XI Integration

A Orientations

let V be a vector space

we say two ordered bases

$v_1, \dots, v_n$ and

$w_1, \dots, w_n$

of V give the same orientation if the unique linear map $L : V \to V$ satisfying $L v_i = w_i$ has positive determinant

exercise: 1) "gives the same orientation" is an equivalence relation

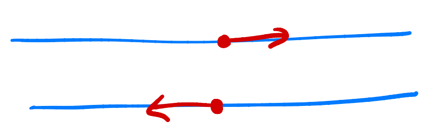
2) $\exists$ exactly 2 equivalence classes of bases

a choice of equivalence class is called an orientation on V

(for 0-dim'l V we just call a choice of $\pm$ for $\{0\}$ an orientation)

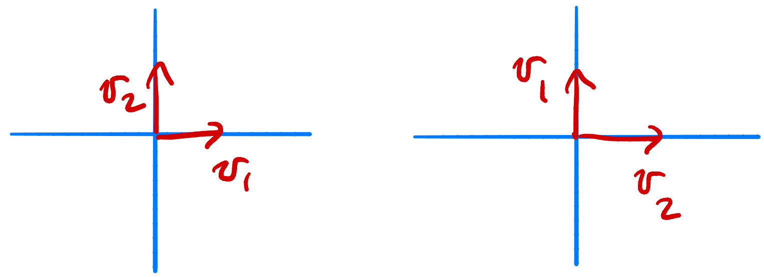
examples:

$\mathbb{R}^1$



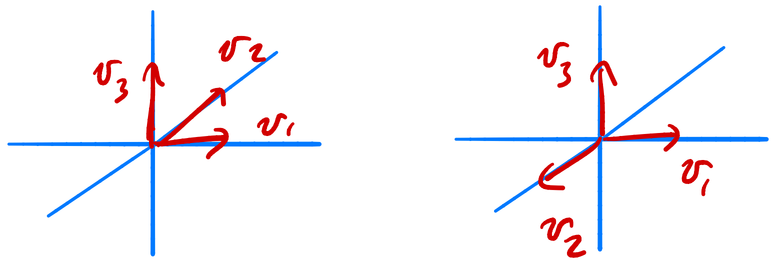
represent the two orientations

$\mathbb{R}^2$



represent the two orientations

$\mathbb{R}^3$



represent the two orientations

<u>note:</u> a basis for V gives a basis for V^*

so an orientation on V gives one on V^*

similarly for $T^k_\ell V$ and $\Lambda^\ell V$

if $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$ are bases for V

and $L : V \to V$ is the linear map $L f_i = e_i$

then using the basis $\{f_i\}$ we can

express L as a matrix $M = (m^i_j)$ where

$$e_i = \sum m^j_i f_j$$

we saw in lemma X.2 that

$$e^1 \wedge \ldots \wedge e^n = (\det M) f^1 \wedge \ldots \wedge f^n$$

exercise: given $\omega \in \Lambda^n V$ with $\omega \neq 0$, the set of ordered bases $e_1, \dots, e_n$ such that $\omega(e_1, \dots, e_n) > 0$ form an orientation on V

Recall: $\Lambda^n V \cong \mathbb{R}$

so a choice of component of

$$\Lambda^n V - \{0\}$$

is equivalent to an orientation on $\Lambda^n V$ and hence on V

now consider a manifold M

$$\text{let } \mathcal{O} = \Lambda^n M - \{\mathbb{O}\}$$

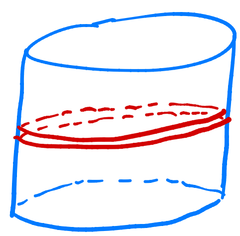
image of zero section

$$= \coprod_{x \in M} \left(\Lambda^n (T_x M) - \{0\} \right)$$

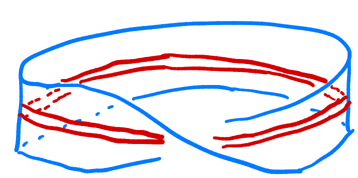
$\mathcal{O}$ is an $(\mathbb{R} - \{0\})$-bundle over M

fiberwise $\mathcal{O}$ has 2 components, so if M is connected then $\mathcal{O}$ has either 1 or 2 components

example:



2 components



1 component

if $\mathcal{O}$ has 2 components then we say M is orientable and a choice of component is called an orientation on M (note this gives an orientation on each $T_x M$ for all $x \in M$)

note: if M is connected and orientable, then an orientation on M is determined by a choice of component of $\mathcal{O}$ above one point $x \in M$

i.e. by a nonzero elt of $\Lambda^n(T_x M)$

i.e. by a choice of oriented basis of $T_x M$

If M, N are oriented n-manifolds, then a map $f: M \to N$ is called orientation preserving if it takes the component of $\mathcal{O}_N$ defining the orientation to the component of $\mathcal{O}_M$ defining the orientation

i.e. df takes a correctly oriented basis for $T_x M$ to a correctly oriented basis for $T_{f(x)} N$.

Th$^{\underline{m}}$ 1:

M a smooth manifold. The Following Are Equivalent

(a) M orientable

(b) $\exists$ a collection of charts $\{\phi_\alpha : U_\alpha \to V_\alpha\}$ for M such that $\{U_\alpha\}$ cover M

and $\det\left[d(\phi_\beta \circ \phi_\alpha^{-1})\right] > 0$

whenever defined

(c) $\exists$ a nonzero n-form on M

Proof:

(a) $\Rightarrow$ (b)

let $\{\phi_\alpha : U_\alpha \to V_\alpha\}$ be all charts s.t. ϕ_α is orientation preserving where $V_\alpha \subset \mathbb{R}^n$ has the orientation $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$

note: $\{U_\alpha\}$ clearly cover M since given any $\psi : U \to V$ that is not orientation preserving consider $\tilde{\psi} = r \circ \psi : U \to r(V)$ where $r(x^1, \dots, x^n) = (x^1, \dots, x^n, x^{n-1})$

clearly $\det(d(r \circ \psi)) = (\det dr)(\det d\psi) > 0$

but now we clearly have $\det(d\, \phi_\alpha^{-1} \circ \phi_\beta) > 0$ ✓

(b) $\Rightarrow$ (c)

let $\{\phi_\alpha : U_\alpha \to V_\alpha\}$ be a collection of charts as in (b)

recall we can find a partition of unity $\{\psi_\alpha : U_\alpha \to \mathbb{R}\}$ subordinate to $\{U_\alpha\}$

i.e. • $0 \le \psi_\alpha(x) \le 1$,

• support $\psi_\alpha \subset U_\alpha$,

• $\forall x, \exists$ nbhd V_x s.t. $\psi_\alpha|_{V_x} = 0$ for all but finitely many α, and

• $\sum_\alpha \psi_\alpha(x) = 1$

now $dx^1 \wedge \dots \wedge dx^n$ is nonzero n-form on $\mathbb{R}^n$

$\therefore$ on V_α for all α

and

$$\psi_\alpha \left[\phi_\alpha^* (dx^1 \wedge \dots \wedge dx^n) \right]$$

is an n-form on M

set

$$\omega = \sum_\alpha \psi_\alpha \left[\phi_\alpha^* (dx^1 \wedge \dots \wedge dx^n) \right]$$

this is clearly an n-form on M

exercise: ω is never zero

Hint: $\omega(x)$ is the sum of finitely many nonzero forms

the condition $d(\phi_\alpha \circ \phi_\beta^{-1}) > 0$ says you are summing positive things

(c) $\Rightarrow$ (a)

given ω as in (c) let

$$\Lambda^+ = \bigcup_{x \in M} \{ a\,\omega(x) \mid a \in \mathbb{R} \;\; a > 0 \}$$

$$\Lambda^- = \bigcup_{x \in M} \{ a\,\omega(x) \mid a \in \mathbb{R} \;\; a < 0 \}$$

