X. Forms

A. Linear Algebra

recall $\underbrace{V^* \otimes \dots \otimes V^*}_{k} = \text{Bilin}(\underbrace{V \times \dots \times V}_{k}, \mathbb{R})$

we say a map $\phi \in \text{Bilin}(V \times \dots V, \mathbb{R})$ is alternating if

$$\phi(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = \text{sgn}(\sigma)\, \phi(v_1, \dots, v_k)$$

where $\sigma \in S_k$ an element of the symmetric group

and $\text{sgn}(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is even \# of permutations} \\ -1 & \text{if " " odd " "} \end{cases}$

example: $\phi(v_1, \dots, v_i, \dots, v_j, \dots v_k) = -\phi(v_1, \dots, v_j, \dots, v_i, \dots v_k)$

let $\Lambda^k(V) = \{\phi \in \text{Bilin}(V \times \dots \times V, \mathbb{R}) : \phi \text{ is alternating}\}$

so $\Lambda^k(V) \subset \underbrace{V^* \otimes \dots \otimes V^*}_{k}$

lemma 1:

$\phi \in \text{Bilin}(V \times \dots \times V, \mathbb{R})$ then
The following are equivalent

1. $\phi \in \Lambda^k(V)$
2. $\phi(v_1, \dots, v_i, \dots, v_i, \dots v_k) = 0$ (zero when entry repeated)
3. $\phi(v_1, \dots, v_k) = 0$ whenever $v_1, \dots, v_k$ not lin. independent

Proof: ① $\Rightarrow$ ②

$\phi \in \Lambda^k(V)$ then

$$\phi(v_1 \dots v_i \dots v_i \dots v_k) = -\phi(v_1 \dots \overset{\text{switch}}{v_i \dots v_i} \dots v_k)$$

so $2\,\phi(v_1 \dots v_i \dots v_i \dots v_k) = 0$

so $\phi(v_1 \dots v_i \dots v_i \dots v_k) = 0$

② $\Rightarrow$ ③

can assume $v_1 = a_2 v_2 + \dots + a_k v_k$

so $$\phi(v_1, \dots v_k) = \sum_{i=2}^{k} a_i\, \phi(v_i, v_2, \dots v_k) = 0$$

③ $\Rightarrow$ ② ✓

② $\Rightarrow$ ①

$$\phi(v_1, \dots, v_i + v_j, \dots, v_i + v_j, \dots, v_k) = 0$$

$$\parallel$$

$$\cancelto{0}{\phi(v_1, \dots v_i, \dots v_i, \dots, v_k)} + \phi(v_1, \dots v_i, \dots v_j, \dots, v_k)$$

$$+ \phi(v_1, \dots v_j, \dots v_i, \dots, v_k) + \cancelto{0}{\phi(v_1, \dots, v_j, \dots, v_j, \dots v_k)} = 0$$

we define a map

$$\mathrm{Alt} : \mathrm{Bilin}(V \times \dots \times V, \mathbb{R}) \to \Lambda^k(V)$$
$$\phi \longmapsto \frac{1}{k!} \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma)\, \phi^{\sigma}$$

where $\phi^{\sigma}(v_1, \dots, v_k) = \phi(v_{\sigma(1)}, \dots, v_{\sigma(k)})$

example:

1) $\phi \in V^* \otimes V^*$

$$(\mathrm{Alt}\,\phi)(v_1, v_2) = \frac{1}{2}[\phi(v_1, v_2) - \phi(v_2, v_1)]$$

2) $\phi \in V^* \otimes V^* \otimes V^*$

$$(\mathrm{Alt}\,\phi)(v_1, v_2, v_3) = \frac{1}{6}[\phi(v_1, v_2, v_3) + \phi(v_2, v_3, v_1) + \phi(v_3, v_1, v_2) - \phi(v_2, v_1, v_3) - \phi(v_1, v_3, v_2) - \phi(v_3, v_2, v_1)]$$

exercise:

1) Alt is a linear map
2) for any $\phi \in T^k(V)$, $\mathrm{Alt}(\phi)$ is alternating
3) if $\phi \in \Lambda^k(V)$, then $\mathrm{Alt}\,\phi = \phi$

(i.e. $\mathrm{Alt}(\mathrm{Alt}) = \mathrm{Alt}$

so Alt is a projection)

now given $\omega \in \Lambda^k(V)$ and $\eta \in \Lambda^\ell(V)$, then we define their wedge product to be

$$\omega \wedge \eta = \frac{(k+\ell)!}{k!\,\ell!}\mathrm{Alt}(\omega \otimes \eta)$$

example:

1) $\omega, \eta \in \Lambda^1(V) = V^*$

$$\omega \wedge \eta = \frac{2!}{1!\,1!}\mathrm{Alt}(\omega \otimes \eta) = 2\,\frac{1}{2}(\omega \otimes \eta - \eta \otimes \omega) = \omega \otimes \eta - \eta \otimes \omega$$

2) $\omega \in \Lambda^2(V)$, $\eta \in \Lambda^1(V)$

$$\omega \wedge \eta\,(v_1, v_2, v_3) = \frac{3!}{2!\,1!}\,\mathrm{Alt}(\omega \otimes \eta)(v_1, v_2, v_3)$$

$$= 3\Big[\tfrac{1}{6}\big[\omega(v_1,v_2)\eta(v_3) - \omega(v_1,v_3)\eta(v_2) - \omega(v_2,v_1)\eta(v_3)$$

$$+ \omega(v_3,v_1)\eta(v_2) + \omega(v_2,v_3)\eta(v_1) - \omega(v_3,v_2)\eta(v_1)\big]\Big]$$

$$= \tfrac{1}{2}\big[2\,\omega(v_1,v_2)\,\eta(v_3) + 2\,\omega(v_2,v_3)\,\eta(v_1)$$

$$+ 2\,\omega(v_1,v_3)\,\eta(v_2)\big]$$

$$= \omega(v_1,v_2)\,\eta(v_3) + \omega(v_2,v_3)\,\eta(v_1) + \omega(v_3,v_1)\,\eta(v_2)$$

recall if $e_1, \dots, e_n$ is a basis for V, then

$$\{e^{i_1} \otimes \dots \otimes e^{i_k}\}_{1 \le i_j \le n}$$

is a basis for $\underbrace{V^* \otimes \dots \otimes V^*}_{k}$

so $\mathrm{Alt}(\{e^{i_1} \otimes \dots \otimes e^{i_k}\}) = \{e^{i_1} \wedge \dots \wedge e^{i_k}\}$ <u>spans</u> $\Lambda^k(V)$

but not independent e.g. $e^1 \wedge e^2 \wedge \dots = -e^2 \wedge e^1 \wedge \dots$

but if we demand $i_1 < i_2 < \dots < i_k$ then

<u>exercise</u>: 1) $\{e^{i_1} \wedge \dots \wedge e^{i_k}\}_{1 \le i_1 < \dots < i_k \le n}$ is a basis for $\Lambda^k V$

