D DeRham Cohomology

Recall, from Section X C, on an n-manifold M we have

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \to \Omega^n(M)$$

and $$d^2 = 0$$

so we get

$$H^k_{DR}(M) = \frac{\ker(d : \Omega^k(M) \to \Omega^{k+1}(M))}{\operatorname{im}(d : \Omega^{k-1}(M) \to \Omega^k(M))}$$

$$= \frac{\text{closed } k\text{-forms}}{\text{exact } k\text{-forms}}$$

and given

$$f : M \to N$$

we get a map

$$f^* : H^k_{DR}(N) \to H^k_{DR}(M)$$
$$[\eta] \mapsto [f^*\eta]$$

since $f^* \circ d = d \circ f^*$

from the properties of f^* and Ω^k we saw

$$(f \circ g)^* = g^* \circ f^* \quad \text{and}$$

$$(id_M)^* = id_{H^k_{DR}(M)}$$

so diffeomorphic manifolds have the same cohomology

we also saw

a) $H^0_{DR}(M) \cong \mathbb{R}^{\#\text{ components of } M}$

(= locally constant functions)

b) $H^k_{DR}(M) = 0$ if $k > n$ or $k < 0$

Another computation we can do is for $M = \mathbb{R}$

$$0 \to \underset{\substack{\parallel \\ C^\infty(M)}}{\Omega^0(M)} \to \underset{\substack{\parallel \\ \{f(x)dx\}}}{\Omega^1(M)} \to 0$$

so if $\alpha \in \Omega^1(M)$ then $d\alpha = 0$ (since $\Omega^2(M) = 0$)

so closed forms $= \Omega^1(M)$

now given $\omega = f(x)dx$

let $g(x) = \int_{[0,x]} \omega = \int_0^x f(x)\,dx$

clearly $g : \mathbb{R} \to \mathbb{R}$ is in $\Omega^0(M)$

and $dg = g'(x)\,dx = f(x)dx = \alpha$

fundamental theorem of calculus

so all $\omega \in \Omega^1(M)$ are exact i.e. $H^1_{DR}(\mathbb{R}) = 0$

and we have

$$H^k_{DR}(\mathbb{R}) \cong \begin{cases} \mathbb{R} & k=0 \\ 0 & k \neq 0 \end{cases}$$

In your homework you showed

$$H^k_{DR}(S^1) \cong \begin{cases} \mathbb{R} & k=0,1 \\ 0 & k \neq 0,1 \end{cases}$$

to go further we need

Thm 8:

$$\boxed{\text{If } f,g: M \to N \text{ are homotopic maps, then } f^* = g^* : H^k_{DR}(N) \to H^k_{DR}(M)}$$

$f: M \to N$ is called a homotopy equivalence if it has a homotopy inverse, that is a map $g: N \to M$ such that $f \circ g \simeq id_N$ and $g \circ f \simeq id_M$

homotopic

exercise:

1) $\mathbb{R}^n$ is homotopy equivalent to a point

2) if M, N are homotopy equivalent then

$$H^k_{DR}(M) \cong H^k_{DR}(N)$$

(note $f^* \circ g^*$ and $g^* \circ f^*$ bijective)

so $H^k_{DR}(\mathbb{R}^n) \cong \begin{cases} \mathbb{R} & k=0 \\ 0 & k \neq 0 \end{cases}$

this is called the Poincaré lemma

and says if $\omega \in \Omega^k(\mathbb{R}^n)$ with

$d\omega = 0$ then $\exists\, \eta \in \Omega^{k-1}(\mathbb{R}^n)$

such that $d\eta = \omega$

i.e. can solve the PDE $d\eta = \omega$

for η if (and only if) $d\omega = 0$

3) $S^1 \times \mathbb{R}^n \simeq S^1$

homotopy equivalent

so $H^k_{DR}(S^1 \times \mathbb{R}^n) \cong \begin{cases} \mathbb{R} & k=0,1 \\ 0 & k \neq 0,1 \end{cases}$

