VIII Cotangent Bundles and 1-forms

A. Linear Algebra:

let V be a vector space

the dual space of V is

$$V^* = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$$
$$= \{\mathbb{R}\text{-linear maps } L : V \to \mathbb{R}\}$$

if $e_1, \dots, e_n$ is a basis for V then set

$$e^i : V \to \mathbb{R} : e_j \mapsto \delta^i_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

exercise:

1) $e^1, \dots, e^n$ is a basis for V^*

(called the dual basis to $e_1, \dots, e_n$)

2) $\dim V^* = \dim V$

3) if $a \in V^*$ then we can write

$$a = \sum_{i=1}^{n} a_i e^i$$

and

$$v = \sum_{i=1}^{n} v^i e_i$$

show

$$a(v) = \sum_{i=1}^{n} a_i v^i \quad \in \mathbb{R}$$

4) if $a = \sum_{i=1}^{n} a_i e^i \in V^i$ then

$$a_i = a(e_i)$$

If $L : V \to W$ is a linear map, then you get

$$L^* : W^* \to V^*$$
$$a \longmapsto a \circ L$$

exercise:

1) $(T \circ S)^* = S^* \circ T^*$

2) $id_V^* = id_{V^*} : V^* \to V^*$

3) if $e_1, \ldots, e_n$ a basis for V

$f_1, \ldots, f_m$ a basis for W

then $L(e_i) = \sum_{i=1}^{m} L_i^j f_j$

i.e. the matrix (L_i^j) represents L in these bases

Show $T^*(e^i) = \sum_{j=1}^{n} L_j^i f^j$

i.e. $(T^*)_j^i = T_i^j$

transpose

expressed as matricies $T^* = T^t$

B. Cotangent bundle

let M be a manifold and $p \in M$

the cotangent space of M at p is

$$T_p^*M = (T_pM)^* = \mathrm{Hom}(T_pM, \mathbb{R})$$

elements are called covectors

note: if $f: M \to \mathbb{R}$, then

$$\begin{array}{ccccc} df_p : T_pM & \to & T_{f(p)}\mathbb{R} & \overset{\text{canonically}}{\cong} & \mathbb{R} \\ \Psi & & \Psi & & \Psi \\ v & \longmapsto & df_p(v) & = & v \cdot f \end{array}$$

so $df_p \in T_p^*M$

now given a coordinate chart $\phi: U \to V$ around p

$q \mapsto (x^1(q), \dots, x^n(q))$

then we get a basis $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$ for T_pM

let $dx^1, \dots, dx^n$ be the dual basis

i.e. $dx^i\left(\frac{\partial}{\partial x^j}\right) = \delta^i_j$

Recall: $\frac{\partial}{\partial x^i}$ really means "$\frac{\partial}{\partial x^i}$" $= d(\phi^{-1})_{\phi(p)} \frac{\partial}{\partial x^i}$

so we should say "dx^i" but again we wont usually do this if $f: M \to N$ is a smooth map and

$\hat{\phi}: \hat{U} \to \hat{V}$ is a coordinate chart around $f(p)$

$q \mapsto (y^1(q), \dots, y^m(q))$

then $\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^m}$ is a basis for $T_{f(p)}N$ and

$dy^1, \dots, dy^m$ " " $T^*_{f(p)}N$

we know that $df_p: T_pM \to T_{f(p)}N$ can be expressed in local coordinates as the matrix

$$\left(\frac{\partial f^j}{\partial x^i}\right)$$

where $f(x^1, \dots, x^n) = (f^1(x^1, \dots, x^n), \dots, f^m(x^1, \dots, x^n))$ (locally)

i.e. if $v \in T_pM$ is $v = \sum_{i=1}^n v^i \frac{\partial}{\partial x^i}$

then $df_p(v) = \sum_{i,j} v^i \frac{\partial f^j}{\partial x^i} \frac{\partial}{\partial y^j}$

<u>Recall:</u> the dual map

$$df_p^*: T^*_{f(p)}N \to T_p^*M$$

is defined by $df_p^*(a) = a \circ df_p$

we denote this map $f^*_{f(p)}$

recall further that $f^*_{f(p)}$ satisfies

1) $(f \circ g)^*_{f(g(p))} = g^*_{g(x)} \circ f^*_{f(g(p))}$

2) $(id_M)^*_p = id_{T_p^* M}$

3) in local coordinates $f^*_{f(p)}$ is given by

$$\left(\frac{\partial f^j}{\partial x^i}\right)^t$$

that is if $a = \sum_{i=1}^{m} a_i \, dy^i \in T_{f(p)} N$

then
$$f^*_{f(p)} a = \sum a_i \frac{\partial f^i}{\partial x^j} dx^j$$

<ins>Remark:</ins> consider

$$f : \mathbb{R}^n \to \mathbb{R}$$

$$df = \sum \frac{\partial f}{\partial x^i} dx^i$$

this looks like the gradient in vector calculus so why is it <ins>not</ins> a vector here?