<u>Hint</u>: assume dependent and evaluate on k vectors

2) so $\dim \Lambda^k V = \binom{n}{k} = \dfrac{n!}{k!\,(n-k)!}$ $\quad 0 \le k \le n$

<u>note</u>: $\Lambda^k V = \{0\}$ for $k < 0$ or $k > n$

we define

$$\Lambda(V) = \Lambda^0(V) \oplus \dots \oplus \Lambda^n(V) \qquad (\dim V = n)$$

and call it the exterior algebra of V

note: $\dim \Lambda(V) = 2^n$

lemma 2:

$$\boxed{\begin{array}{l} 1)\ (a\omega + a'\omega') \wedge \eta = a\omega\wedge\eta + a'\omega'\wedge\eta \\ \quad\ \eta\wedge(a\omega + a'\omega') = a\,\eta\wedge\omega + a'\,\eta\wedge\omega' \\ 2)\ \omega\wedge(\eta\wedge\xi) = (\omega\wedge\eta)\wedge\xi \\ 3)\ \omega\wedge\eta = (-1)^{kl}\,\eta\wedge\omega \quad \text{where } \omega\in\Lambda^k(V),\ \eta\in\Lambda^l(V) \\ 4)\ (\omega^1\wedge\dots\wedge\omega^k)(v_1,\dots,v_k) = \det(\omega^j(v_i)) \\ \qquad \text{where } \omega^i \in V^*,\ v_i \in V \end{array}}$$

Proof: 1) $\otimes$ has this property and Alt is linear ✓

2) we first note if $\omega \in T_k V^*$, $\eta \in T_l V^*$ then

$$\mathrm{Alt}(\omega \otimes \mathrm{Alt}\,\eta) = \frac{1}{(k+l)!}\sum_{\sigma\in S_{k+l}} (\operatorname{sgn}\sigma)\Big(\omega\otimes\frac{1}{l!}\sum_{\tau\in S_l}\operatorname{sgn}\,\eta^\tau\Big)^\sigma$$

think of $S_l \subset S_{k+l}$ so that $\tau \in S_l$ only moves last l elements

$$= \frac{1}{(k+l)!\,l!}\sum_{\sigma\in S_{k+l}}\sum_{\tau\in S_l}(\operatorname{sgn}\sigma)(\operatorname{sgn}\tau)(\omega\otimes\eta)^{\sigma\circ\tau}$$

set $\mu = \sigma\circ\tau$ for each $\mu \in S_{k+l}$ there are $l!$ ways to write μ as $\sigma\circ\tau$

$$= \frac{1}{(k+l)!} \sum_{\mu \in S_{k+l}} (\operatorname{sgn} \mu) (\omega \otimes \eta)^{\mu}$$

$$= \operatorname{Alt}(\omega \otimes \eta)$$

similarly $\operatorname{Alt}((\operatorname{Alt} \omega) \otimes \eta) = \operatorname{Alt}(\omega \otimes \eta)$

so $\overset{k}{\omega} \wedge (\overset{l}{\eta} \wedge \overset{m}{\xi}) = \omega \wedge \left(\frac{(l+m)!}{l!\,m!} \operatorname{Alt}(\eta \otimes \xi) \right)$

$$= \frac{(k+l+m)!}{k!\,(l+m)!} \frac{(l+m)!}{l!\,m!} \operatorname{Alt}(\omega \otimes \operatorname{Alt}(\eta \otimes \xi))$$

$$= \frac{(k+l+m)!}{k!\,l!\,m!} \operatorname{Alt}(\omega \otimes (\eta \otimes \xi))$$

$$= \frac{(k+l+m)!}{k!\,l!\,m!} \operatorname{Alt}((\omega \otimes \eta) \otimes \xi)$$

$$= (\omega \wedge \eta) \wedge \xi$$

similarly ✓

3) let $\tau = \begin{bmatrix} 1 & \cdots & l & l+1 & \cdots & l+k \\ k+1 & \cdots & k+l & 1 & \cdots & k \end{bmatrix}$

note: $\sigma(1) = \sigma(\tau(l+1))$

$\vdots$

$\sigma(k) = \sigma(\tau(l+k))$

$\sigma(k+1) = \sigma(\tau(1))$

$\vdots$

$\sigma(k+l) = \sigma(\tau(l))$

so $\operatorname{Alt}(\omega \otimes \eta)(v_1, \ldots, v_{k+l})$

$$= \frac{(k+l)!}{k!\,l!} \sum_{\sigma \in S_{k+l}} (\operatorname{sgn} \sigma)\, \omega(v_{\sigma(1)} \cdots v_{\sigma(k)})\, \eta(v_{\sigma(k+1)} \cdots v_{\sigma(k+l)})$$

$$= \frac{(k+l)!}{k!\,l!} \sum_{\sigma \in S_{k+l}} (\operatorname{sgn} \sigma)\ \omega(v_{\sigma(\tau(l+1))} \cdots v_{\sigma(\tau(l+k))})\, \eta(v_{\sigma(\tau(1))} \cdots v_{\sigma(\tau(l))})$$

$$= \frac{(k+l)!}{k!\,l!} (\operatorname{sgn} \tau) \sum_{\sigma \in S_{k+l}} (\operatorname{sgn} \sigma\circ\tau)\, \omega(v_{\sigma(\tau(l+1))} \cdots v_{\sigma(\tau(l+k))})\, \eta(v_{\sigma(\tau(1))} \cdots v_{\sigma(\tau(l))})$$

$$= (\operatorname{sgn} \tau) \frac{(k+l)!}{k!\,l!} \sum_{\sigma' \in S_{k+l}} (\operatorname{sgn} \sigma')\ \omega(v_{\sigma'(l+1)} \cdots v_{\sigma'(l+k)})\, \eta(v_{\sigma'(1)} \cdots v_{\sigma'(l)})$$

$$= (\operatorname{sgn} \tau)\ \operatorname{Alt}(\eta \otimes \omega)(v_1, \dots, v_{k+l})$$

so $\quad \omega \wedge \eta = (-1)^{kl}\, \eta \wedge \omega \qquad (\text{check } \operatorname{sgn} \tau = (-1)^{kl})$