<u>note</u>: to see $f^* = g^*$ in theorem we need to

see if $\omega \in \Omega^k(N)$ and $d\omega = 0$

then $f^*\omega = g^*\omega + d\eta$ for some

$\eta \in \Omega^{k-1}(M)$

η will depend on ω so we want

a map $h_k : \Omega^k(N) \to \Omega^{k-1}(M)$

such that
$$f^*\omega = g^*\omega + d\,h_k(\omega)$$
or we could also have
$$\boxed{f^*\omega - g^*\omega = d(h_k(\omega)) + h_{k+1}(d(\omega))}$$
such $\{h_k\}$ are called a chain homotopy from f^* to g^*

a simple case of this is

lemma 9:

let $i_t : M \to M \times [0,1]$
$x \longmapsto (x,t)$

then $\exists\, h_k : \Omega^k(M \times [0,1]) \to \Omega^{k-1}(M)$

s.t. $i_0^* - i_1^* = d \circ h_k + h_{k+1} \circ d$

Proof: for $\omega \in \Omega^k(M \times [0,1])$

let $h_k(\omega) = \int_0^1 (\iota_{\frac{\partial}{\partial t}} \omega)\, dt$

we know ω is a sum of terms

(I) $f(x,t)\, dt \wedge dx^{i_1} \wedge \dots \wedge dx^{i_{k-1}}$ and

(II) $f(x,t)\, dx^{i_1}\wedge\dots\wedge dx^{i_k}$

the integral above means for (I)

$$\left(\int_0^1 f(x,t)\,dt\right) dx^{i_1}\wedge\dots\wedge dx^{i_{k-1}}$$

and for (II) the integral is 0.

If we show the formula holds in each case we will be done

<u>Case (I):</u>

$$d_M(h_k\omega) = d_M\left[\left(\int_0^1 f(x,t)\,dt\right) dx^{i_1}\wedge\dots\wedge dx^{i_{k-1}}\right]$$

$$= \sum_{i=1}^n \left(\int_0^1 \frac{\partial f}{\partial x^i}\,dt\right) dx^i\wedge dx^{i_1}\wedge\dots\wedge dx^{i_{k-1}}$$

and

$$h_{k+1}(d\omega) = h_{k+1}\left(\sum_{i=1}^n \frac{\partial f}{\partial x^i}\, dx^i\wedge dt\wedge dx^{i_1}\dots\wedge dx^{i_{k-1}}\right)$$

$$= \sum_{i=1}^n \left(\int_0^1 -\frac{\partial f}{\partial x^i}\,dt\right) dx^i\wedge dx^{i_1}\wedge\dots\wedge dx^{i_{k-1}}$$

so $(d\circ h_k + h_{k+1}\circ d)\,\omega = 0$

and $(i_j^*\omega)(f\,dt\wedge dx^{i_1}\dots\wedge dx^{i_{k-1}}) = 0$

since $\frac{\partial}{\partial t}$ not in $T_x M$

so done ✓

Case II:

$$d_M(h_k\,\omega) = 0$$

$$h_{k+1}(d\omega) = h_{k+1}\Big[\frac{\partial f}{\partial t}\,dt\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_k} + \sum_{i=1}^{n}\frac{\partial f}{\partial x^i}\,dx^i\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_k}\Big]$$

$$= \left(\int_0^1 \frac{\partial f}{\partial t}\,dt\right) dx^{i_1}\wedge\cdots\wedge dx^{i_k}$$

$$= \big(f(x,1) - f(x,0)\big)\,dx^{i_1}\wedge\cdots\wedge dx^{i_k}$$

$$= i_1^{*}\omega - i_0^{*}\omega \qquad \checkmark$$

Proof of Th^m 8:

let $H: M\times[0,1]\to N$ be the homotopy f to g

<u>note</u>: $f = H\circ i_0$ and $g = H\circ i_1$ (i_j as in lemma 9)

so we get

lemma 9

$$\begin{tikzcd} \Omega^k(N) \arrow[r, "H^*"] \arrow[rr, bend right, "\tilde{h}_k"'] & \Omega^k(M\times[0,1]) \arrow[r, "h_k"] & \Omega^k(M) \end{tikzcd}$$

now $\tilde{h}(d\omega) + d\tilde{h}\omega = h\circ H^*(d\omega) + d\circ h(H^*\omega) = h\circ d(H^*\omega) + d\circ h(H^*\omega)$

lemma 9

$$= i_1^*(H^*\omega) - i_0^*(H^*\omega) = (H\circ i_1)^*\omega - (H\circ i_0)^*\omega$$

$$= g^*\omega - f^*\omega$$

as discussed above if $[\omega]\in H^k_{DR}(N)$

then $d\omega = 0$ so

$$g^*\omega - f^*\omega = d(\tilde{h}_k \omega)$$

and thus $[g^*\omega] = [f^*\omega]$ in $H^k_{DR}(M)$ $\blacksquare$

Thm 10 (Mayer-Vietoris):

