : \|f(x) - \tilde{f}(x)\| < \delta(x)\}$

this is an open set (check this!) containing A

we now use the continuity of f to construct a "nice cover" of M

for each $x \in M - A$, $\exists$ a nbhd U_x of x in $M - A$ st

$$\left.\begin{array}{l} \delta(y) > \frac{1}{2}\delta(x) \text{ and} \\ \|f(y) - f(x)\| < \frac{1}{2}\delta(x) \quad \forall y \in U_x \end{array}\right\} \text{continuity!}$$



$\{U_x\}$ is a cover of $M - A$

since M is second countable $\exists$ a countable subcover $\{U_{x_i}\}_{i=1}^{\infty}$

let $\{\psi_0, \psi_i\}$ be a partition of unity subord. to $\{U_0, U_{x_i}\}_{i=1}^{\infty}$

set
$$\hat{f}(y) = \psi_0(y)\tilde{f}(y) + \sum_{i=1}^{\infty} \psi_i(y) f(x_i)$$

$\hat{f}$ is smooth and equal to f on A

now
$$\begin{aligned}
\|\hat{f}(y) - f(y)\| &= \Big\| \psi_0(y)\tilde{f}(y) + \sum_{i=1}^{\infty}\psi_i(y) f(x_i) - \Big(\psi_0(y) + \sum_{i=1}^{\infty}\psi_i(y)\Big) f(y) \Big\| \\
&\le \psi_0(y)\|\tilde{f}(y) - f(y)\| + \sum_{i=1}^{\infty}\psi_i(y)\|f(x_i) - f(y)\| \\
&< \psi_0(y)\,\delta(y) + \sum_{i=1}^{\infty}\psi_i(y)\underbrace{\tfrac{1}{2}\delta(x_i)}_{>\delta(y)} \\
&< \Big(\psi_0(y) + \sum_{i=1}^{\infty}\psi_i(y)\Big)\delta(y) = \delta(y)
\end{aligned}$$

We would like a similar theorem for $f: M \to N$ but no notion of distance in N so need something new!

B. Normal bundles and tubular neighborhoods

to continue our discussion of approximations we need <u>normal bundles</u>

given a vector bundle
$$\begin{tikzcd} E \arrow[d, "p"] \\ M \end{tikzcd}$$
and a subbundle
$$\begin{tikzcd} G \arrow[r, phantom, "\subset"] \arrow[dr, "p"'] & E \arrow[d] \\ & M \end{tikzcd}$$

with fiber dimension of E equal to n and of G equal to k

then set $E/G = \bigcup_{x \in M} E_x / G_x$

here E_x is the fiber of E above x
and similarly for G_x

let $q : E/G \to M$ be the obvious projection

to see E/G is a vector bundle we need local trivializations

let
$$\begin{tikzcd} U \times \mathbb{R}^n \arrow[r, "\phi"] \arrow[dr, "p_1"'] & p^{-1}(U) \arrow[d, "p"] \\ & U \end{tikzcd}$$
be a local trivialization of E

note for each $x \in U$, $\phi^{-1}(G_x)$ is a k-dimensional subspace of $\mathbb{R}^n$ and there is a matrix A_x such that $A_x(\mathbb{R}^k \times \{0\}) = \phi^{-1}(G_x)$

exercise: $A : U \to GL(n, \mathbb{R}) : x \mapsto A_x$ is smooth

so
$$\begin{tikzcd} U \times \mathbb{R}^k \arrow[r] \arrow[d, phantom, "\cap" description] & (p|_G)^{-1}(U) \arrow[d, phantom, "\cap" description] \\ U \times \mathbb{R}^n \arrow[r, "\tilde{\phi}"] \arrow[dr, "p_1"'] & p^{-1}(U) \arrow[d, "p"] \\ & U \end{tikzcd}$$