4) $(\omega^1 \wedge \dots \wedge \omega^k)(v_1, \dots, v_k) = \frac{(1+\dots+1)!}{1!\cdots 1!} \operatorname{Alt}(\omega^1 \otimes \dots \otimes \omega^k)(v_1, \dots, v_k)$

$$= k!\, \frac{1}{k!} \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma)\, \omega^1(v_{\sigma(1)}) \cdots \omega^k(v_{\sigma(k)})$$

<u>Claim</u>:

if $M = (m^i_j)$ is a $k \times k$ matrix, then

$$\det M = \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma)\, m^1_{\sigma(1)} \cdots m^k_{\sigma(k)}$$

Clearly Claim $\Rightarrow$ 4

<u>Proof of Claim</u>

<u>exercise</u>: Show $\exists!$ function $f : \operatorname{Mat}(k \times k) \to \mathbb{R}$

s.t. 1) $f(I_k) = 1$

2) $f(M) = -f(M')$ if M' is obtained from M by switching adjacent columns

3) $f(M) = f(M')$ if M' is obtained from M by adding a multiple of one column to another

4) $f(M') = n f(M)$ if M' is obtained from M by multiplying one column by n

exercise: Show Det and formula above satisfy 1) – 4)

Cor 3:

$\boxed{\text{ }}$

let $L : V^n \to W^n$ be a linear map

$e_1, \dots, e_n$ a basis for V

$f_1, \dots, f_n$ " " W

express L as a matrix $M_L = (m_i^j)$ in these bases

(i.e. $L e_i = \sum m_i^j f_j$)

then $L^*(f^1 \wedge \dots \wedge f^n) = (\det M_L)\, e^1 \wedge \dots \wedge e^n$

Proof: immediate from lemma 2 part 4)

(evaluate both sides on $e_1, \dots, e_n$)

B. k-forms

for a manifold set

$$\Lambda^k M = \coprod_{p \in M} \Lambda^k (T_p M)$$

exercise: Show $\Lambda^k M$ is a manifold and a vector bundle over M with fiber $\Lambda^k(T_p M) \subset T_k(T_p^*(M))$

let $\Omega^k(M) = \Gamma(\Lambda^k M)$ sections of $\Lambda^k M$

$\alpha \in \Omega^k(M)$ is called a k-form

note: 1) $\Lambda^1 M = T^* M$

so $\Omega^1(M)$ agrees with earlier definition

2) $\Lambda^0 M = M \times \mathbb{R}$

so $\Omega^0(M) \cong C^\infty(M)$

$$\begin{tikzcd} M \times \mathbb{R} \arrow[d] \\ M \arrow[u, bend right, "\sigma"'] \end{tikzcd} \qquad \sigma(p) = (p, f(p))$$

$$f : M \to \mathbb{R}$$

now given $f : M \to N$

we get $f^* : \Omega^k(N) \to \Omega^k(M)$

by restricting $f^* : \Gamma(T^k N) \to \Gamma(T^k M)$

to $\Omega^k(N)$

Lemma 4:

$$\boxed{f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta}$$

Proof: we know

$$f^*(\omega \otimes \eta) = f^*\omega \otimes f^*\eta$$

the result follows ∎

In local coords

$$f^*\left(\sum \omega_{i_1 \dots i_k} dy^{i_1} \wedge \dots \wedge dy^{i_k}\right)$$

$$= \sum (\omega_{i_1 \dots i_k} \circ f)\, d(y^{i_1} \circ f) \wedge \dots \wedge d(y^{i_k} \circ f)$$

example: $f(r, \theta) = (r\cos\theta, r\sin\theta)$

$$f^*(dx \wedge dy) = d(r\cos\theta) \wedge d(r\sin\theta)$$

$$= (\cos\theta\, dr - r\sin\theta\, d\theta) \wedge (\sin\theta\, dr + r\cos\theta\, d\theta)$$

$$= -r\sin^2\theta\, d\theta \wedge dr + r\cos\theta\, dr \wedge d\theta$$

$$= r\, dr \wedge d\theta$$

Thm 5:

$f: M^n \to N^n$ smooth map

$\phi: U \to V$ local coords for M $(x^1, \dots, x^n)$

$\phi': U' \to V'$ " " N $(y^1, \dots, y^n)$

s.t. $f(U) \subset U'$

set $F = \phi' \circ f \circ \phi^{-1}$

Then $F^*(dy^1 \wedge \dots \wedge dy^n) = \det(DF)\, dx^1 \wedge \dots \wedge dx^n$

Proof: by Cor 3 since $(Df)^i_j = \frac{\partial f^i}{\partial x^j}$ ▦

C Exterior Derivative

Th$^{\underline{m}}$6:

M a smooth manifold of dimension n

$\exists!$ map

$$d : \Omega^k(M) \to \Omega^{k+1}(M)$$

for $k = 0, \dots, n$

such that

1) $d(a\alpha + b\beta) = a\,d\alpha + b\,d\beta \quad a, b \in \mathbb{R}$
2) $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{|\alpha|} \alpha \wedge d\beta$ where $\alpha \in \Omega^{|\alpha|}(M)$
3) $d^2 = 0$
4) df is the exterior derivative if $f \in \Omega^0(M) = C^\infty(M)$

d is called the exterior derivative on forms

Proof: if $\omega \in \Omega^k(\mathbb{R}^n)$ $\quad x^1, \dots, x^n$ coordinates

then

$$\omega = \sum \omega_{i_1 \dots i_k}\, dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

where $\omega_{i_1 \dots i_k} : \mathbb{R}^n \to \mathbb{R}$

to simplify notation we use multi-index notation

i.e. $I = (i_1, \dots, i_k)$ then

$$dx^I = dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

<u>note</u>: $d\omega_I \in \Omega^1(\mathbb{R}^n)$

so we define $$d\omega = \sum (d\omega_I) \wedge dx^I$$

clearly:
- $d\omega \in \Omega^{k+1}(M)$
- $d(a\omega + b\eta) = a\,d\omega + b\,d\eta$
- df same as before for $f \in \Omega^0(\mathbb{R}^n)$

