VII Transversality

A. Genericity of transversality

we know what it means for a map

$$f: M \to N$$

to be transverse to a submanifold S of N
and that if f is transverse to S (write $f \pitchfork S$)
then $f^{-1}(S)$ is a submanifold of M of
codimension = codim S in N.

exercise: if $\partial M \neq \emptyset$ and $f|_{\partial M}: \partial M \to N$ is $\pitchfork$ to S
and $f: M \to N$ is $\pitchfork$ to S
then $f^{-1}(S)$ is a submanifold of M
with $\partial f^{-1}(S) \subset \partial M$

How easy is it to find $\pitchfork$ maps?

Th$^{\underline{m}}$ 1:

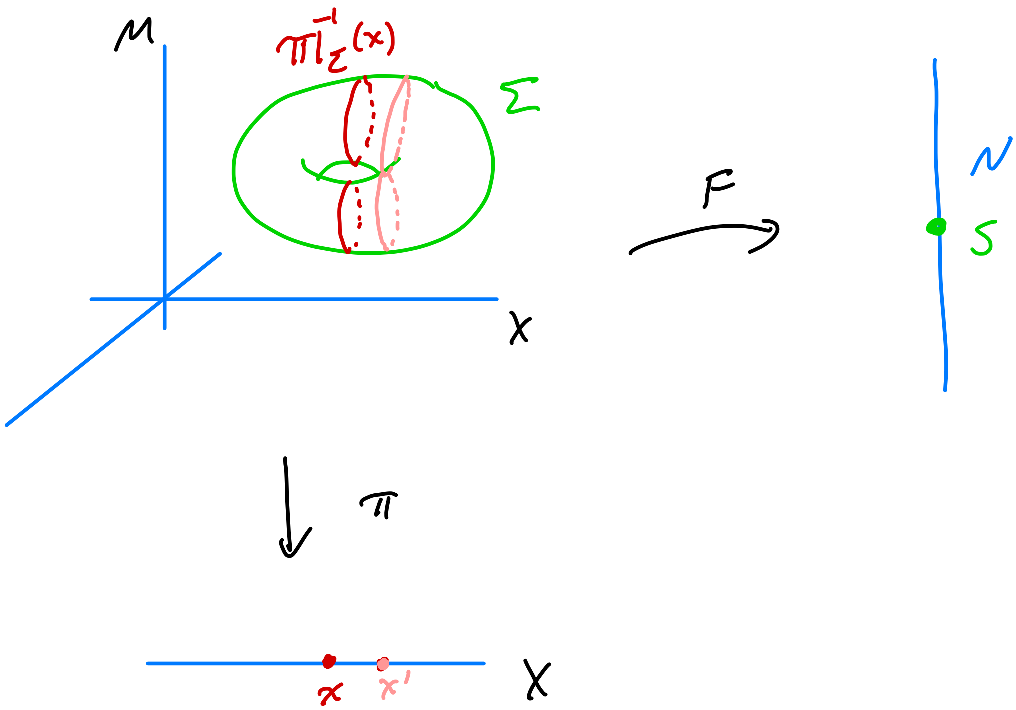
let M, N, X be smooth manifolds
(M can have boundary)
let $S \subset N$ be a submanifold
if $F: M \times X \to N$ is transverse to S
(and $F|_{\partial M \times X}$ $\pitchfork$ to S)
then except for a set of measure zero
$f_x: M \to N: p \mapsto F(p, x)$ is $\pitchfork$ to S

Proof: we assume $\partial M = \emptyset$ and leave $\partial M \neq \emptyset$ as an exercise

note: $\Sigma = F^{-1}(S)$ is a submanifold of $M \times X$

let $\pi : M \times X \to X : (p,x) \mapsto x$

Claim: x a regular value of $\pi|_S \Rightarrow f_x \pitchfork S$



given the claim we are done by Sard's Th$^{\underline{m}}$!