$\tilde{\phi}(x, v) = \phi(x, A_x(v))$

and $\tilde{\Phi}$ thus induces a map

$$\begin{tikzcd}
U \times \mathbb{R}^{n-k} \arrow[d, phantom, "\cong" description] & & \\
U \times \mathbb{R}^n/\mathbb{R}^k \arrow[rr, "\hat{\phi}"] \arrow[dr, "p_1"'] & & q^{-1}(U) \arrow[dl, "q"] \\
& U &
\end{tikzcd}$$

so E/G is an $(n-k)$-vector bundle

exercise: if G' is a subbundle of E s.t. $G_x \oplus G_x' = E_x$ $\forall x \in M$

then $\exists$ an isomorphism

$$\begin{tikzcd}
\psi : G' \arrow[rr] \arrow[dr] & \circ & E/G \arrow[dl] \\
& M &
\end{tikzcd}$$

now if S is a submanifold of M then TS is a subbundle of $TM|_S$

we define the normal bundle of S in M to be

$$\nu_M(S) = TM|_S / TS$$

note: if M is a submanifold of $\mathbb{R}^n$ then set

$$TM^\perp = \{v \in T_x\mathbb{R}^n \mid x \in M \text{ and } v \perp T_xM\}$$

Euclidean inner product on $\mathbb{R}^k$

clearly at each point $x \in M$

$$T_x\mathbb{R}^n = T_xM \oplus (TM^\perp)_x$$

so by the above $\nu_{\mathbb{R}^k}(M) \cong TM^{\perp}$

this is why we call it the normal bundle

(you can do this for any S in M but need M to have an inner product on each T_xM i.e. a Riemannian metric)

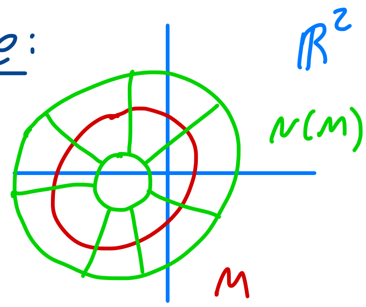
Th^m 4:

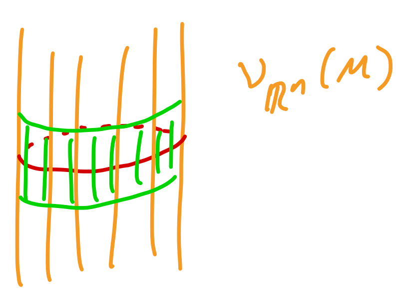
> if M^m is a compact submanifold of $\mathbb{R}^n$ then in any open neighborhood U of M, $\exists$ an open set $N(M)$ containing M s.t. $N(M)$ is an open disk bundle over M:
>
> $$\begin{array}{ccc} \mathring{B}^{n-m} & \to & N(M) \\ & & \downarrow q \\ & & M \end{array}$$
>
> and $\exists$ a subdisk bundle $N' \subset \nu_{\mathbb{R}^n}(M)$ s.t.
>
> $$N' \cong N(M)$$
>
> In particular, $q: N(M) \to M$ is the identity on M and q is a submersion

Remarks: 1) true if M is non-compact too

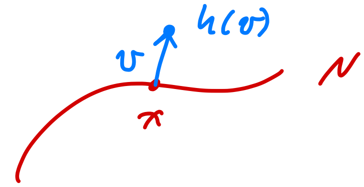
2) true for any submanifold S of M

example:





Proof: recall $\nu(M) \cong TM^\perp$



let $h: TM^\perp \to \mathbb{R}^n$

$$v \in (T_x M)^\perp \mapsto \underset{\substack{\shortparallel \\ x}}{p(v)} + v$$

recall $p(x) \in M \subset \mathbb{R}^n$

$v \in T_x \mathbb{R}^n = \mathbb{R}^n$

let $Z = \{\text{zero vectors in } TM^\perp\}$

<u>note</u>: any $v \in Z$ is a regular point of h

indeed: $T_{p(v)}\mathbb{R}^n = \mathbb{R}^n = T_{p(v)}M \oplus (TM)^\perp_{p(v)}$