now
- $\alpha = \sum \alpha_I\, dx^I$ and $\beta = \sum \beta_J\, dx^J$

  then

$$\begin{aligned} d(\alpha \wedge \beta) &= d \sum (\beta_J \alpha_I\, dx^I \wedge dx^J) \\ &= \sum (\beta_J\, d\alpha_I + \alpha_I\, d\beta_J) \wedge dx^I \wedge dx^J \\ &= \sum \big(d\alpha_I \wedge dx^I \wedge (\beta_J\, dx^J) \\ &\qquad + (-1)^{|\alpha|} \alpha_I\, dx^I \wedge (d\beta_J \wedge dx^J)\big) \\ &= d\alpha \wedge \beta + (-1)^{|\alpha|} \alpha \wedge d\beta \end{aligned}$$

and
- $d\omega = \sum d\omega_I \wedge dx^I$

  so $$d^2\omega = \sum d^2\omega_I \wedge dx^I - d\omega_I \wedge \cancel{d(dx^I)}$$ by def^n

now $f \in \Omega^0(\mathbb{R}^n)$ then

$$df = \sum \frac{\partial f}{\partial x^i}\, dx^i$$

so $$d^2 f = \sum d\left(\frac{\partial f}{\partial x^i}\right) \wedge dx^i$$

$$= \sum \frac{\partial^2 f}{\partial x^j \partial x^i} dx^j \wedge dx^i$$

$$= 0$$ each term appears twice and with opposite sign!

$$\text{so } d^2\omega = 0$$

thus d on $\mathbb{R}^n$ has the all the properties!

now for $\omega \in \Omega^k(M)$

let $\phi: U \to V$ be a coordinate chart

$(\phi^{-1})^* \omega$ is a k-form on $V \subset \mathbb{R}^n$

$d((\phi^{-1})^*\omega)$ is a $(k+1)$-form on $V \subset \mathbb{R}^n$

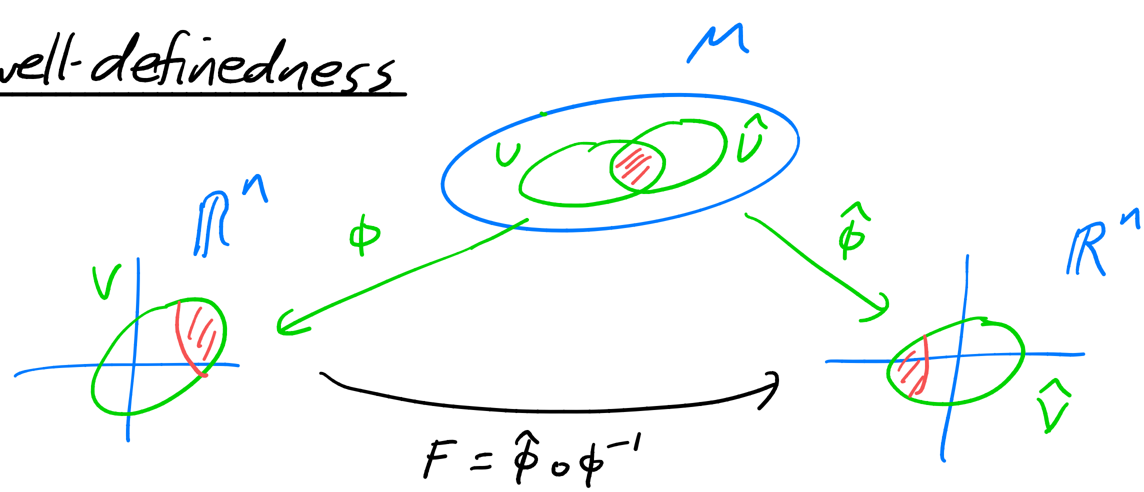
so $\phi^*(d((\phi^{-1})^*\omega))$ is a $(k+1)$-form on $U \subset \mathbb{R}^n$

we define

$$d\omega(x) = \phi^*(d((\phi^{-1})^*\omega))(x) \in \Lambda^{k+1}_x M$$

clearly $d\omega$ satisfies all the properties <u>if</u> it is well-defined

<u>Check well-definedness</u>



need to show

$$\phi^*\left[d((\phi^{-1})^*\omega)\right](x) = \hat{\phi}^*\left[d((\hat{\phi}^{-1})^*\omega)\right](x)$$

for this let $\omega_V = (\phi^{-1})^*\omega$ and

$$\omega_{\hat{V}} = (\hat{\phi}^{-1})^*\omega$$

note: $F^*\omega_{\hat{V}} = (\hat{\phi}\circ\phi^{-1})^*(\hat{\phi}^{-1})^*\omega$

$$= (\phi^{-1})^*\circ\hat{\phi}^*\circ(\hat{\phi}^{-1})^*\omega$$

$$= (\phi^{-1})^*\omega = \omega_V$$

<u>Claim</u>: $dF^* = F^*d$

given this note

$$\phi^*\left[d((\phi^{-1})^*\omega)\right] = \phi^*(d\omega_V) = \phi^*(d(F^*\omega_{\hat{V}}))$$

$$= \phi^*(F^*(d\omega_{\hat{V}})) = \phi^*\circ(\hat{\phi}\circ\phi^{-1})^* d\omega_{\hat{V}}$$

$$= \phi^*\circ(\phi^{-1})^*\circ\hat{\phi}^* d\omega_{\hat{V}}$$

$$= \hat{\phi}^*(d((\hat{\phi}^{-1})^*\omega))$$ done!