let M be a smooth manifold

U, V open subsets of M that cover M

for each k, $\exists$ a linear map

$$\delta^k : H^k_{DR}(U \cap V) \to H^{k+1}_{DR}(M)$$

such that the following sequence is exact

$$\cdots \to H^k_{DR}(M) \xrightarrow{k^* \oplus \ell^*} H^k_{DR}(U) \oplus H^k_{DR}(V) \xrightarrow{i^* - j^*} H^k_{DR}(U \cap V) \xrightarrow{\delta^k} H^{k+1}_{DR}(M) \to \cdots$$

where

$$\begin{tikzcd} & U \arrow[dr, "k"] & \\ U \cap V \arrow[ur, "i"] \arrow[dr, "j"'] & & M \\ & V \arrow[ur, "\ell"'] & \end{tikzcd}$$

are inclusions

exactness means the image of one map is equal to the kernel of the next

Cor 11:

for $n \geq 1$

$$H^k_{DR}(S^n) \cong \begin{cases} \mathbb{R} & k = 0, n \\ 0 & k \neq 0, n \end{cases}$$

note: $H^n_{DR}(S^n)$ is generated by any volume form (since if $\omega = d\eta$, $\int_{S^n}\omega = 0$)

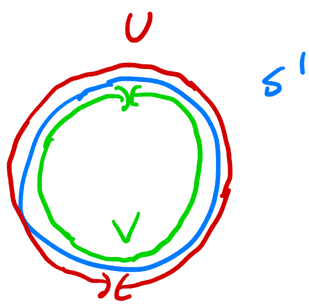
Proof:

we know $H^0_{DR}(S^n) \cong \mathbb{R}$ since S^n connected $(n \geq 1)$

induct on n:

$n=0$: $H^k_{DR}(S^0) \cong \begin{cases} \mathbb{R}\oplus\mathbb{R} & k=0 \\ 0 & k\neq 0 \end{cases}$

$n=1$:



$U = S^1 - \{(0,-1)\}$
$V = S^1 - \{(0,1)\}$

$U \cong V \cong \mathbb{R}$

so $H^k_{DR}(U) \cong H^k_{DR}(V) \cong \begin{cases} \mathbb{R} & k=0 \\ 0 & k\neq 0 \end{cases}$

$U \cap V \cong \mathbb{R} \cup \mathbb{R}$

so $H^k_{DR}(U\cap V) \cong \begin{cases} \mathbb{R}\oplus\mathbb{R} & k=0 \\ 0 & k\neq 0 \end{cases}$

thus we get

$$H^{k-1}_{DR}(S^1) \to H^{k-1}_{DR}(U)\oplus H^{k-1}_{DR}(V) \to H^{k-1}_{DR}(U\cap V) \to H^k_{DR}(S^1) \to H^k_{DR}(U)\oplus H^k_{DR}(V)$$

for $k=1$:

$$\underset{\substack{\uparrow \\ H^{-1}_{DR}(U\cap V)}}{0} \xrightarrow{g} \mathbb{R} \xrightarrow{\phi} \mathbb{R}\oplus\mathbb{R} \xrightarrow{\psi} \mathbb{R}\oplus\mathbb{R} \xrightarrow{f} H^1_{DR}(S^1) \xrightarrow{h} 0$$

ϕ injective since $\ker \phi = \operatorname{im} g = 0$

$\ker \psi = \operatorname{im} \phi = \mathbb{R}$

$\therefore$ image $\psi \cong \mathbb{R}\oplus\mathbb{R}/\ker\psi \cong \mathbb{R} \subset \mathbb{R}\oplus\mathbb{R}$

f is surjective since image $f = \ker h = H^1_{DR}(S^1)$

so $H^1_{DR}(S^1) \cong \operatorname{im} f \cong \mathbb{R}\oplus\mathbb{R}/\ker f \cong \mathbb{R}\oplus\mathbb{R}/\operatorname{im}\psi$

$\cong \mathbb{R}\oplus\mathbb{R}/\mathbb{R} \cong \mathbb{R}$

for $k>1$:

$$0 \to 0 \to 0 \to H^k_{DR}(S^1) \to 0$$

so $H^k_{DR}(S^1) = 0$

<u>$n>1$</u>:

$U = S^n - \{\text{south pole}\}$

$V = S^n - \{\text{north pole}\}$

$U \cong V \cong \mathbb{R}^n$

$U \cap V \cong \mathbb{R}^n - \{0\} \cong S^{n-1} \times \mathbb{R}$

$\simeq S^{n-1}$

so $H^k_{DR}(U) \cong H^k_{DR}(V) \cong \begin{cases} \mathbb{R} & k=0 \\ 0 & k \neq 0 \end{cases}$

by induction

$H^k_{DR}(U \cap V) \cong \begin{cases} \mathbb{R} & k = 0, n-1 \\ 0 & k \neq 0, n-1 \end{cases}$

