E. Products and Poincaré duality

note if $[\omega] \in H^k_{DR}(M)$, $[\eta] \in H^\ell_{DR}(M)$

then 1) $d(\omega \wedge \eta) = \cancel{d\omega} \wedge \eta + (-1)^k \omega \wedge \cancel{d\eta}$
$= 0$

so $\omega \wedge \eta$ gives $k+\ell$ cohomology class

2) if $\omega' = \omega + d\alpha$, then

$$\omega' \wedge \eta = \omega \wedge \eta + d\alpha \wedge \eta$$
$$= \omega \wedge \eta + d(\alpha \wedge \eta)$$

so $[\omega' \wedge \eta] = [\omega \wedge \eta]$

and similarly for $\eta' = \eta + d\beta$

so this product is well-defined!

$$H^k_{DR}(M) \times H^\ell_{DR}(M) \to H^{k+\ell}_{DR}(M)$$
$$([\omega], [\eta]) \mapsto [\omega] \cup [\eta] = [\omega \wedge \eta]$$

lemma 15:

if M is an n-manifold, then
$$H^*_{DR}(M) = \bigoplus_{i=0}^{n} H^i_{DR}(M)$$
is a graded ring with product $\cup$ satisfying

$$[\alpha] \cup [\beta] = (-1)^{|\alpha||\beta|} [\beta] \cup [\alpha]$$

and $f: M \to N$ induces a ring homomorphism

$$f^*: H^*_{DR}(N) \to H^*_{DR}(M)$$

Proof: follows from ring structure on $\Omega^*(M)$ and properties of $\wedge$ on $\Omega^*(M)$ ▦

example: cohomology ring of T^2

we know $H^k(T^2) \cong \begin{cases} \mathbb{R} & k = 0, 2 \\ \mathbb{R} \oplus \mathbb{R} & k = 1 \\ 0 & \text{otherwise} \end{cases}$

note: $T^2 = \mathbb{R}^2 / \mathbb{Z}^2$ where $\mathbb{Z}^2$ acts by $(a,b) \in \mathbb{Z}^2$

$$T_{(a,b)}: \mathbb{R}^2 \to \mathbb{R}^2$$
$$(x,y) \mapsto (x+a, y+b)$$

we see $T^*_{(a,b)} dx = dx$, $T^*_{(a,b)} dy = dy$

so dx, dy give 1-forms on T^2

$dx \wedge dy$ gives 2-form on T^2

clearly dx, dy, and $dx \wedge dy$ closed

$\therefore [dx], [dy] \in H^1_{DR}(T^2)$

note $[dx] \cup [dy] = [dx \wedge dy]$

and $dx \wedge dy$ a volume form

so $[dx \wedge dy] \neq 0$ (generates $H^2_{DR}(T^2)$)

if $[dx] = a[dy]$ some a, then

$$[dx] \wedge [dy] = [a\, dy \wedge dy] = 0 \; \cancel{\otimes}$$

so $[dx], [dy]$ span $H^1_{DR}(T^2)$

and $1, dx, dy, dx \wedge dy$ span $H^*_{DR}(T^2)$

as a ring $\quad H^*(T^2) \cong \Lambda(a,b)$ ← exterior algebra on $\{a,b\}$ $\deg a = \deg b = 1$

$$= \mathbb{R}[a,b] \big/ \langle a^2, b^2, ab + ba \rangle$$

<u>new cohomology</u>:

$$\Omega^k_c(M) = \{ \alpha \in \Omega^k(M) \mid \operatorname{supp} \alpha \text{ is compact} \}$$

<u>note</u>: 1) $\Omega^k_c(M) = \Omega^k(M)$ if M compact

2) but on, for example, $\mathbb{R}^k$ different

3) Integration

$$\Omega^n_c(M^n) \to \mathbb{R}$$
$$\omega \longmapsto \int_M \omega$$

always defined

4) $d: \Omega^k_c(M) \to \Omega^{k+1}_c(M)$

and we still have $d^2 = 0$

we define the de Rham cohomology of M with compact support to be

$$H^k_c(M) = \frac{\ker(d : \Omega^k_c(M) \to \Omega^{k+1}_c(M))}{\operatorname{im}(d : \Omega^{k-1}_c(M) \to \Omega^k_c(M))}$$

examples:

1) if M non-compact and connected, then

$$d : \Omega^0_c(M) \to \Omega^1_c(M)$$
$$f \mapsto df$$

so if $df = 0$, f is locally constant

but $f = 0$ outside a compact subset

so $f \equiv 0$

$\therefore$ $H^0_c(M) = 0$ if M non-compact

2) $\mathbb{R}^1$, given $\omega \in \Omega^1_c(\mathbb{R})$

$$\omega = f(x)\,dx$$

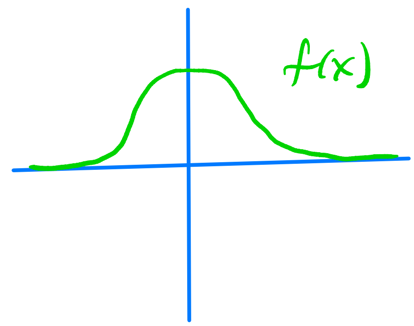
if $\int_{\mathbb{R}} \omega = 0$, then set

$$F(x) = \int_{-\infty}^{0} \omega$$

as we saw earlier F has compact sup.

and $dF = \omega$

now let $\omega_0 = f(x)dx$ with



such that $\int_{\mathbb{R}} \omega_0 = 1$

we have $\omega_0 \in \Omega^1_c(\mathbb{R})$ and $d\omega_0 = 0$

since $\int_{\mathbb{R}} \omega \neq 0$ we see $[\omega_0] \neq 0$ in $H^1_c(\mathbb{R})$

also if $\omega \in \Omega^1_c(\mathbb{R})$ then let $c = \int_{\mathbb{R}} \omega$

so $\int_{\mathbb{R}} \omega - c\omega_0 = 0$

and thus $\exists\, F \in \Omega^0_c(\mathbb{R})$ st.

$$\omega = c\omega_0 + dF$$

i.e. $[\omega] = c[\omega_0]$

and $H^1_c(\mathbb{R}) \cong \mathbb{R}$

i.e. $H^k_c(\mathbb{R}) \cong \begin{cases} \mathbb{R} & k = 1 \\ 0 & k \neq 1 \end{cases}$

Remark: $H^k_c(M)$ is <u>not</u> a homotopy invariant!

since $\mathbb{R} \simeq \{0\}$ but $H^k_c(\{0\}) = \begin{cases} \mathbb{R} & k=0 \\ 0 & k \neq 0 \end{cases}$

this is unlike $H^k_{DR}(M)$

note: 1) if $f: M \to N$ is proper (i.e. $f^{-1}(C)$ is compact for any compact C), then

$$f^*: \Omega^k_c(N) \to \Omega^k_c(M)$$

and as before we get

$$f^*: H^k_c(N) \to H^k_c(M)$$

2) if $i: U \to M$ is inclusion of an open set then any $\omega \in \Omega^k_c(U)$ can be extended to a form on M by zero

$$i_* \omega \in \Omega^k_c(M)$$

exercise: Show if $M = U \cup V$, U, V open then

$$0 \to \Omega^k_c(U \cap V) \xrightarrow{i_* \ominus j_*} \Omega^k_c(U) \oplus \Omega^k_c(V) \xrightarrow{h_* + l_*} \Omega^k_c(M) \to 0$$

is exact for all k where

$$\begin{tikzcd} & U \arrow[dr, "k"] & \\ U \cap V \arrow[ur, "i"] \arrow[dr, "j"'] & & M \\ & V \arrow[ur, "l"'] & \end{tikzcd}$$

are inclusions

$\therefore$ just as in lemma 12 we get Mayer-Viëtorus for cohomology with compact support

$$\cdots \to H^k_c(U\cap V) \xrightarrow{(\imath_*\ominus \jmath_*)} H^k_c(U)\oplus H^k_c(V) \xrightarrow{k_* + l_*} H^k_c(M) \xrightarrow{\delta} H^{k+1}_c(U\cap V) \to \cdots$$

<u>exercise</u>: If M has a finite good cover show $H^k_c(M)$ is finite dimensional <u>Hint</u>: Th^{m} 14

let's consider $\pi : M\times\mathbb{R} \to M$
$$(p,x) \longmapsto p$$

<u>note</u>: π^* does <u>not</u> give $\Omega^k_c(M) \to \Omega^k_c(M\times\mathbb{R})$

but we can "integrate"

i.e. locally any k-form on $M\times\mathbb{R}$ with compact support is a sum of terms of the form

I $f(x,t)\, dt\wedge dx^1\wedge\ldots\wedge dx^{k-1}$

II $f(x,t)\, dx^1\wedge\ldots\wedge dx^k$

so globally any $\omega\in\Omega^k(M\times\mathbb{R})$ is a sum of terms of the form