Pf of claim:

suppose x is a regular value of $\pi|_\Sigma$

need to see $f_x \pitchfork S$

suppose $(p,x) \in \Sigma$ so $F(p,x) = s \in S$

$F \pitchfork S \Rightarrow \forall v \in T_s N,\ \exists w \in T_{(p,x)}(M \times X)$

s.t. $v - dF_{(p,x)}(w) \in T_s S$

we need to find $u \in T_p M$ s.t.

$$v - d(f_x)_p(u) \in T_{s = f_x(p)} S$$

now $T_{(p,x)}(M \times X) = T_p M \times T_x X$

$$\underset{w}{\overset{\psi}{}} \quad = \quad \underset{(a,b)}{\overset{\psi}{}}$$

if $b = 0$, then consider

$$M \xrightarrow{i} M \times X$$
$$p \longmapsto (p, x)$$

we know $di_p(a) = (a, 0)$

$$\begin{aligned} \text{so } (df_x)_p(a) &= d(F \circ i)_p(a) \\ &= dF_{(p,x)}(di_p(a)) \\ &= dF_{(p,x)}(a, 0) = dF_{(p,x)}(a,b) \\ &= dF_{(p,x)}(w) \end{aligned}$$

and so $v - (df_x)_p(a) \in T_s S$

and we are done

now in general, we know that

$$d\pi_{(p,x)}(T_{(p,x)} \Sigma) = T_x X$$

since x is a regular value

and $d\pi_{(p,x)} : T_{(p,x)}(M \times X) \to T_x X$

$$T_{(p,x)}(M \times X) = T_p M \times T_x X$$

is just projection

so $\exists\ (c,b) \in T_{(p,x)}\Sigma$ s.t. $d\pi_{(p,x)}(c,b) = b$

<u>note</u>: since $(c,b) \in T_{(p,x)}\Sigma$ we know

$$dF_{(p,x)}(c,b) \in T_s S$$

$$\text{so } v - dF_{(p,x)}(a-c, 0) = v - dF_{(p,x)}((a,b) - (c,b))$$
$$= \underbrace{\left(v - dF_{(p,x)}(a,b)\right)}_{\in T_s S} + \underbrace{dF_{(p,x)}(c,b)}_{\in T_s S}$$

$\therefore\ v - dF_{(p,x)}(a-c,0) \in T_s S$

and we are done by above

<u>Thm 2:</u>

> let M, N be smooth manifolds (M possibly with ∂) and S a submanifold of N
> for any smooth map $f : M \to N$ there is a homotopy to a smooth map $\hat{f} : M \to N$ s.t. $\hat{f} \pitchfork S$ and $\hat{f}|_{\partial M} \pitchfork S$

<u>Proof</u>: we can think of $N \subset \mathbb{R}^k$ for some k

let X be the unit ball in $\mathbb{R}^k$

now set
$$F : M \times X \to N$$
$$(y,x) \mapsto q(\underbrace{f(y)+x}_{\text{call } \tilde{F}(y,x)})$$

where $q : N(N) \to N$

tubular nbhd of N in $\mathbb{R}^k$

note: $T_{(y,x)}(M\times X) = T_y M \times \underset{\substack{\cong \\ \mathbb{R}^k}}{T_x X} \xrightarrow{d\tilde{F}} \underset{\substack{\cong \\ \mathbb{R}^k}}{T_{f(y)+x}\mathbb{R}^k} \xrightarrow{dq} T_{F(y,x)} N$

clearly $d\tilde{F}$ is a submersion (just consider vectors in $T_x X$)

and by Th$^{\underline{m}}$ VI.4 dq is a submersion

$\therefore$ dF is a submersion!

so F is transverse to <u>any</u> submanifold of N

$\therefore$ by Th$^{\underline{m}}$ 1 $\exists$ a dense set of $x \in X$ s.t.