$\underline{k \geq 2:}$

$$\underbrace{H^{k-1}_{DR}(U)\oplus H^{k-1}_{DR}(V)}_{\cong\, 0} \to H^{k-1}_{DR}(U\cap V) \xrightarrow{\psi} H^{k}_{DR}(S^n) \to \underbrace{H^{k}_{DR}(U)\oplus H^{k}_{DR}(V)}_{\cong\, 0}$$

so ψ is surjective and injective

so $\cong$

$\therefore H^k(S^n) \cong H^{k-1}(S^{n-1})$

$\underline{k=1}$

$$\begin{tikzcd}
H^{-1}_{DR}(S^{n-1}) \arrow[r] \arrow[d, phantom, "\shortparallel"] & H^{0}_{DR}(S^n) \arrow[r] \arrow[d, phantom, "\cong"] & H^{0}_{DR}(U)\oplus H^{0}_{DR}(V) \arrow[r] \arrow[d, phantom, "\cong"] & H^{0}_{DR}(U\cap V) \arrow[r] \arrow[d, phantom, "\cong"] & H^{1}_{DR}(S^n) \arrow[dr] & \\
0 \arrow[r] & \mathbb{R} \arrow[r] & \mathbb{R}\oplus\mathbb{R} \arrow[r] & \mathbb{R} \arrow[ur] & & 0 \\
& & & & & H^1(U)\oplus H^1(V) \arrow[u]
\end{tikzcd}$$

same argument as above shows

$H^1_{DR}(S^n) = 0$

so $H^k_{DR}(S^n) \cong \begin{cases} \mathbb{R} & k = 0, n \\ 0 & k \neq 0, n \end{cases}$

to prove Mayer-Vietoris we need

lemma 12:

with maps from Th$^{\underline{m}}$ 10 we have the following exact sequence for all k

$$0 \to \Omega^k(M) \xrightarrow{h^* + l^*} \Omega^k(U) \oplus \Omega^k(V) \xrightarrow{i^* - j^*} \Omega^k(U \cap V) \to 0$$

Proof of Th$^{\underline{m}}$ 10 given lemma 12:

consider

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
\Omega^{k-1}(M) \arrow[r, "d"] \arrow[d, "h^* \oplus l^*"] & \Omega^k(M) \arrow[r, "d"] \arrow[d, "h^* \oplus l^*"] & \Omega^{k+1}(M) \arrow[d, "h^* \oplus l^*"] \\
\Omega^{k-1}(U) \oplus \Omega^{k-1}(V) \arrow[r, "d"] \arrow[d, "i^* - j^*"] & \Omega^k(U) \oplus \Omega^k(V) \arrow[r, "d"] \arrow[d, "i^* - j^*"] & \Omega^{k+1}(U) \oplus \Omega^{k+1}(V) \arrow[d, "i^* - j^*"] \\
\Omega^{k-1}(U \cap V) \arrow[r, "d"] \arrow[d] & \Omega^k(U \cap V) \arrow[r, "d"] \arrow[d] & \Omega^{k+1}(U \cap V) \arrow[d] \\
0 & 0 & 0
\end{tikzcd}$$

Diagram Chase: to define $H^k(U \cap V) \to H^{k+1}(M)$

need to start with $c \in \Omega^k(U \cap V)$

s.t. $dc = 0$ and get $a \in \Omega^{k+1}(M)$

with $da = 0$

note only one thing to do at every step

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
\Omega^{k-1}(M) \arrow[r, "d"] \arrow[d, "h^* \oplus l^*"] & \Omega^k(M) \arrow[r, "d"] \arrow[d, "h^* \oplus l^*"] & \Omega^{k+1}(M) \arrow[d, "h^* \oplus l^*"] \\
\Omega^{k-1}(U) \oplus \Omega^{k-1}(V) \arrow[r, "d"] \arrow[d, "i^* - j^*"] & \Omega^k(U) \oplus \Omega^k(V) \arrow[r, "d"] \arrow[d, "i^* - j^*"] & \Omega^{k+1}(U) \oplus \Omega^{k+1}(V) \arrow[d, "i^* - j^*"] \\
\Omega^{k-1}(U \cap V) \arrow[r, "d"] \arrow[d] & \Omega^k(U \cap V) \arrow[r, "d"] \arrow[d] & \Omega^{k+1}(U \cap V) \arrow[d] \\
0 & 0 & 0
\end{tikzcd}$$

a

$b \mapsto db$

$c \mapsto dc = 0$

given $c \in \Omega^k(U \cap V)$ with $dc = 0$

$i^* - j^*$ surjective so

$\exists\, b \in \Omega^k(U) \oplus \Omega^k(V)$ s.t.