clearly $\Lambda^+ \cap \Lambda^- = \varnothing$

$$\mathcal{O}_M = \Lambda^+ \amalg \Lambda^-$$

so $\mathcal{O}_M$ is disconnected

exercise: $f: M \to M$

$\omega \in \Omega^n(M)$ gives an orientation on M

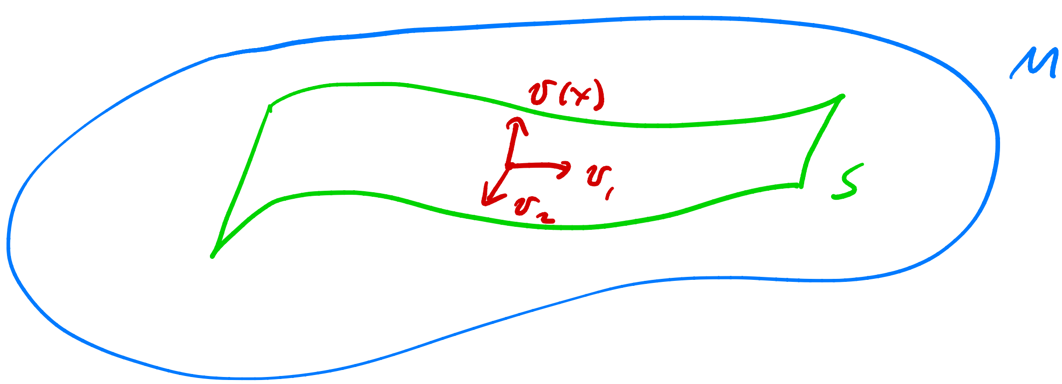
then f is orientation preserving

$\iff$

$f^*\omega(x)$ is a positive multiple $\omega_{f(x)}$

lemma 2:

> M a smooth oriented n-manifold
>
> S a smooth submanifold of M of codim 1
>
> v a vector field along S that is transverse to S
>
> (i.e. v is a section of $TM|_S$ and $T_x S$ and $v(x)$ span $T_x M$)
>
> Then $\exists$ a unique orientation on S such that
>
> $(v_1, \dots, v_{n-1})$ is an oriented basis for $T_x S$
>
> $\iff$
>
> $(v(x), v_1, \dots, v_{n-1})$ is an oriented basis for $T_x M$



Proof: let Ω be an n-form on M that gives the orientation on M

let $\omega = (\iota_v \Omega)|_S$ this is an $(n-1)$-form on S

one can easily check $\omega \neq 0$ on S

moreover $\omega(v_1, \dots, v_{n-1}) > 0 \iff \Omega(v(x), v_1, \dots, v_{n-1}) > 0$

examples:

1) $S^n \subset \mathbb{R}^{n+1}$

let $v = \sum x^i \frac{\partial}{\partial x^i}$

v is transverse to S^n

$\Omega = dx^1 \wedge \dots \wedge dx^{n+1}$ orients $\mathbb{R}^{n+1}$

so $\omega = \iota_v \Omega = \sum (-1)^{i-1} dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^{n+1}$

orients S^n

exercise:

let $r: \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} : p \mapsto -p$

(note $r|_{S^n} : S^n \to S^n$)

show $r^* \omega = (-1)^{n+1} \omega$

so $r|_{S^n}$ is orientation preserving

$\iff$

n is odd

2) $\mathbb{RP}^2$ is not oriented

If it were then there is a nonzero 2-form ω giving orientation

let $p: S^2 \longrightarrow \mathbb{R}P^2$

$(x^1, x^2, x^3) \longmapsto [x^1 : x^2 : x^3]$

(note: $p^{-1}(x) = \{y, r(y)\}$ for y s.t. $p(y) = x$

so if we define equivalence relation on S^2 by

$x \sim y \Leftrightarrow r(x) = y$

then $\mathbb{R}P^2 = S^2/\sim$)

now $\tilde{\omega} = p^*\omega$ is a nonzero 2-form on S^2

(check this, but p is a local diffeo. so clear)

from above

$$\begin{tikzcd} S^2 \arrow[rr, "r"] \arrow[dr, "p"'] & & S^2 \arrow[dl, "p"] \\ & \mathbb{R}P^2 & \end{tikzcd}$$

so $p^* = (p \circ r)^* = r^* \circ p^*$

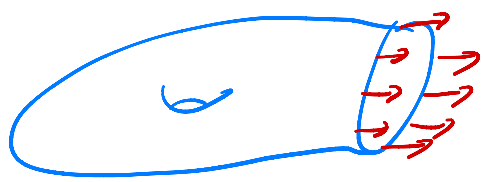
$\therefore \tilde{\omega} = p^*\omega = r^*(p^*\omega) = r^*\tilde{\omega}$

$\therefore$ r is orientation preserving

$\otimes$ above exercise !