suppose $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is a coordinate change

$$\phi(x^1, \dots, x^n) = (y^1(x^1, \dots, x^n), \dots, y^n(x^1 \dots x^n))$$

then
$$d\phi\Big(\underbrace{\sum \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^i}}_{\nabla f}\Big) = \sum \frac{\partial f}{\partial x^i} d\phi\left(\frac{\partial}{\partial x^i}\right)$$

$$= \underbrace{\sum \frac{\partial f}{\partial x^i} \frac{\partial y^j}{\partial x^i} \frac{\partial}{\partial y^j}}_{\text{note index problem}}$$

but f in coords $(y^1, \dots, y^n)$ is $f \circ \phi^{-1}$

so $\frac{\partial f}{\partial y^i} = \sum \frac{\partial f}{\partial x^j} \frac{\partial x^j}{\partial y^i} \frac{\partial}{\partial y^i} \neq d\phi(\nabla f)$

but $(\phi^{-1})^*(df) = (\phi^{-1})^*\left(\sum \frac{\partial f}{\partial x^i} dx^i\right)$

$$= \sum \frac{\partial f}{\partial x^i} (\phi^{-1})^*(dx^i)$$

$$= \sum \frac{\partial f}{\partial x^i} \frac{\partial x^i}{\partial y^j} dy^j = d(f \circ \phi^{-1})$$

so the "gradient" transforms as a covector under coordinate change, not as a vector

Cotangent Bundles:

set $T^*M = \coprod_{p \in M} T_p^*M$

we can put a topology and bundle structure on this as we did for TM

i.e. given coord. chart $\phi : U \to V$

note $V \subset \mathbb{R}^n$ so $TV = V \times \mathbb{R}^n$

and $T^*V = V \times \mathbb{R}^n$

now $\phi^* : V \times \mathbb{R}^n \to T^*U = T^*M|_U$

is a bijection

so coordinate charts on M give these coord charts on T^*M and local trivializations a section of T^*M is called a 1-form

(or covector field)

$$\begin{tikzcd} T^*M \arrow[d, "\pi"'] \\ M \arrow[u, bend right, "\alpha"'] \end{tikzcd}$$

denote the space of sections $\Omega^1(M) = \Gamma(T^*M)$

note: given $\alpha \in \Omega^1(M)$ this gives a linear map

$$\mathfrak{X}(M) \xrightarrow{\phi_\alpha} C^\infty(M)$$
$$v \longmapsto \alpha(v)$$

exercise: $\Omega^1(M) \longrightarrow \mathrm{Hom}_{C^\infty(M)}(\mathfrak{X}(M), C^\infty(M))$
$$\alpha \longmapsto \phi_\alpha$$

is an isomorphism

(this is not so easy!)

note: from above we have

$$d : C^\infty(M) \to \Omega^1(M)$$
$$f \longmapsto df$$

this is called the exterior derivative

exercise:

1) $d(af+bg) = a\,df + b\,dg \qquad a,b \in \mathbb{R}$

2) $d(fg) = f\,dg + g\,df$

3) if $\Phi: M \to N$ a smooth map

define $\phi^*: C^\infty(N) \to C^\infty(M)$
$$f \longmapsto f \circ \phi$$

then

$$\begin{array}{ccc} C^\infty(N) & \xrightarrow{\phi^*} & C^\infty(M) \\ d \downarrow & & \downarrow d \\ \Omega^1(N) & \xrightarrow{\phi^*} & \Omega^1(M) \end{array}$$ commutes

4) $f \in C^\infty(N)$, $\alpha \in \Omega^1(N)$, then $\Phi^*(f\alpha) = (\phi^* f)(\phi^*\alpha)$

examples:

1) $f(x,y,z) = (\overbrace{x^2 y}^{u}, \overbrace{y\sin z}^{v})$

$$\alpha = u^2 dv + v\,du \in \Omega^1(\mathbb{R}^2)$$

now $f^* du = d(u \circ f) = d(x^2 y)$
$$= 2xy\,dx + x^2 dy$$

similarly

$$f^*\alpha = (x^2y)^2 d(y\sin z) + y\sin z\, d(x^2 y)$$
$$= (x^2y)^2(\sin z\,dy + y\cos z\,dz)$$
$$+ y\sin z\,(2xy\,dx + x^2 dy)$$
$$= (2xy^2\sin z)\,dx + (x^4y^2 + x^2y)\sin z\,dy$$
$$+ (x^4y^3\cos z)\,dz$$

2) $\phi(r,\theta) = (\overbrace{r\cos\theta}^{x}, \overbrace{r\sin\theta}^{y})$

$$\alpha = x\,dy - y\,dx$$

$$\begin{aligned}
\phi^*\alpha &= r\cos\theta\, d(r\sin\theta) - r\sin\theta\, d(r\cos\theta) \\
&= r\cos\theta(\cancel{\sin\theta\, dr} + r\cos\theta\, d\theta) \\
&\quad - r\sin\theta(\cancel{\cos\theta\, dr} - r\sin\theta\, d\theta) \\
&= r^2(\cos^2\theta + \sin^2\theta)\,d\theta = r^2\,d\theta
\end{aligned}$$