$i^* - j^*(b) = c$

note $(i^* - j^*)(db) = d(i^* - j^*)b$
$= dc = 0$

by exactness $\exists\, a \in \Omega^{k+1}(M)$

such that $h^* \oplus l^*(a) = dc$

note $k^* \oplus l^*(da) = d(h \oplus l^*(a))$
$= d(dc) = 0$

by exactness at $\Omega^{k+1}(M)$
$da = 0$ so gives
a homology class

define $\delta^k : H^k(U \cap V) \longrightarrow H^{k+1}(M)$
$[c] \longmapsto [a]$

Claim: δ^k well-defined

① need δ^k independent of b

suppose b, b' both map to c

$(i^* - j^*)(b' - b) = 0$

by exactness $\exists\ \tilde{a}$ such that

$(h^* \oplus l^*)(\tilde{a}) = b' - b$

note by exactness $\exists!$ a s.t. $(h^* \oplus l^*)(a) = db$
and $!$ a' s.t. $(k^* \oplus l^*)(a') = db'$

now $(h^* + l^*)(a - a' - d\tilde{a})$

$= db - db' - d(h^* - \ell^*(\tilde{a}))$

$= db - db' - db' + db = 0$

so exactness gives $a - a' - d\tilde{a} = 0$

so $[a] = [a']$ in $H^{k+1}(M)$

② also need δ^k independent of choice of $c, c' \in [c]$

<u>exercise:</u>

1) Show this (diagram chase)

2) Show δ^k is a homomorphism

3) Check exactness of Mayer–Viétoris sequence

<u>Proof of lemma 12:</u>

<u>exactness at $\Omega^k(M)$:</u>

need to see $k^* \oplus \ell^*$ is injective

suppose $\omega \in \Omega^k(M)$ and $k^*(\omega) = \ell^*(\omega) = 0$

i.e. $\omega|_U = 0$, $\omega|_V = 0$

then $\omega = 0$ on $U \cup V = M$

so $\omega = 0$ in $\Omega^k(M)$ ✓

exactness at $\Omega^k(U)\oplus\Omega^k(V)$:

first note $(\imath^*-\jmath^*)(k^*\oplus l^*)(\omega)$

$$= \imath^*(k^*(\omega)) - \jmath^*(l^*(\omega))$$

$$= \omega|_{U\cap V} - \omega|_{U\cap V} = 0$$

so image$(k^*\oplus l^*) \subset \ker(\imath^*-\jmath^*)$

now suppose $(\eta,\eta')\in\ker(\imath^*-\jmath^*)$

so $\imath^*\eta - \jmath^*\eta' = 0$

i.e. $\eta|_{U\cap V} = \eta'|_{U\cap V}$

then define

$$\sigma = \begin{cases} \eta & \text{on } U \\ \eta' & \text{on } V \end{cases}$$

note $\sigma\in\Omega^k(M)$ and

$(k^*\oplus l^*)(\sigma) = (\eta,\eta')$ ✓

exactness at $\Omega^k(U\cap V)$:

we need to see $\imath^*-\jmath^*$ is onto

let $\omega\in\Omega^k(U\cap V)$

let $\{\psi_U,\psi_V\}$ be a partition of unity subordinate to $\{U, V\}$

set $\eta = \begin{cases} \psi_V \omega & U \cap V \\ 0 & U - \text{support } \psi_V \end{cases}$

$\eta' = \begin{cases} -\psi_U \omega & U \cap V \\ 0 & V - \text{support } \psi_U \end{cases}$

note $(\eta, \eta') \in \Omega^k(U) \oplus \Omega^k(V)$

and $(\imath^* - \jmath^*)(\eta, \eta') = \eta|_{U \cap V} - \eta'|_{U \cap V}$

$= \psi_V \omega + \psi_U \omega = \omega$ ▦

Th$^{\underline{m}}$ 13:

M compact, connected, oriented n-manifold with out boundary
Then
$$I : H^n_{DR}(M) \longrightarrow \mathbb{R}$$
$$[\omega] \longmapsto \int_M \omega$$
is an isomorphism
In particular, $H^n_{DR}(M) \cong \mathbb{R}$ and is spanned by a volume form