and
$$\begin{tikzcd} TM^\perp \arrow[d, "p"', shift right] \\ M \arrow[u, "\sigma"', shift right] \end{tikzcd}$$
where $\sigma(x) = 0 \in (TM)^\perp_x$ and $p \circ \sigma = id_M$

so $p|_Z$ and σ are inverses of eachother

so $p|_Z$ is a diffeomorphism $Z \to M$

$\therefore$ along Z dp_v maps onto $T_{p(v)}M$

now $dh|_{TM^\perp_{p(v)}}$ maps onto the linear space $TM^\perp_{p(v)} \subset T_{p(v)}\mathbb{R}^n$

so image $dh_{p(v)} = T_{p(v)}\mathbb{R}^n$ ✓

thus since dim $TM^\perp$ and $\mathbb{R}^n$ are same h is a local diffeomorphism along Z and $\therefore$ in a neighborhood of Z

(any nbhd of Z contains a subdisk bundle of $TM^\perp$)

moreover $h|_Z : Z \to M$ is a diffeomorphism

this implies h is a diffeomorphism on a suitably small nbhd N' of Z

Indeed: let $N_{1/n}$ be points in $TM^\perp \leq \frac{1}{n}$ from Z

for some n $h|_{M_{1/n}}$ an embedding

if not then $\forall n$ $\exists$ $x_n, y_n \in M_{1/n}$ s.t. $x_n \neq y_n$

and $h(x_n) = h(y_n)$

let $x = \lim x_n$ and $y = \lim y_n$ (exist by cptness)

$\therefore$ $h(x) = h(y) \Rightarrow x = y$ (since $x, y \in Z$)

h a local diffeomorphism so $\exists$ nbhd of $x = y$ where h is one-to-one, this contradicts $\exists$ x_n, y_n

C. Approximations of continuous functions II

two continuous functions $f_0, f_1 : X \to Y$ are called homotopic if $\exists$ a continuous map

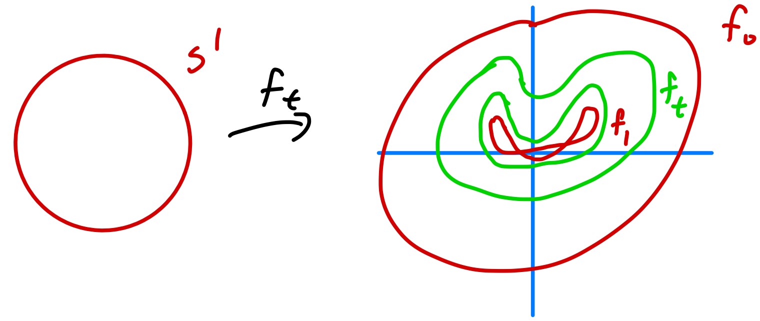
$$F : X \times [0,1] \to Y$$

such that

$$F(x,0) = f_0(x) \quad \forall x \in X$$
$$F(x,1) = f_1(x)$$

we call F a homotopy

if we set $f_t : X \to Y : x \mapsto F(x,t)$ then we can think of F as giving a "continuous deformation" from f_0 to f_1 through the f_t's e.g.



if X and Y are smooth manifolds and F is smooth then we call this a smooth homotopy

exercise: (smooth) homotopy is an equivalence relation

Th$^{\underline{m}}$5:

> Suppose N and M are smooth manifolds (M without boundary)
>
> and $f: N \to M$ is continuous
>
> Then f is homotopic to a smooth map.
>
> If f is already smooth on a closed subset A then the homotopy can be chosen to be fixed on A.

Proof: we can assume $M \subset \mathbb{R}^k$ by the Whitney embedding theorem

so $f: N \to M \subset \mathbb{R}^k$ is continuous

$\exists$ a tubular nbhd $N(M)$ of M in $\mathbb{R}^k$ and $q: N(M) \to M$ its projection

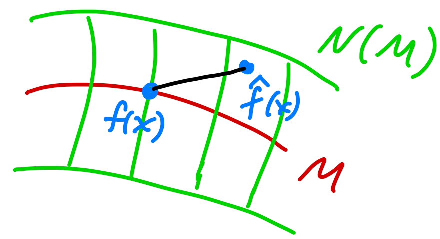
by Th$^{\underline{m}}$1 we know there is a smooth map $\hat{f}: N \to \mathbb{R}^k$ that is arbitrarily close to f

in particular $\operatorname{im} \hat{f} \subset N(M)$

exercise: we can choose $\hat{f}$ s.t.