<u>proof of Claim</u>:

given any $F: \mathbb{R}^n \to \mathbb{R}^n$

$$(x^1, \dots x^n) \mapsto (\underbrace{F^1(x^1, \dots x^n)}_{= y^1}, \dots \underbrace{F^n(x^1, \dots x^n)}_{= y^n})$$

we know $dF\left(\frac{\partial}{\partial x^i}\right) = \sum \frac{\partial y^j}{\partial x^i} \cdot \frac{\partial}{\partial y^j}$

so $\quad F^* dy^j = \sum_i \frac{\partial y^j}{\partial x^i} dx^i$

and thus

$$d(F^* dy^j) = d\Big(\sum_i \frac{\partial y^j}{\partial x^i} dx^i\Big)$$
$$= \sum_i \Big(d \frac{\partial y^j}{\partial x^i}\Big) \wedge dx^i$$
$$= \sum_{i,k} \frac{\partial^2 y^j}{\partial x^k \partial x^i} dx^k \wedge dx^i$$

$dx^i \wedge dx^i = 0$ so

$$= \sum_{i<k} \Big(\frac{\partial^2 y^j}{\partial x^i \partial x^k} - \frac{\partial^2 y^j}{\partial x^k \partial x^i}\Big) dx^i \wedge dx^k$$
$$= 0 = F^*(d(dy^j))$$

and $\quad dF^*(dy^i \wedge dy^j) \underset{\text{lemma 4}}{=} d(F^* dy^i \wedge F^* dy^j)$

$$= \cancel{(dF^* dy^i)}^{\,0} \wedge F^* dy^j - F^* dy^i \wedge \cancel{(dF^* dy^j)}^{\,0}$$
$$= 0 = F^*(d(dy^i \wedge dy^j))$$

$$\therefore \; F^* d(\omega_I dx^I) = F^*(d\omega_I \wedge dx^I)$$
$$= F^* d\omega_I \wedge F^* dx^I$$

exercise from VIII B

$$= d(F^* \omega_I) \wedge F^* dx^I$$
$$= d[(F^* \omega_I) F^* dx^I]$$
$$= d(F^*(\omega_I dx^I)) \checkmark$$

now for uniqueness:

Claim: if $\omega = \omega'$ on an open set U, then $d\omega = d\omega'$ on U

(i.e. d is <u>local</u>)

<u>Proof</u>: let $p \in U$ and $f: M \to \mathbb{R}$ a bump function s.t. $f = \begin{cases} 1 & \text{near } p \\ 0 & \text{outside } U \end{cases}$

so $f(\omega - \omega') = 0$

$\therefore$ linearity of d gives

$$\begin{aligned} 0 &= d(f(\omega-\omega'))(p) \\ &= \cancel{df(p)}^{\,0} \wedge (\omega-\omega')(p) + \cancel{f(p)}^{\,1}(d\omega - d\omega')(p) \\ &= d\omega(p) - d\omega'(p) \checkmark \end{aligned}$$

now in a coordinate chart $\phi: U \to V$

if ω is supported in U then we have the forms "dx^I" $= \phi^* dx^I$

and we can write ω as

$$\omega = \sum \omega_I \text{"} dx^I \text{"}$$

now

$$\begin{aligned} d\omega &\overset{1}{=} \sum d(\omega_I \text{"} dx^I \text{"}) \\ &\overset{2}{=} \sum d\omega_I \wedge \text{"} dx^I \text{"} + d\omega_I \wedge d \text{"} dx^I \text{"} \end{aligned}$$

<u>note</u>: $$\begin{aligned} d \text{"} dx^i \text{"} &= d\phi^*(dx^i) \\ &= d(d(x^i \circ \phi)) \\ &\overset{3}{=} 0 \end{aligned}$$

so $d \text{"} dx^I \text{"} = 0$ too

$= \sum d\omega_I \wedge \text{"}dx^I\text{"}$

since $d\omega_I$ is determined by ④

this expression uniquely determines $d\omega$

finally if $\omega \in \Omega^k(M)$ let $\tilde{\omega} = f\omega$ where f is a bump function with support in coord chart and $f = 1$ near p

so $d\omega(p) = d\tilde{\omega}(p) \leftarrow$ is determined by ①–④

$\therefore d\omega(p)$ unique $\forall p$

examples:

1) $\alpha = dz - y\,dx$ in $\mathbb{R}^3$

then $d\alpha = -dy \wedge dx = dx \wedge dy$

and $\alpha \wedge d\alpha = dx \wedge dy \wedge dz$

2) if $\omega = P\,dx + Q\,dy + R\,dz$

then
$$d\omega = \frac{\partial P}{\partial y} dy \wedge dx + \frac{\partial P}{\partial z} dz \wedge dx + \frac{\partial Q}{\partial x} dx \wedge dy + \frac{\partial Q}{\partial z} dz \wedge dy + \frac{\partial R}{\partial x} dx \wedge dz + \frac{\partial R}{\partial y} dy \wedge dz$$
$$= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy + \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) dx \wedge dz + \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy \wedge dz$$

$$\begin{bmatrix} P \\ Q \\ R \end{bmatrix} \longmapsto \begin{bmatrix} Q_x - P_y \\ R_x - P_z \\ R_y - Q_z \end{bmatrix}$$ looks like curl!

$$\Omega \xrightarrow{d} \Omega^2$$

$$\begin{array}{ll} dx & dx \wedge dy \\ dy & dx \wedge dz \\ dz & dy \wedge dz \end{array}$$

3) $\eta = P\, dx \wedge dy + Q\, dz \wedge dx + R\, dy \wedge dz$

then $d\eta = \left(\frac{\partial P}{\partial z} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial x}\right) dx \wedge dy \wedge dz$

(similar to divergence)

note we have

DeRham complex $\Bigg\{$ $\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \dots \xrightarrow{d} \Omega^n(M)$

and $d^2 = 0$

so $\operatorname{im} d \subset \ker d$

<u>define</u>: $$H^k_{DR}(M) = \frac{\ker(d : \Omega^k(M) \to \Omega^{k+1}(M))}{\operatorname{im}(d : \Omega^{k-1}(M) \to \Omega^k(M))}$$

this is called the <u>k^{th} DeRham cohomology</u> of M

if $d\omega = 0$ we say ω is <u>closed</u>

if $\omega = d\eta$ we say ω is <u>exact</u>

so $H^k_{DR}(M) = \dfrac{\text{closed } k\text{-forms}}{\text{exact } k\text{-forms}}$

note: 1) $H^k_{DR}(M) = 0$ if $k > \dim M$ or $k < 0$

2) $H^0_{DR}(M) = \ker(d : \underset{\underset{C^\infty(M)}{\parallel}}{\Omega^0(M)} \to \Omega^1(M))$