$$f_x(\cdot) = F(\cdot, x) \text{ is } \pitchfork \text{ to } S$$

now let $G: M\times[0,1] \to N$

$(y,t) \longmapsto F(y,tx)$ for x as

then this is a homotopy from $G(y,0) = F(y,0) = f(y)$

to $G(y,x) = F(y,x) = f_x(y) \pitchfork S$ ▦

Th$^{\underline{m}}$ 3:

$$\boxed{\begin{array}{l}\text{let } M, N \text{ be smooth manifolds } (M \text{ possibly with } \partial) \\ \quad S \text{ a smooth submanifold of } N \\ \quad f: M\to N \text{ a smooth map} \\ \text{if } \exists \text{ a closed set } C \text{ of } M \text{ on which } f \text{ is } \pitchfork \text{ to } S \\ \quad (\text{and } f|_{\partial M} \pitchfork S \text{ on } C)\text{, then} \\ \quad f \text{ is smoothly homotopic to } g: M\to N \text{ s.t.} \\ \quad g \pitchfork S \ (\text{and } g|_{\partial M} \pitchfork S) \text{ and } g=f \text{ on an} \\ \quad \text{open subset of } C.\end{array}}$$

Idea of Proof:

Step 1: there is an open neighborhood U of C s.t. f (and $f|_C$) is $\pitchfork$ to S on U

Hint: $\pitchfork$ is an "open" condition see proof of stability th$^{\underline{m}}$

Step 2: there is a function $\gamma : M \to \mathbb{R}$ s.t.

$\exists$ open sets $C \subset U' \subset U'' \subset U$ and

$\gamma = 0$ on $\overline{U}'$ and

$\gamma = 1$ outside U''

Hint: Cor VI.3

Step 3: If $F : M \times X \to Y$ is the function from last proof, then set

$$G : M \times X \longrightarrow N$$
$$(y, x) \longmapsto F(y, \gamma^2(y)\, x)$$

note: 1) where $\gamma = 1$ (outside U''), $G \pitchfork S$ since F is

2) where $\gamma = 0$ (inside U'), $G(y,x) = F(y,0) = f(y) \pitchfork S$ since f is on U'

3) *exercise* G is transverse to S everywhere (i.e. on $U'' - U$)

now proof follows as proof of Th$^{\underline{m}}$ 2

B. 1-Manifolds and applications

Th$^{\underline{m}}$:

$\boxed{\text{every compact connected 1-manifold is diffeomorphic to } [0,1] \text{ or } S^1}$

Remark: we will <u>not</u> prove this

It is not too hard, see Guillemin & Pollack

Th$^{\underline{m}}$ 4:

$\boxed{\begin{array}{l}\text{let } M \text{ be a smooth compact manifold with boundary} \\ \text{there is no continuous retraction of } M \text{ to } \partial M \\ \quad \text{i.e. no map } f: M \to \partial M \text{ s.t. } f = id \text{ on } \partial M\end{array}}$

Proof: Suppose there is a retraction $f: M \to \partial M$

assume f is smooth (for now)

$\exists\, x \in \partial M$ that is a regular value of f and $f|_{\partial M}$

so $f^{-1}(x) = S$ is a submanifold of M and $\partial S \subset \partial M$

$\{x\}$ closed in ∂M so S closed in M

M compact so S is compact

$\therefore$ S = union of intervals and S^1's

<u>note</u>: $f^{-1}(x) \cap \partial M = \{x\}$ since $f = id$ on ∂M

so S has one boundary point!

but compact 1-manifolds must have an even number of boundary points $\otimes$

so f does not exist!

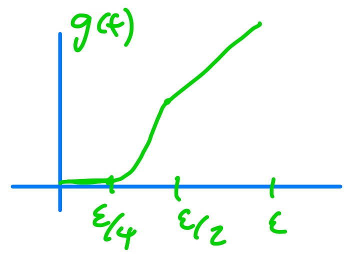
now suppose f is only continuous

by Th$^{\underline{m}}$ V.3 $\exists$ a nbhd of ∂M in M diffeomorphic to $\partial M \times [0, \varepsilon)$

now set $\tilde{f} : \partial M \times [0, \varepsilon) \to \partial M \times [0, \varepsilon)$

$$(x, t) \longmapsto f(x, g(t))$$

where



and extend $\tilde{f}$ to rest of M by using f

<u>note</u>: $\tilde{f}$ is smooth on $\partial M \times [0, \varepsilon/4)$ (just projection to ∂M)