C. Integration

a 1-form α on $[a,b] \subset \mathbb{R}$ can be written

$$\alpha = f(t)\,dt$$

for some function $f: [a,b] \longrightarrow \mathbb{R}$

t is the coord. on $[a,b]$

we define the integral of α on $[a,b]$ to be

$$\int_{[a,b]} \alpha = \int_a^b f(t)\,dt$$

ordinary integral from calculus

suppose $\underset{s}{\phi:[c,d]} \longrightarrow \underset{t}{[a,b]}$ is a diffeomorphism s.t. $\phi(c) = a$

then note

$$\begin{aligned}
\int_{[c,d]} \phi^*\alpha &= \int_c^d \phi^*(f(t)\,dt) \\
&= \int_c^d (f\circ\phi(s))\,\phi'(s)\,ds
\end{aligned}$$

set $t = \phi(s)$

so $dt = \phi'(s)\,ds$

$c \to a \quad d \to b$

$$= \int_a^b f(t)\,dt = \int_{[a,b]} \alpha$$

<u>note</u>: if $\phi(c) = d$ then

$$\int_{[c,d]} \phi^* \alpha = -\int_{[a,b]} \alpha$$

now if $\alpha \in \Omega^1(M)$ and $C \subset M$ is a compact 1-manifold with a direction chosen, then on can parameterize C by some map

$$\gamma : [a,b] \to C \subset M$$

(if $C = S^1$ then $\gamma(a) = \gamma(b)$)

so a t increases, C is traversed in the chosen direction

then define

also write this $\int_\gamma \alpha$

$$\boxed{\int_C \alpha = \int_{[a,b]} \gamma^* \alpha}$$

the computation above says this is well-defined

<u>example</u>: given a 1-form $\alpha = P\,dx + Q\,dy + R\,dz$ in $\mathbb{R}^3$ then the integral over some curve $C \subset \mathbb{R}^3$ is

$$\int_C \alpha = \int_C P\,dx + Q\,dy + R\,dz$$

$$= \int_a^b \left[(P\circ\gamma(t))x'(t) + (Q\circ\gamma(t))y'(t) + (R\circ\gamma(t))z'(t) \right] dt$$

where $\gamma(t) = (x(t), y(t), z(t))$ parameterizes C
$a \le t \le b$

so integrating α over C is just the line integral from vector calculus

exercise:

1) $\int_C \alpha$ is independent of oriented parameterization (essentially done above)

2) $\int_C a\alpha + b\beta = a\int_C \alpha + b\int_C \beta$ $\quad \alpha, \beta \in \Omega^1(M)$, $a, b \in \mathbb{R}$

3) if γ is a constant map then
$$\int_\gamma \alpha = 0$$

4) $a < c < b$ then
$$\int_\gamma \alpha = \int_{\gamma_1} \alpha + \int_{\gamma_2} \alpha$$
where $\gamma_1 = \gamma|_{[a,c]}$ $\quad \gamma_2 = \gamma|_{[c,b]}$

lemma 1:

$$\boxed{\begin{array}{l} \int_C df = f(B) - f(A) \\ \text{where } C \text{ is a path from } A \text{ to } B \end{array}}$$

Proof:

$$\int_C df = \int_a^b \gamma^* df = \int_a^b d(f\circ\gamma)$$
$$= \int_a^b (f\circ\gamma)'(t)\,dt$$
$$= f\circ\gamma(b) - f\circ\gamma(a)$$
$$= f(B) - f(A)$$

$\gamma:[a,b]\to M$ param, C

if $\alpha = df$ for some $f\in C^\infty(M)$ we call α exact

Thm 2:

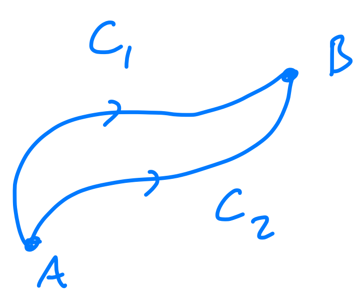
$\alpha\in\Omega^1(M)$ then the following are equivalent

1) α is exact
2) $\int_C \alpha = 0$ for all loops $C\subset M$
3) $\int_C \alpha$ only depends on the end points of C

Proof:

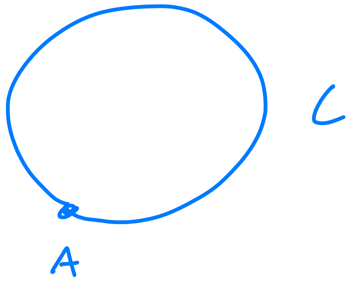
1) $\Rightarrow$ 2) and 3) by lemma 1.

2) $\Rightarrow$ 3)



so $$0 = \int_{C_1 - C_2}\alpha = \int_{C_1}\alpha - \int_{C_2}\alpha$$

3) $\Rightarrow$ 2)



let c_A = constant loop $t \mapsto A$

$$\int_C \alpha = \int_{c_A} \alpha = 0$$

3) $\Rightarrow$ 1) fix $x_0 \in M$ (assume M connected)

$\forall x \in M$ choose a path γ_x from x_0 to x

set $f(x) = \int_{\gamma_x} \alpha$

f well-defined by 3)

exercise: show $df = \alpha$