Proof: clearly I well-defined since if $[\omega]=[\omega']$ in $H^n(M)$, then $\exists\ \eta \in \Omega^{n-1}(M)$ such that

$$\omega = \omega' + d\eta$$

$$\therefore \int_M \omega = \int_M \omega' + d\eta = \int_M \omega' + \int_{\underbrace{\partial M}_{=\emptyset}} \eta = \int_M \omega'$$

M oriented $\Rightarrow \exists$ never-zero n-form Ω on M

and M compact so

$$\int_M \Omega = a > 0$$

$$\therefore I([b\Omega]) = b\int_M \Omega = ba$$

so I is onto

we are left to see $I(\omega)=0 \Leftrightarrow [\omega]=0$

i.e. $I(\omega)=0 \Leftrightarrow \omega = d\eta$

($\Leftarrow$) clear!

for the proof of ($\Rightarrow$) suppose $\int_M \omega = 0$

let $\{U_i\}_{i=1}^n$ be a finite cover of M by open sets diffeomorphic to $\mathbb{R}^n$

set $M_k = U_1 \cup \dots \cup U_k$

since M is connected we can assume U_i ordered so that $M_k \cap U_{k+1} \neq \emptyset$

Claim: if ω is a compactly supported n-form on M_k such that $\int_{M_k} \omega = 0$, then $\exists$ compactly supported $(n-1)$-form η on M_k s.t. $d\eta = \omega$

(note: want compact support so we can integrate)

If claim true, the we are done w/ $k = m$

Proof is by induction on k

$k=1$: $M_1 \cong \mathbb{R}^n$

$d\omega = 0$ and $H^n(\mathbb{R}^n) = 0$ so $\exists \eta'$ s.t. $d\eta' = \omega$

but η' might not have compact support

if not need to fix it

consider $n=1$:

$\omega = f(x)dx$ and f compactly supported

set $F(x) = \int_{-\infty}^{x} f(t)dt$

clearly $dF = F'(x)dx = f(x)dx = \omega$

if support of $f \subset [-R, R]$

then $F(x) = 0$ for $x \leq -R$

and for $x > R \Rightarrow F(x) = \int_{-\infty}^{x} f(t)\,dt$

$$= \int_{-\infty}^{\infty} f(t)\,dt = 0$$

so F has compact support. ✓

consider $n \geq 2$:

let B, B' be open balls in $\mathbb{R}^n$ s.t.

$$\text{supp}\,\omega \subset \overline{B} \subset B'$$

so $0 = \int_{\mathbb{R}^n} \omega = \int_{\overline{B}'} \omega = \int_{\overline{B}'} d\eta' = \int_{\partial \overline{B}'} \eta'$

$\partial \overline{B}' = S^{n-1}$

let $\imath : S^{n-1} \hookrightarrow \mathbb{R}^n - B$ be inclusion

so $\int_{S^{n-1}} \imath^*(\eta') = 0$

by Cor 11 we know $[\imath^*\eta'] = [0]$ in $H^{n-1}_{DR}(S^{n-1}) \cong \mathbb{R}$

$\imath$ is a homotopy equivalence $(\mathbb{R}^n - B \cong S^{n-1} \times [a, \infty))$

so Th$^{\underline{m}}$ 8 gives $H^{n-1}(\mathbb{R}^n - B) \underset{\imath^*}{\xrightarrow{\cong}} H^{n-1}(S^{n-1})$

$\therefore \exists\, \gamma \in \Omega^{n-2}(\mathbb{R}^n - B)$ s.t. $d\gamma = \eta'|_{\mathbb{R}^n - B}$

let ψ be a bump function s.t.

$$\psi = \begin{cases} 1 & \text{on } \mathbb{R}^n - B' \\ 0 & \text{on } B \end{cases}$$