so by Th$^{\underline{m}}$ VI.5 we can homotop $\tilde{f}$ to $\hat{f}$ rel ∂M so that $\hat{f}$ is smooth

thus $\hat{f}$ is a smooth retraction $\otimes$ ▦

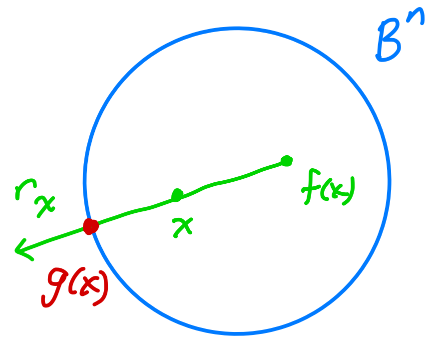
Cor 5 (Brower fixed-point th$^{\underline{m}}$):

any continuous map $f : B^n \to B^n$ (closed unit ball) has at least one fixed pt.

i.e. $x \in B^n$ s.t. $f(x) = x$

Proof: assume not

then $f(x)$ and x define a ray r_x starting at $f(x)$ and going through x



let $g(x)$ = unique point in $r_x \cap \partial B^n$

as shown in picture

Claim: $g(x)$ is continuous

if true then this $\otimes$ Th$^{\underline{m}}$ 4 since $g(x)=x$ for $x \in \partial B^n$

to see the claim is true note r_x is parameterized

by $(1-t)f(x) + tx \qquad t \geq 0$

the ray intersects ∂B^n at

$$(1-t)^2 \|f(x)\|^2 + 2t(1-t)\,(x \cdot f(x)) + t^2\|x\|^2 = 1$$

so

$$\left(\|x\|^2 - 2\,x\cdot f(x) + \|f(x)\|^2\right)t^2 + 2\left(x\cdot f(x) - \|f(x)\|^2\right)t + \|f(x)\|^2 - 1 = 0$$

$$\underbrace{\overbrace{\|x - f(x)\|^2}^{>0}\, t^2 + 2\left(x\cdot f(x) - \|f(x)\|^2\right)t + \left(\|f(x)\|^2 - 1\right)}_{p(t) \,=\, \text{polynomial in } t} = 0$$

so p has two roots

and we know $\quad p(0) = \|f(x)\|^2 - 1 \leq 0$

$\qquad\qquad\qquad\;\; p(1) = \|x\|^2 \leq 0$

but $p(t) > 0$ as $t \to \pm\infty$

so one root ≤ 0 and one ≥ 1

quadratic formula says the positive root t_x

depends continuously on x

$\therefore\ g(x) = (1 - t_x) f(x) + t_x\, x$ is continuous

C. Mod 2 intersection theory

Suppose M, N are manifolds

S a submanifold of N

$\dim M + \dim S = \dim N$

M, S have no boundary

S closed in N

M compact

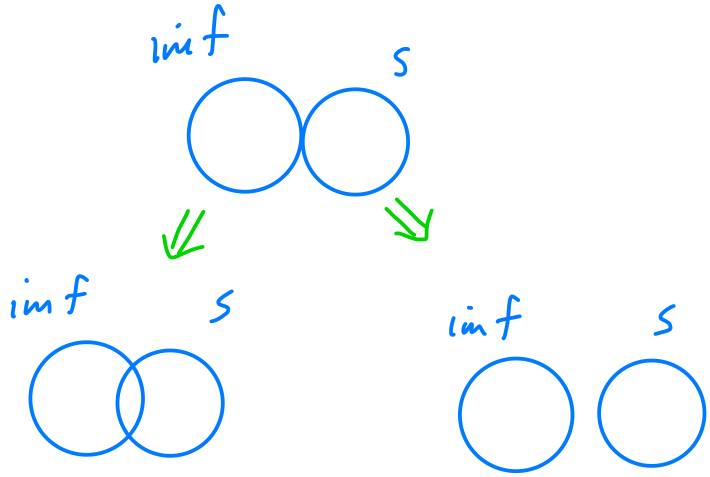
Then given $f: M \to N$, we can use Thm 2 to

homotop f to f_1 so that $f_1 \pitchfork S$

note: $f_1^{-1}(S) = f_1(M) \cap S = \{\text{finite set of points}\}$

0-dim'l submanifold
compact since M is and
S is closed

Example:



The intersection mod 2 of f and S is

$$I_2(f,S) = \# f_1^{-1}(S) \bmod 2$$

$\boxed{\text{Th}^{\underline{m}}\,6\text{: } I_2(f,S) \text{ is well-defined, i.e. if } f_1 \sim f_2 \text{ and } f_1, f_2 \pitchfork S \text{ then } I_2(f_1,S) = I_2(f_2,S)}$

Proof: let $F : M \times [0,1] \to N$ be the homotopy from f_1 to f_2

by Th$^{\underline{m}}$3 we may find a homotopy $\hat{F}$ s.t. $\hat{F} \pitchfork S$

now $\hat{F}^{-1}(S)$ = compact 1-manifold

$$= \underbrace{\coprod S^1}_{\subset\, M \times (0,1)} \cup \underbrace{\gamma_1 \cup \ldots \cup \gamma_n}_{\text{arcs with end points on } M \times \{0,1\}}$$

note: $f_1^{-1}(S) \cup f_2^{-1}(S) = \partial(\hat{F}^{-1}(S))$ ← even # of pts so $\# f_1^{-1}(S) = \# f_2^{-1}(S) \mod 2$

Remarks:

1) given any $f_1, f_2 : M \to N$ homotopic then
$$I_2(f_1, S) = I_2(f_2, S)$$
(since homotopy is an equivalence rel^{n})

2) given S_1, S_2 submanifolds of N s.t.
$$\dim S_1 + \dim S_2 = \dim N$$
(we say S_1, S_2 have complementary dim)

suppose S_1, S_2 closed and compact

then $I_2(S_1, S_2)$ is defined to be $I(i, S_2)$
where $i : S_1 \to N$ is inclusion

Th^{m} 7:

> suppose M, N, S and $f: M \to N$ as above
> if $\exists$ a compact manifold W s.t. $\partial W = M$ and
> f can be extended to $F: W \to N$
> then $I_2(f, S) = 0$

Proof: homotop F to $\hat{F}$ that is transverse to S

<u>note</u>: $I_2(f, S) = I(\hat{F}|_{\partial W}, S)$ since f and $\hat{F}|_{\partial W}$ are homotopic

now $\hat{F}^{-1}(S)$ is a compact 1-manifold

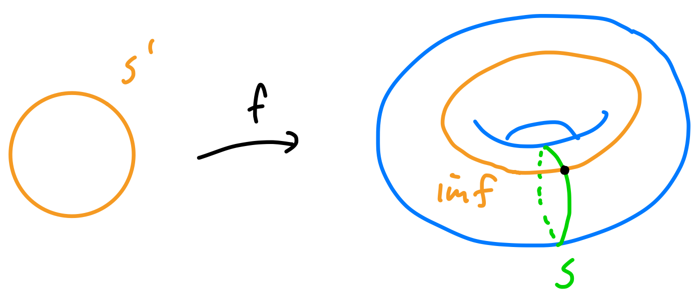
so $\partial \hat{F}^{-1}(S)$ = even # of points
$$\begin{array}{l} \shortparallel \\ (\hat{F}|_{\partial W})^{-1}(S) \end{array}$$

so $I_2(\hat{F}|_{\partial W}, S) = 0$ ▦

example: consider $T^2 = S^1 \times S^1$

let $S = S^1 \times \{pt\}$

and $f: S^1 \longrightarrow T^2$
$$\theta \longmapsto (pt, \theta)$$



note: $f \pitchfork S$ and $f^{-1}(S) = \{pt\}$

so $I_2(f, S) = 1$

thus f does not extend over D^2

(or any compact surface)

Intuitively clear but not easy to show!