$= \text{locally constant functions}$

$= \mathbb{R}^{\#\text{ components of } M}$

Amazing Fact: If M is compact, then

$H^k_{DR}(M)$ is finite dimensional $\forall k$

Th$^{\underline{m}}$ 7:

given $f : M \to N$, then

$$\begin{array}{ccc} \Omega^k(N) & \xrightarrow{f^*} & \Omega^k(M) \\ \downarrow d & \circ & \downarrow d \\ \Omega^{k+1}(N) & \xrightarrow{f^*} & \Omega^{k+1}(M) \end{array}$$

commutes

Remark: if $[\omega] \in H^k_{DR}(N)$ then $d\omega = 0$

so $f^*\omega \in \Omega^k(M)$ and

$d f^*\omega = f^* d\omega = 0$

so $[f^*\omega] \in H^k_{DR}(M)$

if $\omega' = \omega + d\eta$, then

$$f^*\omega' = f^*\omega + d(f^*\eta)$$

so $[f^*\omega'] = [f^*\omega]$

that is we get a linear map

$$f^*: H^k_{DR}(N) \to H^k_{DR}(M)$$

i.e. k^{th} DeRham cohomology is a contravariant functor from smooth manifolds to vector spaces

Proof of Th$\underline{m}$ 7:

need $(d f^*\omega)(x) = (f^* d\omega)(x)$ for all x

since d is local we can just check in coordinate charts, but we did this in the proof of Th$\underline{m}$ 6

a useful lemma is

lemma 8:

$\omega \in \Omega^k(M)$, $v_1, \dots, v_{k+1}$ vector fields, then

$$d\omega(v_1, \dots, v_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i-1} v_i \cdot (\omega(v_1, \dots, \hat{v}_i, \dots v_{k+1})) + \sum_{1 \le i < j \le k+1} (-1)^{i+j} \omega([v_i, v_j], v_1, \dots \hat{v}_i, \dots \hat{v}_j, \dots v_{k+1})$$

examples:

1) $\omega \in \Omega^0(M)$, then

$$d\omega(v) = v \cdot \omega$$

2) $\omega \in \Omega^1(M)$, then

$$d\omega(v_1, v_2) = v_1 \cdot \omega(v_2) - v_2 \cdot \omega(v_1) - \omega([v_1, v_2])$$

Proof: let $D(v_1, \dots, v_{k+1}) =$ R.H.S.

note: 1) both $D\omega$ and $d\omega$ can be computed locally so we just check in coord. charts

2) both are linear so just need to check $D\omega = d\omega$ for $\omega = f\,dx^I$

3) $d\omega(v_1, \dots, f v_i, \dots, v_{k+1}) = f\,d\omega(v_1, \dots, v_i, \dots, v_{k+1})$ since $d\omega$ a $(k+1)$-tensor

exercise: $D\omega(v_1, \dots, f v_i, \dots, v_{k+1}) = f\,D\omega(v_1, \dots, v_i, \dots, v_{k+1})$

so by linearity $D\omega = d\omega$

if $D\omega(\frac{\partial}{\partial x^{j_1}}, \dots, \frac{\partial}{\partial x^{j_{k+1}}}) = d\omega(\frac{\partial}{\partial x^{j_1}}, \dots, \frac{\partial}{\partial x^{j_{k+1}}})$

for all $1 \le j_1 < \dots < j_{k+1} \le n$

to see this note

$$d(f\,dx^I) = \sum_\ell \frac{\partial f}{\partial x^\ell} dx^\ell \wedge dx^I$$

$$I = (i_1, \dots, i_k)$$

so $\quad d(f\,dx^I)\left(\frac{\partial}{\partial x^{j_1}}, \dots, \frac{\partial}{\partial x^{j_{k+1}}}\right) = \sum \frac{\partial f}{\partial x^\ell}\, \delta_J^{\ell I}$

where $\delta_J^{\ell I} = \begin{cases} 1 & \text{if } \{\ell\}\cup I = J \text{ upto even permutation} \\ -1 & \text{if } \{\ell\}\cup I = J \text{ '' '' odd '' '' } \\ 0 & \text{otherwise} \end{cases}$

if non-zero then $j_{p_\ell} = \ell$ some $1 \le p_\ell \le k+1$

clearly $\delta_J^{\ell I} = (-1)^{p_\ell - 1}\delta^I_{\hat{J}_{p_\ell}}$ where $\hat{J}_{p_\ell} = J$ with p_ℓ removed

so $\quad d\omega\left(\frac{\partial}{\partial x^{j_1}}, \dots, \frac{\partial}{\partial x^{j_{k+1}}}\right) = \sum_\ell (-1)^{p_\ell - 1}\frac{\partial f}{\partial x^\ell}\, \delta^I_{\hat{J}_{p_\ell}}$

now $\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0$ so

$$D\omega\left(\frac{\partial}{\partial x^{j_1}}, \dots, \frac{\partial}{\partial x^{j_{k+1}}}\right) = \sum_{i=1}^{k+1} (-1)^{i-1}\frac{\partial}{\partial x^{j_i}}\left(f\,dx^I\left(\frac{\partial}{\partial x^{j_1}}, \dots \widehat{\frac{\partial}{\partial x^{j_i}}}, \dots, \frac{\partial}{\partial x^{j_{k+1}}}\right)\right)$$

$$= \sum (-1)^{i-1}\frac{\partial f}{\partial x^{j_i}}\, \delta^I_{\hat{J}_{j_i}}$$

$$= d\omega\left(\frac{\partial}{\partial x^{j_1}}, \dots, \frac{\partial}{\partial x^{j_{k+1}}}\right)$$

D. Lie Derivatives

generalizing the Lie derivative we get

$\omega \in \Omega^k(M)$

$v \in \mathcal{X}(M)$

ϕ_t the flow of v

then $\quad \mathcal{L}_v\omega(x) = \lim_{t\to 0} \dfrac{\phi_t^*\omega_{\phi_t(x)} - \omega_x}{t}$

$$= \frac{d}{dt}\left(\phi_t^* \omega_t\right)_{t=0}$$

lemma 9:

$\mathcal{L}_v : \Omega^k(M) \to \Omega^k(M)$ is

1) Linear
2) $\mathcal{L}_v(\omega \wedge \eta) = (\mathcal{L}_v \omega) \wedge \eta + \omega \wedge (\mathcal{L}_v \eta)$
3) $\mathcal{L}_v(\iota_w \omega) = \iota_{\mathcal{L}_v w}\, \omega + \iota_w \mathcal{L}_v \omega$
4) if $v_1, \dots, v_k$ are vector fields, then

$$\mathcal{L}_v(\omega(v_1, \dots, v_k)) = (\mathcal{L}_v \omega)(v_1, \dots, v_k) + \sum_{i=1}^{k} \omega(v_1, \dots, \mathcal{L}_v v_i, \dots, v_k)$$

recall here $\iota_v \omega$ is a $(k-1)$-form defined by $\iota_v \omega(v_1, \dots, v_{k-1}) = \omega(v, v_1, \dots, v_{k-1})$

and we know

$$\iota_v(\omega \wedge \eta) = (\iota_v \omega) \wedge \eta + (-1)^{|\omega|} \omega \wedge \iota_v \eta$$

Proof: 1) evaluation and pull-backs are linear.