$\therefore$ $\mathbb{RP}^2$ is not oriented

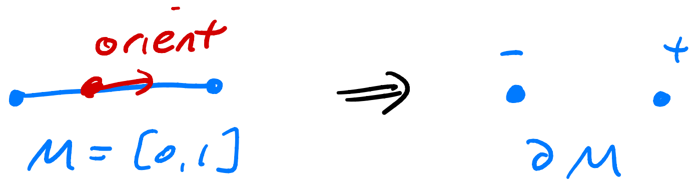
Recall in the proof of Thm V.3 we saw that if M is a manifold with boundary then there is a vector field v on M such that $v(x)$ points <u>out</u> of M for all $x \in \partial M$

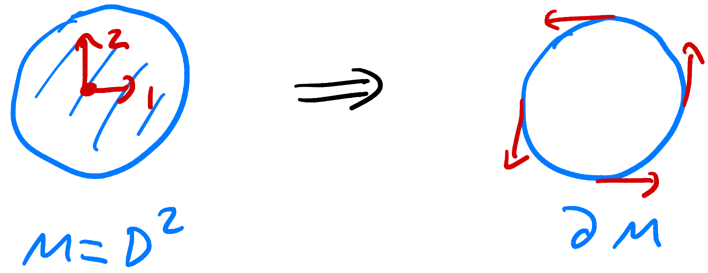


thus if M is an oriented manifold with boundary, then there is an orientation induced on ∂M using an outward pointing vector field along ∂M (as in lemma 2)

i.e. if Ω gives an orientation on M and v is an outward pointing vector field then $\omega = \iota_v \Omega|_{\partial M}$ orients ∂M

examples:





B. Integration

let $x^1, \dots, x^n$ be standard coordinates on $\mathbb{R}^n$

$\Lambda^n \mathbb{R}^n \cong \mathbb{R}$ so an n-form looks like

$$\omega = f(x^1, \dots, x^n)\, dx^1 \wedge \dots \wedge dx^n$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a function

for any set $A \subset \mathbb{R}^n$ we define

$$\int_A \omega = \underbrace{\int_A f \, d\mathrm{vol}}_{\text{from vector calculus}}$$

<u>note</u>: if $\underset{x^1 \dots x^n}{\phi : \mathbb{R}^n} \to \underset{y^1 \dots y^n}{\mathbb{R}^n}$ is a coord change (i.e. local diffeo)

and $\omega = f(y)\, dy^1 \wedge \dots \wedge dy^n$ then

$$\int_{\phi(A)} \omega = \int_{\phi(A)} f \, d\mathrm{vol}_{y^1 \dots y^n}$$

vector calc change of variables

$$= \int_A (f\circ\phi)\,|\text{Jacobian } \phi|\, dvol_{x^1\dots x^n}$$

depending of ϕ orientation preserving or reversing

$$= \pm \int_A (f\circ\phi)\ \det(d\phi)\ dvol_{x^1\dots x^n}$$

$$= \pm \int_A (f\circ\phi)(\det d\phi)\ dx^1\wedge\dots\wedge dx^n$$

$$= \pm \int_A f^*\omega$$

so if ϕ is orientation preserving, then

$$\int_{\phi(A)} \omega = \int_A \phi^*\omega$$

now suppose $\omega \in \Omega^n(M)$

and M is oriented

if $\operatorname{supp}(\omega) \subset$ coord chart U, $\phi: U \to V$

then define

$$\int_M \omega = \int_{\mathbb{R}^n} (\phi^{-1})^*\omega$$

from above this is well-defined as long as ϕ is orientation preserving

now if M is oriented a $\{\phi_\alpha : U_\alpha \to V_\alpha\}_{\alpha\in A}$

is a set of oriented charts s.t.

$\{U_\alpha\}_{\alpha \in A}$ cover M (exist by Th$^{\underline{m}}$ 1)

then let $\{\psi_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$

for any $\omega \in \Omega^n(M)$ define

$$\int_M \omega = \sum_{\alpha \in A} \int_M \psi_\alpha \omega$$

supported in coord chart!

$$= \sum_{\alpha \in A} \int_{\mathbb{R}^n} (\phi_\alpha^{-1})^* (\psi_\alpha \omega)$$