exercise: 1) Show S^2 is not diffeomorphic to T^2 by showing any $f: S^1 \to S^2$ can be extended to D^2

2) Show $\mathbb{R}P^2$ is not diffeomorphic to S^2

<u>Hint</u>: Consider "self-intersection" of

$$\begin{tikzcd} S^1 \arrow[r, hook, "\text{equator}"'] \arrow[rr, bend right] & S^2 \arrow[r, "\text{quot.}"'] & \mathbb{R}P^2 \end{tikzcd}$$

Suppose now M, N are two manifolds of the same dimension and without boundary

assume M is compact and N is connected

given any map $f: M \to N$ we call $I_2(f, \{p\})$ for any $p \in N$ the <u>degree mod 2</u> of f and denote it $\deg_2(f)$

<u>Th^m 8</u>:

$\boxed{\text{For any two points } p_1, p_2 \in N, \ I_2(f, \{p_1\}) = I_2(f, \{p_2\}) \ (\text{ie } \deg_2(f) \text{ is well-defined})}$

for the proof we need a few observations

first since $\deg_2(f)$ is defined in terms of $I_2(f, \{p\})$ and $I_2(f, \{p\})$ is unchanged if we homotop f we see

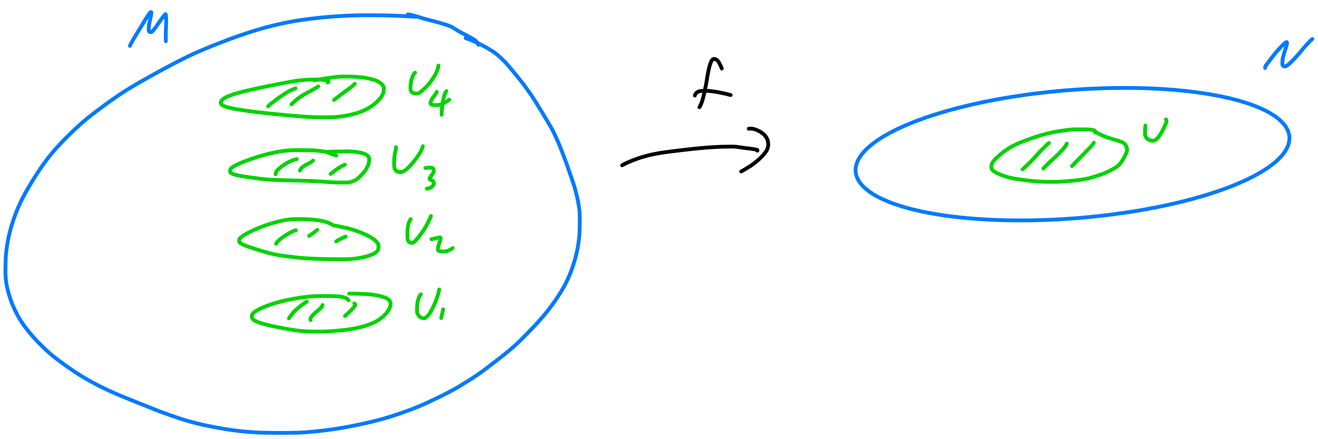
<u>Cor 9</u>:

$\boxed{\text{homotopic maps have the same degree}}$

we also need

lemma 10:

> With notation as above, for any regular value p of f $\exists$ an open nbhd U of p s.t.
> $$f^{-1}(U) = U_1 \cup \ldots \cup U_k$$
> where the U_k are disjoint and $f: U_i \to U$ is a diffeomorphism.