and $0 \le \psi \le 1$

set $\eta = \eta' - d(\psi\gamma)$

on $\mathbb{R}^n - B'$ $\quad d(\psi\gamma) = d\gamma = \eta'$

so $\eta = 0$ on $\mathbb{R}^n - B'$

$\therefore$ η is compactly supported and

$d\eta = d\eta' - d(d\psi\gamma) = \underline{\omega}$ ✓

<u>inductive step $k \ge 2$:</u>

let ω be compactly supported in

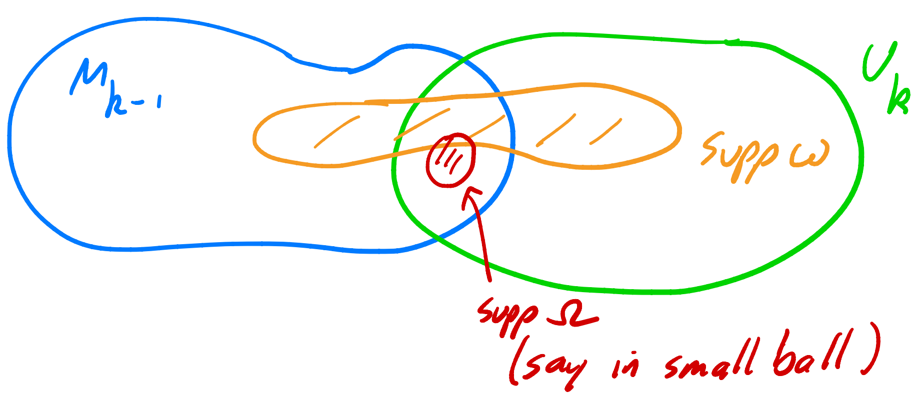
$$M_k = M_{k-1} \cup U_k$$

and $\int_{M_k} \omega = 0$

choose any form $\Omega \in \Omega^n(U_k \cap M_{k-1})$

that is compactly supported

and $\int_{U_k} \Omega = \int_{M_{k-1}} \Omega = 1$



let $\{\phi, \psi\}$ be a partition of unity subordinate to $\{M_{k-1}, U_k\}$

set $$c = \int_{M_k} \phi\,\omega = \int_{M_{k-1}} \phi\,\omega$$

<u>note:</u>

1) $$\int_{M_{k-1}} (\phi\,\omega - c\,\Omega) = 0$$

$\phi\,\omega - c\,\Omega$ is compactly supported in M_{k-1} so by induction $\exists\, \alpha \in \Omega^{n-1}(M_{k-1})$ with compact support s.t.

$$d\alpha = \phi\,\omega - c\,\Omega$$

2) $$\int_{U_k} \psi\,\omega + c\,\Omega = \int_{M_k} (1-\phi)\,\omega + c\,\Omega$$

$$= \underbrace{\cancel{\int_{M_k} \omega}}_{0 \text{ by hypothesis}} + \overbrace{\cancel{\int_{M_k} c\,\Omega - \phi\,\omega}}^{0 \text{ by 1)}}$$

$$= 0$$

also $\psi\,\omega + c\,\Omega$ is compactly supported in U_k so by base case $\exists\, \beta \in \Omega^{n-1}(U_k)$ with compact support and $d\beta = \psi\omega + c\,\Omega$

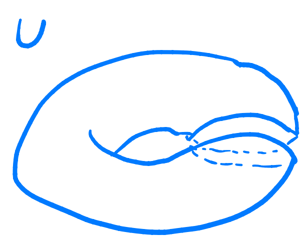
now $\alpha + \beta$ has compact support in M_k and
$$d(\alpha+\beta) = (\phi\omega - c\,\Omega) + (\psi\omega + c\,\Omega)$$
$$= (\phi + \psi)\,\omega = \omega$$

with the above theorem we can now use Mayer-Veitoris to make more computations

example: compute $H^k_{DR}(T^2)$

let $U = T^2 - (\{p\} \times S^1)$ where $T^2 = S^1 \times S^1$

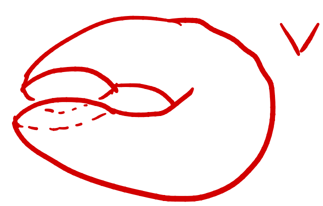
$V = T^2 - (\{-p\} \times S^1)$



so $U \cong V \cong S^1 \times I \simeq S^1$

$U \cap V \cong S^1 \times (I_1 \cup I_2)$

$\simeq S^1 \cup S^1$



so $H^k(U) \cong H^k(V) \cong \begin{cases} \mathbb{R} & k = 0,1 \\ 0 & k \neq 0,1 \end{cases}$