2) $\phi_t^*(\omega \wedge \eta)_{\phi_t(x)} = \left(\phi_t^* \omega_{\phi_t(x)}\right) \wedge \left(\phi_t^* \eta_{\phi_t(x)}\right)$

by lemma 4

so $$\mathcal{L}_v(\omega\wedge\eta) = \lim_{t\to 0} \frac{\phi_t^*\omega_{\phi_t(x)}\wedge\phi_t^*\eta_{\phi_t(x)} - \omega_x\wedge\eta_x}{t}$$

$$= \lim_{t\to 0} \frac{\phi_t^*\omega_{\phi_t(x)}\wedge\phi_t^*\eta_{\phi_t(x)} - \omega_x\wedge\phi_t^*\eta_{\phi_t(x)} + \omega_x\wedge\phi_t^*\eta_{\phi_t(x)} - \omega_x\wedge\eta_x}{t}$$

$$= \lim_{t\to 0}\left[\left(\frac{\phi_t^*\omega_{\phi_t(x)} - \omega_x}{t}\right)\wedge\phi_t^*\eta_{\phi_t(x)} + \omega_x\wedge\left(\frac{\phi_t\eta_{\phi_t(x)} - \eta_x}{2}\right)\right]$$

$$= \mathcal{L}_v\omega\wedge\eta + \omega\wedge\mathcal{L}_v\eta$$

3) $$\mathcal{L}_v(\iota_w\omega) = \lim_{t\to 0}\frac{\phi_t^*(\iota_w\omega)_{\phi_t(x)} - (\iota_w\omega)_x}{t}$$

$$= \lim_{t\to 0}\frac{\phi_t^*(\iota_w\omega)_{\phi_t(x)} - \iota_w\phi_t^*\omega_{\phi_t(x)} + \iota_w\phi_t^*\omega_{\phi_t(x)} - (\iota_w\omega)_x}{t}$$

<u>note</u>: $\iota_w\phi_t^*\omega_{\phi(x)}(v_1,\dots,v_{k-1})$

$$= \omega_{\phi_t(x)}(d\phi_t(w), d\phi_t(v_1),\dots, d\phi_t(v_{k-1}))$$

$$= \phi_t^*\left(\iota_{d\phi_t(w)}\omega\right)(v_1,\dots,v_{k-1})$$

$$= \lim_{t\to 0}\left[\phi_t^*\left(\iota_{\left(\frac{w-d\phi_t(w)}{t}\right)}\right)\omega_x + \iota_w\left(\frac{\phi_t^*\omega_{\phi_t(x)} - \omega_x}{t}\right)\right]$$

$$= \iota_{\mathcal{L}_v w}\,\omega + \iota_w\mathcal{L}_v\omega$$

recall $\mathcal{L}_v w = \dfrac{(d\phi_{-t})(w) - w}{2}$

but note $\psi_t = \phi_{-t}$ is flow of $-v$

so the above is $-\mathcal{L}_{-v} w = \mathcal{L}_v w$ ✓

4) <u>by induction:</u>

function

$k=0$: $\mathcal{L}_v \omega = \mathcal{L}_v \omega$

$k=1$: $\mathcal{L}_v(\omega(v_1)) = \mathcal{L}_v(\iota_{v_1} \omega)$

$$= \iota_{\mathcal{L}_v v_1} \omega + \iota_{v_1} \mathcal{L}_v \omega$$

$$= (\mathcal{L}_v \omega)(v_1) + \omega(\mathcal{L}_v v_1)$$

assume true for $k-1$, now let ω be a k-form

$$\mathcal{L}_v(\omega(v_1, \dots, v_k)) = \mathcal{L}_v((\iota_{v_1} \omega)(v_2, \dots, v_k))$$

by induction

$$= (\mathcal{L}_v(\iota_{v_1} \omega))(v_2, \dots, v_k) + \sum_{i=2}^{k} (\iota_{v_1} \omega)(v_2, \dots \mathcal{L}_v v_i, \dots v_k)$$

by 3)

$$= (\iota_{\mathcal{L}_v v_1} \omega)(v_2, \dots v_k) + \iota_{v_1}(\mathcal{L}_v \omega)(v_2, \dots, v_k) + \sum_{i=2}^{k} (\iota_{v_1} \omega)(v_2, \dots, \mathcal{L}_v v_i, \dots, v_k)$$

$$= \mathcal{L}_v \omega(v_1, v_2, \dots, v_k) + \sum_{i=1}^{k} \omega(v_1, \dots, \mathcal{L}_v v_i, \dots, v_k)$$

▦

<u>Cor 10:</u>

for functions f

$$\boxed{\mathcal{L}_v(df) = d \mathcal{L}_v f}$$

Proof: by lemma 9.4) we know

$$\big(\mathcal{L}_v(df)\big)(w) = \mathcal{L}_v(df(w)) - df([v,w])$$
$$= v.w.f - \big[v.(w.f) - w.(v.f)\big]$$
$$= d(\mathcal{L}_v f)(w)$$

Thm 11:

> for any k-form ω and vector field v
> $$\mathcal{L}_v \omega = d\iota_v \omega + \iota_v d\omega$$