Proof of Thm 8:

given $p \in N$ we can homotop f to be transverse to p (so p is a regular value of f)

now we have $U_1, \ldots, U_k$ as in lemma 9

so for any $q \in U$, $f^{-1}(q)$ has k points

$\therefore$ the function

$$\begin{aligned} N &\to \mathbb{Z} \\ p &\mapsto I_2(f, \{p\}) \end{aligned}$$

is locally constant

since N is connected it is thus constant

(exercise)

Proof of Lemma 9:

let $\{p_1, \dots, p_k\} = f^{-1}(p)$ (we know $f^{-1}(p)$ is a compact 0-manifold so a finite number of points)

each p_i has a neighborhood W_i such that

$$f|_{W_i} : W_i \to f(W_i)$$

is a diffeomorphism (by inverse function th^{m})

since M is Hausdorff we can assume W_i are disjoint

<u>note:</u> $X = f(M - \bigcup_{i=1}^{k} W_i)$ is compact (since M is)

set $U = (f(W_1) \cap \dots \cap f(W_k)) - X$

need to remove X since some pts in $M - \bigcup W_i$ might map to $\bigcap f(W_i)$

this is an open set and $p \in U$

now set $U_i = W_i \cap f^{-1}(U)$

<u>Clearly:</u> $f|_{U_i} : U_i \to U$ is a diffeomorphism

and $\bigcup U_i \subseteq f^{-1}(U)$

now if $x \in f^{-1}(U)$ then $f(x) \in U$

so $f(x) \notin X = f(M - \bigcup_{i=1}^{k} W_i)$

$\therefore x \in \bigcup W_i \Rightarrow x \in W_i$ some i

and so $x \in W_i \cap f^{-1}(U) = U_i$

thus $\bigcup U_i = f^{-1}(U)$

Thm 11:

$\boxed{\text{If } M = \partial W,\ W \text{ compact, and } f: M \to N \text{ can be extented to a map } F: W \to N \text{ then } \deg_2(f) = 0}$

Proof: take a regular value p of f

so the $\deg_2(f) = \#(f^{-1}(p)) \bmod 2$

we can homotop F rel ∂W (i.e. don't change f)

so that $F \pitchfork \{p\}$

now $F^{-1}(p)$ is a compact 1-manifold with boundary

$f^{-1}(p)$ so $f^{-1}(p)$ is an even number of pts ▦

Cool application:

Thm 12:

$\boxed{\text{every complex polynomial of odd degree has a root}}$

Remarks:

1) Similar proof with non-mod 2 degree proves general result (any positive degree)

2) The proof is a simple case of a very powerful idea
<u>The Continuity Method</u>: if you can't solve an

equation, homotop it to one you can solve and then argue original one solvable too!

Proof: let $p(z)$ be an odd degree polynomial

if $p(z)$ has no zeros then

$$\frac{p(z)}{|p(z)|} : \mathbb{C} \to S^1 \text{ is well-defined}$$

now consider

$$\begin{aligned} p_t(z) &= t\,p(z) + (1-t)z^m \\ &= t(z^m + a_{m-1}t^{m-1} + \dots + a_0) + (1-t)z^m \end{aligned}$$

(can assume p is monic)

note: away from origin p_t has a zero

$$\Longleftrightarrow$$

$$\frac{p_t(z)}{z^m} \text{ does}$$

but $$\frac{p_t(z)}{z^m} = 1 + t\underbrace{\left(a_{m-1}\frac{1}{z} + \dots + a_0\frac{1}{z^m}\right)}_{\circledast}$$

as $z \to \infty$, $\circledast \to 0$

so if D_r is the disk of radius r about origin in $\mathbb{C}$

then for large r, $|\circledast| \ll 1$ on ∂D_r

so $p_t(z) \neq 0$ on ∂D_r

$$\therefore \frac{p_t(z)}{|p_t(z)|} : \underset{\overset{\shortparallel}{S^1}}{\partial D_r} \longrightarrow S^1 \text{ is well-defined}$$

and gives a homotopy from $\left.\frac{p(z)}{|p(z)|}\right|_{\partial D_r}$ to $\left.\frac{z^m}{|z^m|}\right|_{\partial D_r}$

you can easily check $\deg_2\left(\frac{z^m}{|z^m|}\right) = m \not\equiv 0 \mod 2$
if m odd

so $\deg_2\left(\left.\frac{p(z)}{|p(z)|}\right|_{\partial D_r}\right) \not= 0$

and $\left.\frac{p(z)}{|p(z)|}\right|_{\partial D_r}$ can't be extended over D_r

so from above $p(z)$ must have a root in D_r