$(1-t) f(x) + t \hat{f}(x)$ is in $N(M)$

$\forall t \in [0,1]$ and $x \in N$



so $F : N \times [0,1] \to M$

$(x,t) \mapsto g((1-t) f(x) + t \hat{f}(x))$

is a homotopy from f to $g \circ \hat{f}$ and $g \circ \hat{f}$ is smooth!

moreover $\hat{f} = f$ on a closed set A where f is smooth so homotopy fixed on A

Cor 6:

> let $f_0, f_1 : N \to M$ be two smooth maps
>
> Then f_0 and f_1 are homotopic
>
> $\Leftrightarrow$
>
> they are smoothly homotopic

Proof: $(\Leftarrow)$ ✓

$(\Rightarrow)$ let $F : N \times [0,1] \to M$ be a homotopy

(smooth on $N \times \{0,1\}$)

Th$^{\underline{m}}$ 5 gives $\tilde{F} : N \times [0,1] \to M$ smooth

and $\tilde{F} = F$ on $N \times \{0,1\}$

D. Homotopy and Stability

frequently in math (and more so in physics...) we are interested in properties that are "stable" i.e. don't change under "small perterbations"

more precisely we say a property of a map

$$f_0 : M \to N$$

is stable if for any smooth homotopy f_t of f_0 there is some $\varepsilon > 0$ s.t. f_t also has this property for all $t < \varepsilon$.

Th$^{\underline{m}}$ 7 (Stability Th$^{\underline{m}}$):

The following properties of smooth maps from a compact manifold M to a manifold N are stable:

(1) local diffeomorphism

(2) immersions

(3) submersions

(4) maps transverse to a fixed closed submanifold $S \subset N$

(5) embeddings

(6) diffeomorphisms

recall (from homework) that $f: M \to N$ is

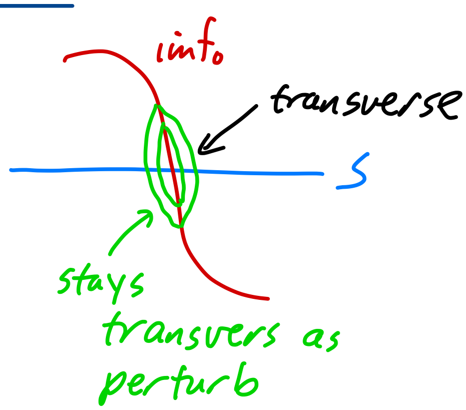
<u>transverse</u> to $S \subset N$ if $\forall p \in f^{-1}(S)$ we have

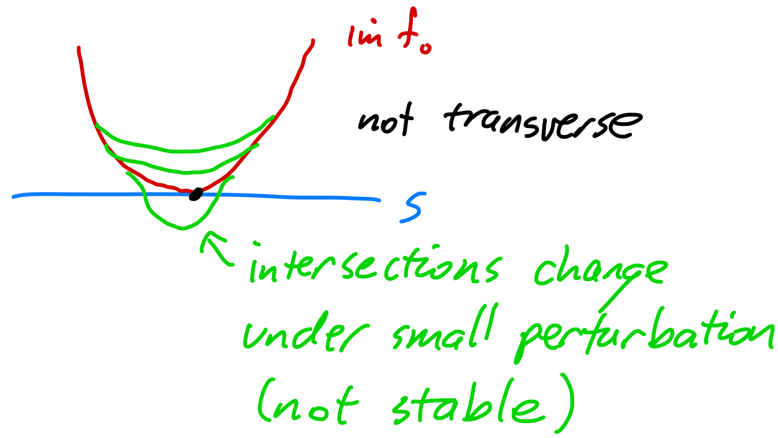
$T_{f(p)} N$ is spanned by $\mathrm{im}(df_p)$ and $T_{f(p)} S$