I $f(x,t)\, dt\wedge\pi^*\eta \qquad \eta\in\Omega^{k-1}(M)$

II $f(x,t)\,\pi^*\eta \qquad \eta\in\Omega^k(M)$

if $f(x,t)$ has compact support then define for type (I) terms

$$\pi_*\left(f(x,t)\,dt\wedge\pi^*\eta\right)=\left(\int_{-\infty}^{\infty}f(x,t)\,dt\right)\eta$$

and for terms of type (II)

$$\pi_*\left(f(x,t)\,\pi^*\eta\right)=0$$

exercise: $d\circ\pi_*=\pi_*\circ d$

so π_* induces a map

$$\pi_*: H_c^*(M\times\mathbb{R})\to H_c^*(M)$$

now let $e=e(t)\,dt$ be a 1-form on $\mathbb{R}^1$ with compact support with

$$\int_{\mathbb{R}}e(t)\,dt=1$$

(note $de=0$)

we can pull e back to $M\times\mathbb{R}$ by

we can pull e back to $M \times \mathbb{R}$ by $M \times \mathbb{R} \to M$
but still call it e

define

$$e_* : \Omega^k_c(M) \to \Omega^{k+1}_c(M \times \mathbb{R})$$
$$\eta \longmapsto (\pi^* \eta) \wedge e$$

exercise:

1) $e_* \circ d = d \circ e_*$

so e_* induces a map

$$e_* : H^k_c(M) \to H^{k+1}_c(M \times \mathbb{R})$$

2) $\pi_* \circ e_* = 1$ on $\Omega^k_c(M)$

Thm 16:

e_* and π_* induce isomorphisms on cohomology $\quad H^{k+1}_c(M \times \mathbb{R}) \cong H^k_c(M)$

Imeediately from this thm an computation of $H^k_c(\mathbb{R})$
we get

Cor 17:

$$H^k_c(\mathbb{R}^n) \cong \begin{cases} \mathbb{R} & k = n \\ 0 & k \neq n \end{cases}$$

note: $H^k_c(\mathbb{R}^n) \cong H^{n-k}_{DR}(\mathbb{R}^n)$ special case of Poincaré duality

Proof: define $K: \Omega^k_c(M\times\mathbb{R}) \to \Omega^{k-1}_c(M\times\mathbb{R})$ by

(I) $(\pi^*\eta)\wedge f(x,t)\,dt \longmapsto \pi^*\eta\left[\int_{-\infty}^{t} f(x,s)\,ds - \int_{-\infty}^{\infty} f(x,s)\,ds \int_{-\infty}^{t} e(s)\,ds\right]$

(II) $\pi^*\eta \longmapsto 0$

Claim: $1 - e_*\circ\pi_* = (-1)^{q-1}(d\circ K - K\circ d)$ on $\Omega^k_c(M\times\mathbb{R})$

note: claim says 1 is chain homotopic to $e_*\circ\pi_*$

so as in our proof of lemma 9, $e_*\circ\pi_* = 1$

on $H^k_c(M\times\mathbb{R})$

since we already know $\pi_*\circ e_* = 1$ on $H^k_c(M)$

we see both π_* and e_* are isomorphisms

Pf of Claim in case (I):

$$(1-e_*\circ\pi_*)\left((\pi^*\eta)\wedge f(x,t)\,dt\right) = (\pi^*\eta)\wedge f(x,t)\,dt - \pi^*\eta\left(\int_{-\infty}^{\infty} f(x,t)\,dt\right)\wedge e$$

$$d\circ K((\pi^*\eta)\wedge f(x,t)\,dt) = d\left[(\pi^*\eta)\left(\int_{-\infty}^{t} f(x,s)\,ds - \int_{-\infty}^{\infty} f(x,s)\,ds\int_{-\infty}^{t} e(s)\,ds\right)\right]$$

$$= (\pi^* d\eta)\left(\int_{-\infty}^{t} f(x,s)\,ds - \int_{-\infty}^{\infty} f(x,s)\,ds\int_{-\infty}^{t} e(s)\,ds\right)$$

$$+(-1)^k(\pi^*\eta)\wedge\left(\sum_{i=1}^{n}\left(\int_{-\infty}^{t}\frac{\partial f}{\partial x^i}\,ds\right) - \int_{-\infty}^{\infty}\frac{\partial f}{\partial x^i}\,ds\int_{-\infty}^{t} e(s)\,ds\right)dx^i$$

$$+(-1)^k(\pi^*\eta)\wedge\left(f(x,t) - \left(\int_{-\infty}^{\infty} f(x,s)\,ds\right)e(t)\right)dt$$

$$K\circ d\big((\pi^*\eta)\wedge f(x,t)\,dt\big) = K\Big((\pi^* d\eta)\wedge f(x,t)\,dt + (-1)^k \sum_{i=1}^n (\pi^*\eta)\wedge \frac{\partial f}{\partial x^i}\,dx^i \wedge dt\Big)$$