<u>lemma 3:</u>

$\int_M \omega$ is independent of cover $\{U_\alpha\}$ and partition of unity

i.e. $\int_M \omega$ is well-defined

<u>Proof</u>: let $\{\tilde{\phi}_\beta : U_\beta \to V_\beta\}$ and $\{\tilde{\psi}_\beta\}$ be another oriented cover of M and partition of unity

$$\int_M \psi_\alpha \omega = \int_M \Big(\sum_\beta \tilde{\psi}_\beta\Big) \psi_\alpha \omega = \sum_\beta \int_M \tilde{\psi}_\beta \psi_\alpha \omega$$

so $\int_M \omega = \sum_{\alpha,\beta} \int_M \tilde{\varphi}_\beta \varphi_\alpha \omega$

note $\tilde{\varphi}_\beta \varphi_\alpha \omega$ is supported in U_α and $\tilde{U}_\beta$

so $\int_M \tilde{\varphi}_\beta \varphi_\alpha \omega$ is the same wheather we use U_α and $\tilde{U}_\beta$ chart. So $\int_M \omega$ independent of cover and partition

we can also define for $A \subset M$

$$\int_A \omega = \sum_\alpha \int_{\phi_\alpha(U_\alpha \cap A)} (\phi_\alpha^{-1})^* \varphi_\alpha \omega$$

Thm 4:

integrals satisfy

1) $\int_M a\omega + b\eta = a\int_M \omega + b\int_M \eta$

2) if $\overline{M}$ is M with its opposite orientation then $\int_{\overline{M}} \omega = -\int_M \omega$

3) if $f: N \to M$ is an orientation preserving diffeomorphism, then $\int_N f^*\omega = \int_M \omega$

4) if $M = M_1 \cup M_2$ and $M_1 \cap M_2$ is a $(\dim M)-1$ manifold in M_1 then

$$\int_M \omega = \int_{M_1} \omega + \int_{M_2} \omega$$

5) If $A \subset M$ has measure zero then

$$\int_M \omega = \int_{M-A} \omega$$

Proof: exercise (most are almost obvious)

example: let $\alpha = x\,dy - y\,dx$ on $\mathbb{R}^2$

and restrict α to $S^1 \subset \mathbb{R}^2$ unit S^1

compute $\int_{S^1} \alpha$

recall sets of measure zero can be ignored so

$$\int_{S^1} \alpha = \int_{S^1 - \{(1,0)\}} \alpha$$

now $f : (0, 2\pi) \to S^1$

$\theta \mapsto (\cos\theta, \sin\theta)$

gives a coordinate chart (actually f^{-1} is) so $$\int_{S^1} \alpha = \int_{S^1 - \{(1,0)\}} \alpha = \int_{(0,2\pi)} f^* \alpha$$

$$= \int_{(0,2\pi)} \cos\theta (\cos\theta) d\theta + \sin\theta (\sin\theta) d\theta$$

$$= \int_0^{2\pi} 1 \, d\theta = 2\pi$$

to integrate on M^n we need M to be oriented, an orientation is given by a never-zero n-form

we call a never-zero n-form ω a volume form on M

(it is sometimes denoted dvol)

bad notation since not necessarily d of something but standard notation

now given a volume form dvol on M we can integrate functions $f: M \to \mathbb{R}$ as follows
$$\int_M f = \int_M f \, dvol$$

we say the volume of M is

$$\int_M d\mathrm{vol} \quad \left(\text{or } \mathrm{Vol}_\omega(M) = \int_M \omega\right)$$

now given a k-form $\alpha \in \Omega^k(M)$ on M we cannot integrate α on M but given any k-dimensional oriented submanifold Σ of M let

$$\iota : \Sigma \to M$$

be the inclusion map and define

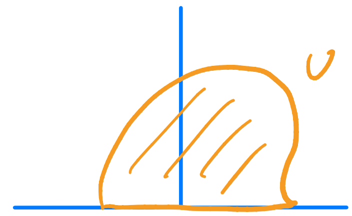
$$\int_\Sigma \alpha = \int_\Sigma \iota^* \alpha$$

C. Stokes' Theorem

Thm 5 (Stokes' Theorem):

> let M be a smooth oriented n-manifold with boundary
> let $\beta \in \Omega^{n-1}(M)$ be an $(n-1)$-form on M that is compactly supported, then
> $$\int_M d\beta = \int_{\partial M} \beta$$