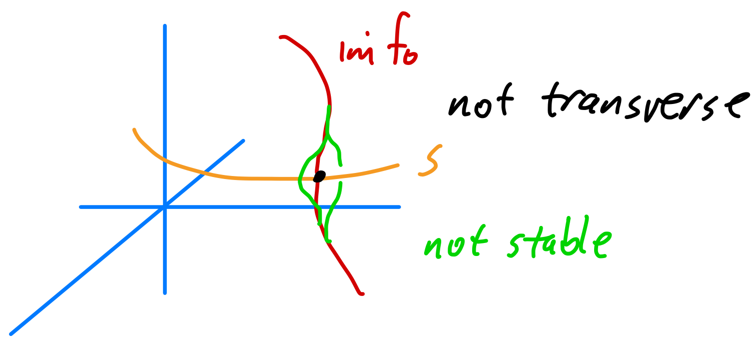
and if f is transverse to S then $f^{-1}(S)$ is

a manifold of codim = codim S in N

example:







Proof: (2) let $F: M \times [0,1] \to N$ be a smooth homotopy

and $f_0(x) = F(x,0)$ be an immersion

we must find an $\varepsilon > 0$ s.t. $\forall (x,t) \in M \times [0,\varepsilon)$

$(df_t)_x$ is injective

recall lemma II.9 says if a linear map has

max rank then near by linear maps do too so for each $x \in M$, $\exists$ a nbhd U_x of $(x,0)$ in $M \times [0,1]$

s.t. $(df_t)_y$ is max rank $\forall$ $(y,t) \in U_x$

now $U = \bigcup_{x \in M} U_x$ is an open nbhd of $M \times \{0\}$

so th$^{\underline{m}}$ follows!

items (1) and (3) have almost the same proof

as does (4) (once you do your homework)

now for (5) ((6) is similar)

F a homotopy as above with f_0 an embedding

we know $\exists\, \varepsilon > 0$ s.t. f_t immersion for $t \in [0, \varepsilon)$

since M is compact we are done if we show

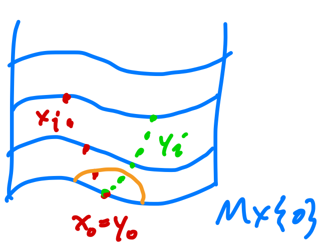
$\exists\, \varepsilon' > 0$ s.t. f_t is injective $\forall\, t \in [0, \varepsilon')$

to this end, assume not, so $\exists$ a sequence

$$t_i \to 0 \quad \text{(decreasing)}$$

and $x_i \neq y_i \in M$ s.t. $f_{t_i}(x_i) = f_{t_i}(y_i)$

by compactness we have $x_i \to x$, $y_i \to y$ (after passing to a subsequence)



now $F(x_0, 0) = \lim_{i \to \infty} F(x_i, t_i) = \lim_{i \to \infty} F(y_i, t_i) = F(y_0, 0)$

since f_0 injective $x_0 = y_0$

<u>note:</u> if $G : M \times I \to N \times I$

$(x,t) \longmapsto (F(x,t), t)$

then $dG_{(x_0, 0)} = \left(\begin{array}{c|c} (df_0)_{x_0} & \begin{array}{c} a_1 \\ \vdots \\ a_n \end{array} \\ \hline 0 \cdots 0 & 1 \end{array} \right)$

f_0 an immersion says df_0 has rank $= \dim M = m$

so $dG_{(x,0)}$ has rank $m+1 = \dim(M \times [0,1])$

so G is an immersion at $(x_0, 0)$

$\therefore$ Thm II.6 $\Rightarrow \exists$ nbhd about $(x_0, 0)$

s.t. G is one-to-one $\cancel{\emptyset}$ $\exists (x_i, t_i)$ and (y_i, t_i)