Cartan's Magic Formula

Proof: induct on $k = |\omega|$

$k=0$: $\iota_v df + d\cancel{\iota_v f}^{\,0} = df(v) = v.f = \mathcal{L}_v f$

$k=1$: locally any 1-form is a sum of terms

$f\,dg$ for functions f, g

thus since both sides are linear and local

we just need to check for $f\,dg$

$$\mathcal{L}_v(f\,dg) \underset{\text{by 9.2)}}{=} (\mathcal{L}_v f)\,dg + f(\mathcal{L}_v dg)$$

$$\underset{\text{by 10}}{=} (\mathcal{L}_v f)\,dg + f\,d(\mathcal{L}_v g)$$

$$= (v.f)\,dg + f\,d(v.g)$$

and we have

$$\iota_v\, d(f\,dg) + d\,\iota_v(f\,dg) = \iota_v(df\wedge dg) + d(f\,dg(v))$$

$$= df(v)\,dg - \cancel{dg(v)}\,df + \cancel{dg(v)}\,df + f\,d(dg(v))$$

$$= (v\cdot f)\,dg + f\,d(v\cdot g) = \mathcal{L}_v(f\,dg)$$

now for $k>1$

locally $\omega = \sum \omega_I\, dx^I$

set $\alpha = \omega_I\, dx^{i_1}$, $\beta = dx^{i_2}\wedge\dots\wedge dx^{i_k}$

then ω is a sum of terms of the form

$$\underbrace{\alpha}_{1\text{-form}} \wedge \underbrace{\beta}_{k-1\text{ form}}$$

so by linearity and locality suffices to check on terms $\alpha\wedge\beta$

$$(\iota_v d + d\iota_v)\,\alpha\wedge\beta = \iota_v(d\alpha\wedge\beta - \alpha\wedge d\beta) + d\big((\iota_v\alpha)\beta - \alpha\wedge\iota_v\beta\big)$$

$$= (\iota_v d\alpha)\wedge\beta + \cancel{d\alpha\wedge\iota_v\beta} - \cancel{(\iota_v\alpha)\wedge d\beta} + \alpha\wedge\iota_v d\beta$$
$$+ (d(\iota_v\alpha))\wedge\beta + \cancel{(\iota_v\alpha)\wedge d\beta} - \cancel{d\alpha\wedge\iota_v\beta} + \alpha\wedge d(\iota_v\beta)$$

$$\underset{\text{induction}}{=} \mathcal{L}_v\alpha\wedge\beta + \alpha\wedge\mathcal{L}_v\beta = \mathcal{L}_v(\alpha\wedge\beta)$$

Cor 12: $\boxed{\mathcal{L}_v d = d\,\mathcal{L}_v}$

Proof:

$$\mathcal{L}_v(d\omega) = \iota_v \cancelto{0}{d(d\omega)} + d\iota_v(d\omega)$$
$$= d\iota_v(d\omega) + \cancelto{0}{dd\iota_v}\,\omega$$
$$= d(\mathcal{L}_v\omega) \qquad \blacksquare$$

Exercise:

$$\mathcal{L}_v\mathcal{L}_w - \mathcal{L}_w\mathcal{L}_v = \mathcal{L}_{[v,w]}$$

to "geometrically" see what the Lie derivative is telling us we have

Th$^{\underline{m}}$ 13:

$$\boxed{\begin{array}{c}\text{let } v \text{ be a vector field on a manifold} \\ \text{Then a } k\text{-form } \omega \text{ is invariant under} \\ \text{the flow of } v \text{ (i.e. } \phi_t^*\omega = \omega) \\ \Longleftrightarrow \\ \mathcal{L}_v\omega = 0\end{array}}$$

Proof:

($\Rightarrow$) obviously $\frac{d}{dt}\Big|_{t=0} \phi_t^*\,\omega_{\phi_t(x)} = \frac{d}{dt}\Big|_{t=0}\omega_x = 0$ ✓

($\Leftarrow$) need a lemma

lemma 14:

let $v \in \mathfrak{X}(M)$ and $\phi_t : M \to M$ its flow
for $\alpha \in \Omega^k(M)$ (or even $T^k(M)$) we have

$$\frac{d}{dt}\Big|_{t=t_0}\left(\phi_t^* \alpha_{\phi_t(x)}\right) = \phi_{t_0}^*\left(\left(\mathcal{L}_v \alpha\right)\Big|_{\phi_{t_0}(x)}\right)$$

we first finish proof of Th^m 13:

if $\mathcal{L}_v \omega = 0$, then

$$\frac{d}{dt}\Big|_{t=s} \phi_t^* \omega_{\phi_t(x)} = \phi_s^*(0) = 0$$

thus for a fixed $x \in M$ constant

let $\gamma : (-\varepsilon, \varepsilon) \longrightarrow \Lambda^k(T_x^* M)$ ← vector space

$$t \longmapsto \phi_t^* \omega_{\phi_t(x)}$$

from above we see

$$\gamma'(s) = \frac{d}{dt}\Big|_{t=s} \phi_t^* \omega_{\phi_t(x)} = 0$$

$\therefore \gamma'$ is constant i.e. $\phi_t^* \omega = \omega$

Proof of lemma 14:

$$\frac{d}{dt}\Big|_{t=t_0} \phi_t^* \alpha_{\phi_t(x)} \overset{t=t_0+s}{=} \frac{d}{ds}\Big|_{s=0} \phi^*_{s+t_0} \alpha_{\phi_{s+t_0}(x)}$$

$$= \frac{d}{ds}\Big|_{s=0} \phi_{t_0}^* \left(\phi_s^* \alpha_{\phi_s(\phi_{t_0}(x))} \right)$$

$$= \phi_{t_0}^* \frac{d}{ds}\Big|_{s=0} \left(\phi_s^* \alpha_{\phi_s(\phi_{t_0}(x))} \right)$$

$$= \phi_{t_0}^* (\mathcal{L}_v \alpha)_{\phi_{t_0}(x)}$$

<u>example:</u>

set $$v = \frac{1}{r + \cos r \sin r} \left((\sin r + r\cos r) \frac{\partial}{\partial z} + \sin r \frac{\partial}{\partial \theta} \right)$$

and $\alpha = \cos r \, dz + r \sin r \, d\theta$

note: $$\mathcal{L}_v \alpha = d\iota_v \alpha + \iota_v d\alpha$$
$$= d(1) + \iota_v(-\sin r \, dr \wedge dz + (\sin r + r\cos r) dr \wedge d\theta)$$
$$= 0$$

so the flow of v preserves α

<u>exercise:</u> Show if α_t is a time dependent 1-form, then

$$\frac{d}{dt} \phi_t^* \alpha_t = \phi_t^* \left(\mathcal{L}_v \alpha + \frac{d\alpha_t}{dt} \right)$$