Proof: consider the case where $M = \mathbb{R}^n_{\geq 0}$ and β has compact support



suppose $\operatorname{supp}\beta \cap \partial(\mathbb{R}^n_{\geq 0}) = \varnothing$

then clearly $\int_{\partial M} \beta = 0$

and we have

$$\int_M d\beta = \int_{\mathbb{R}^n_{\geq 0}} d\left(\sum \beta_i \, dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^n\right)$$

$$= \int_{\mathbb{R}^n_{\geq 0}} \sum (-1)^{i-1} \frac{\partial \beta_i}{\partial x^i} \, dx^1 \wedge \dots \wedge dx^n$$

$$= \sum (-1)^{i-1} \underbrace{\int_{\mathbb{R}^n_{\geq 0}} \frac{\partial \beta_i}{\partial x^i} \, dx^1 \wedge \dots \wedge dx^n}$$

by Fubini's Th$^{\underline{m}}$ integrate w.r.t. x^i first then the fundamental theorem of calculus gives

$$\int_{-\infty}^{\infty} \frac{\partial \beta_i}{\partial x^i} \, dx^i = 0$$

$$= \underline{0} \checkmark$$

in general we have

as above all $i \neq n$ terms 0

$$\int_M d\beta = (-1)^{n-1} \int_{\mathbb{R}^n_{\geq 0}} \frac{\partial \beta_n}{\partial x^n}\, dx^1 \wedge \dots \wedge dx^n$$

$$= (-1)^{n-1} \int_{\mathbb{R}^{n-1}} \Big(\underbrace{\int_0^\infty \frac{\partial \beta_n}{\partial x^n}\, dx^n}_{\substack{\shortparallel \\ \cancel{\beta_n(x^1, \dots, x^{n-1}, \infty)} \to 0 \; - \; \beta_n(x^1, \dots, x^{n-1}, 0)}} \Big)\, dx^1 \wedge \dots \wedge dx^{n-1}$$

$$= (-1)^n \int_{\mathbb{R}^{n-1}} \beta_n(x^1, \dots x^{n-1}, 0)\, dx^1 \wedge \dots \wedge dx^{n-1}$$

Recall: $dx^1 \wedge \dots \wedge dx^n$ orients $\mathbb{R}^n_{\geq 0}$

so orientation on $\partial \mathbb{R}^n_{\geq 0}$ is given by "outward normal first"

i.e. $\iota_{-\frac{\partial}{\partial x^n}} dx^1 \wedge \dots \wedge dx^n = (-1)(-1)^{n-1} dx^1 \wedge \dots \wedge dx^{n-1}$

$$= (-1)^n dx^1 \wedge \dots \wedge dx^{n-1}$$

but orientation on $\mathbb{R}^{n-1}$ given by

$dx^1 \wedge \dots \wedge dx^{n-1}$

so to compute $\int_{\partial M} \beta = \int_{\partial \mathbb{R}^n_{\geq 0}} \beta$ use chart

$f : V \longrightarrow (\partial \mathbb{R}^n_{\geq 0} \cap M) : (y^1, \dots, y^{n-1}) \mapsto (y^1, \dots, y^{n-1}, 0)$

and multiply by $(-1)^n$ to account for the or$^{\underline{n}}$

so $$\int_{\partial M}\beta = \int_V f^*\beta = \int_V f^*\left(\sum \beta_i\, dx^1\wedge\cdots\wedge\widehat{dx^i}\wedge\cdots\wedge dx^n\right)$$

$$= \int_{\mathbb{R}^{n-1}} (-1)^n \beta_n(y^1,\dots,y^{n-1},0)\, dy^1\wedge\cdots\wedge dy^{n-1}$$

all terms but $i=n$ are zero

$$= \int_M d\beta$$

now if β is compactly supported in a coordinate chart of M, then the above computation shows $\int_M d\beta = \int_{\partial M}\beta$

for a general β, choose a collection of coordinate charts $\{\phi_i : U_i \to \mathbb{R}^n_{\geq 0}\}$ such that $M = \bigcup U_i$ and

a partition of unity $\{\psi_i : M \to \mathbb{R}\}$ supordinate to the cover $\{U_i\}$

now
$$\int_{\partial M}\beta = \sum_i \int_{\partial M}\psi_i\beta = \sum_i \int_M d(\psi_i\beta)$$

$$= \sum_i \int_M d\psi_i\wedge\beta + \psi_i\, d\beta$$

$$= \int_M \Big(d\underbrace{\left(\sum\psi_i\right)}_{=1}\wedge\beta + \left(\sum\psi_i\right)d\beta\Big)$$

$$= \int_M 0 + \int_M d\beta = \int_M d\beta$$

Cor 6:

M a smooth compact n-manifold

1) if $\partial M = \emptyset$, then

$$\int_M d\omega = 0 \quad \forall \omega \in \Omega^{n-1}(M)$$

2) if ω is a closed $(n-1)$-form on M, then

$$\int_{\partial M} \omega = 0$$

3) if ω is a closed k-form on M and S is an oriented k-dimensional submanifold of M w/o boundary then

$$\int_S \omega \neq 0$$

implies ω is <u>not</u> exact and S does <u>not</u> bound a $(k+1)$ dimensional submanifold of M

Proof:

1) $\int_M d\omega = \int_{\partial M} \omega = \int_{\emptyset} \omega = 0$

2) $\int_{\partial M} \omega = \int_M d\omega = \int_M 0 = 0$

3) if ω is exact then $\exists \alpha$ s.t. $d\alpha = \omega$

so $\int_S \omega = \int_S d\alpha = \int_{\partial S} \alpha = \int_{\emptyset} \alpha = 0$

if $S = \partial N$, then
$$\int_S \omega = \int_N d\omega = \int_N 0 = 0$$

<u>note:</u> if $f : \Sigma^k \to M^n$ is a smooth map, then we can consider $\int_\Sigma f^* \alpha$ for any $\alpha \in \Omega^k(M)$

<u>Cor 7:</u>

> if $f_0, f_1 : \Sigma \to M$ are smooth maps (Σ an oriented k-manifold)
> that are homotopic rel $\partial \Sigma$
> and α is a closed k-form on M
> then $$\int_\Sigma f_0^* \omega = \int_\Sigma f_1^* \omega$$

<u>Proof:</u> f_0, f_1 homotopic $\Rightarrow \exists$ smooth homotopy
$$F : \Sigma \times [0,1] \to M$$
s.t. $F(x, i) = f_i(x) \qquad i = 0, 1$

note $\Sigma \times [0,1]$ is oriented by (or$^{\underline{n}}$ on Σ, $\frac{\partial}{\partial t}$)

and $\partial(\Sigma \times [0,1]) = (-\Sigma \times \{0\} \cup \Sigma \times \{1\}) \cup (\partial \Sigma \times [0,1])$

so $0 = \int_{\Sigma \times [0,1]} 0 = \int_{\Sigma \times [0,1]} F^* d\alpha = \int_{\Sigma \times [0,1]} d F^* \alpha$

$$= \int_{\partial(\Sigma \times [0,1])} F^* \alpha$$

$$= \int_{-\Sigma \times \{0\}} F^* \alpha + \int_{\Sigma \times \{1\}} F^* \alpha + \int_{\partial\Sigma \times [0,1]} F^* \alpha$$

<u>note:</u> $dF(\frac{\partial}{\partial t}) = 0$ along $\partial\Sigma \times [0,1]$ since homotopy doesn't move $\partial\Sigma$, thus $F^*\alpha = 0$ on $\partial\Sigma \times [0,1]$

$$= \int_{-\Sigma} f_0^* \alpha + \int_{\Sigma} f_1^* \alpha$$

$$= -\int_{\Sigma} f_0^* \alpha + \int_{\Sigma} f_1^* \alpha$$

<u>example:</u>

on $S^1 \times S^1$ consider $\alpha = d\theta$
(θ, ϕ)

$\cap$

$\mathbb{R}^2 \times \mathbb{R}^2$
(x,y) (u,v)

$\theta = \tan^{-1} y/x \qquad \phi = \tan^{-1} v/u$

$$d\theta = \frac{-y/x^2}{1+(y/x)^2}\,dx + \frac{1/x}{1+(y/x)^2}\,dy$$

$$= \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$

on S^1 = unit circle

$$= -y\,dx + x\,dy$$

<u>note</u>: on S^1 $d\alpha = 0$ since its a 2-form

so on $S^1 \times S^1$ $d\alpha = 0$

let $f : S^1 \to S^1 \times S^1 : \psi \mapsto (\psi, (1,0))$

now $f^*\alpha = d\psi$

so $\int_{S^1} f^*\alpha = \int_{S^1} d\psi = 2\pi \neq 0$

so f is <u>not</u> homotopic to a constant function

<u>note</u>: we have once again shown T^2 is not simply connected and thus not $\cong S^2$