$H^k(U \cap V) \cong \begin{cases} \mathbb{R} \oplus \mathbb{R} & k = 0,1 \\ 0 & k \neq 0,1 \end{cases}$

$$\begin{tikzcd}
0 \arrow[r] & H^0(T^2) \arrow[r, "f"] \arrow[d, phantom, "\cong"] & H^0(U)\oplus H^0(V) \arrow[r, "g"] \arrow[d, phantom, "\cong"] & H^0(U\cap V) \arrow[r, "h"] \arrow[d, phantom, "\cong"] & H^1(T^2) \arrow[dd, bend left, "k"] \arrow[ddd, bend right] & & & \\
& \mathbb{R} \arrow[r] & \mathbb{R}\oplus\mathbb{R} \arrow[r] & \mathbb{R}\oplus\mathbb{R} \arrow[ur] & & & & \\
& & & & H^1(U)\oplus H^1(V) \arrow[r, "\phi"] \arrow[d, phantom, "\cong"] & H^1(U\cap V) \arrow[r, "\psi"] \arrow[d, phantom, "\cong"] & H^2(T^2) \arrow[r] & 0 \\
& & & & \mathbb{R}\oplus\mathbb{R} \arrow[r] & \mathbb{R}\oplus\mathbb{R} \arrow[ur] & &
\end{tikzcd}$$

f injective so $\mathbb{R} \cong \operatorname{im} f = \ker g$

$$\operatorname{im} g \cong \frac{\operatorname{domain} g}{\ker g} \cong \frac{\mathbb{R}\oplus\mathbb{R}}{\mathbb{R}} \cong \mathbb{R}$$

similarly $\operatorname{im} h \cong \mathbb{R} \subset H^1(T^2)$

from Thm 13 we know $H^2(T^2) \cong \mathbb{R}$

so $\mathbb{R} \cong H^2(T^2) = \operatorname{im}(\psi) = H^1(U\cap V)/\ker\psi$

so $\ker\psi \cong \mathbb{R}$ $\therefore$ $\operatorname{im}\phi \cong \mathbb{R}$

and as above $\ker\phi \cong \mathbb{R}$

and $\operatorname{im} k = \ker\phi \cong \mathbb{R}$

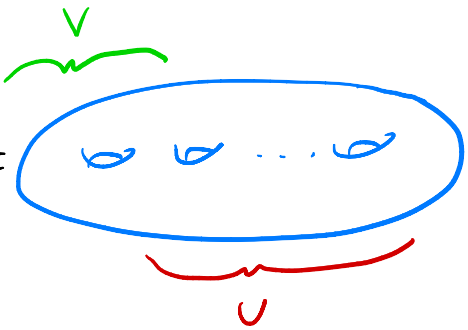
also know $\ker k = \operatorname{im} g \cong \mathbb{R}$

so $$H^1(T^2)/\ker k \cong \operatorname{Im} k$$

that is $H^1(T^2) \cong \ker k \oplus \operatorname{im} k \cong \mathbb{R}\oplus\mathbb{R}$

$$\therefore \quad H^k_{DR}(T^2) \cong \begin{cases} \mathbb{R} & k = 0, 2 \\ \mathbb{R}\oplus\mathbb{R} & k = 1 \\ 0 & \text{otherwise} \end{cases}$$

exercise: Compute $H^k_{DR}(\Sigma_g)$ where

$\Sigma_g =$ 

another application of Mayer-Vietoris

Th$^{\underline{m}}$ 14:

$\boxed{\text{If } M \text{ is a compact manifold, then its cohomology is finite}}$

we need following

called good cover

Fact:

$\boxed{\text{any } n\text{-manifold has a cover } \{U_\alpha\} \text{ such that all finite intersections are } \emptyset \text{ or diffeo to } \mathbb{R}^n}$

easiest way to prove this is using Riemannian geometry (don't strictly need this but proof is nicer with it!)

Proof: note Mayer-Vietoris gives

$$H^{q-1}(U\cap V) \xrightarrow{\delta} H^q(U\cup V) \xrightarrow{i^*\oplus j^*} H^q(U)\oplus H^q(V)$$

so $H^q(U \cup V) \cong \mathrm{im}(i^* \oplus j^*) \oplus \ker(i^* \oplus j^*)$

$$= \mathrm{im}(i^* \oplus j^*) \oplus \mathrm{im}(\delta)$$

$\therefore$ if $H^{q-1}(U \cap V)$, $H^q(U)$, $H^q(V)$ finite dimensional then so is $H^q(U \cup V)$

Claim: if M has a finite good cover then $H^k_{DR}(M)$ is finite dimensional $\forall k$

to see this we induct on number of elements in good cover

base case: $U_1 \cong \mathbb{R}^n$ done!

assume true for all manifolds with good cover with $\leq l$ elements

suppose $M = U_1 \cup \ldots \cup U_{l+1}$

set $M' = U_1 \cup \ldots \cup U_l$

homology of M' and U_{l+1} finite dim'l by induction

as is $M' \cap U_{l+1} = \underbrace{(U_1 \cap U_{l+1}) \cup \ldots \cup (U_l \cap U_{l+1})}_{\leq l \text{ sets}}$

$\therefore$ done by above observation