$$= (\pi^* d\eta)\Big(\int_{-\infty}^t f(x,s)\,ds - \int_{-\infty}^{\infty} f(x,s)\,ds \int_{-\infty}^t e(s)\,ds\Big)$$

$$+ (-1)^k \sum_{i=1}^n \pi^*\eta \wedge dx^i \Big(\int_{-\infty}^t \frac{\partial f}{\partial x^i}\,ds - \int_{-\infty}^{\infty} \frac{\partial f}{\partial x^i}\,ds \int_{-\infty}^t e(s)\,ds\Big)$$

so $(d\circ K - K\circ d)\big((\pi^*\eta)\wedge f(x,t)\,dt\big)$

$$= (-1)^{k+1}\Big[(\pi^*\eta)\wedge f(x,t)\,dt - \big((\pi^*\eta)\wedge e\big)\int_{-\infty}^{\infty} f(x,s)\,ds\Big]$$

$$= (-1)^{k+1}\big(1 - e_*\circ \pi_*\big)\big((\pi^*\eta)\wedge f(x,t)\,dt\big) \checkmark$$

Pf of Claim in case II:

$$(1 - e_*\circ \pi_*)(\pi^*\eta) f(x,t) = (\pi^*\eta) f(x,t)$$

$$(d\circ K - K\circ d)(\pi^*\eta) = -K\Big[(\pi^* d\eta) f(x,t) + (-1)^k \pi^*\eta \wedge \sum \frac{\partial f}{\partial x^i}\,dx^i + (-1)^k \pi^*\eta \wedge \frac{\partial f}{\partial t}\,dt\Big]$$

$$= (-1)^{k+1}\pi^*\eta\Big(\int_{-\infty}^t \frac{\partial f}{\partial t}(x,s)\,ds - \cancel{\int_{-\infty}^{\infty} \frac{\partial f}{\partial t}\,ds}^{\,0} \int_{-\infty}^t e(s)\,ds\Big)$$

$$= (-1)^{k+1}\pi^*\eta\ f(x,t) \quad \blacksquare$$

Consider vector spaces V, W let

$$\langle \cdot , \cdot \rangle : V \times W \to \mathbb{R}$$

be bilinear map

we say it is <u>non-degenerate</u> if

$$\langle v, w \rangle = 0 \ \forall w \in W \Rightarrow v = 0$$
$$\langle v, w \rangle = 0 \ \forall v \in V \Rightarrow w = 0$$

<u>lemma 18:</u>

V, W finite dimensional

Then a pairing $\langle \cdot , \cdot \rangle : V \times W \to \mathbb{R}$ is non-degenerate

$\Leftrightarrow$

$v \mapsto \langle v, \cdot \rangle$ is an isomorphism $V \to W^*$ and
$w \mapsto \langle \cdot , w \rangle$ " " $W \to V^*$

<u>Proof:</u> $(\Rightarrow)$ $\phi_V : V \to W^* : v \mapsto \langle v, \cdot \rangle$ is injective

since $\phi(v) = 0 \Rightarrow \langle v, w \rangle = 0 \ \forall w \in W$
$\Rightarrow v = 0$

and $\phi_W : W \to V^*$ injective

so $\dim V \le \dim W^* = \dim W \le \dim V^* = \dim V$

so ϕ an isomorphism,

$(\Leftarrow)$ if $\langle v, w \rangle = 0 \ \forall w \in W$, then $\phi_V(v) = 0$ so $v = 0$

similarly for $\langle v, w\rangle = 0 \ \forall v \in V$

for an orientable n-manifold M define

$$\langle \cdot , \cdot \rangle : H^k_{DR}(M) \times H^{n-k}_c(M) \longrightarrow \mathbb{R}$$

$$([\omega], [\eta]) \longmapsto \langle [\omega], [\eta] \rangle = \int_M \omega \wedge \eta$$

note $\omega \wedge \eta$ has compact support so integral is finite

if $\partial M = \emptyset$, then note

$$\eta' = \eta + d\alpha \Rightarrow \int_M \omega \wedge \eta' = \int_M \omega \wedge \eta + \int_M \omega \wedge d\alpha$$

$$= \int_M \omega \wedge \eta + \int_M d(\omega \wedge \alpha)$$

since $d\omega = 0$

$$= \int_M \omega \wedge \eta + \int_{\underbrace{\partial M}_{=\emptyset}} \omega \wedge \alpha$$

$$= \int_M \omega \wedge \eta$$

similarly for $\omega' = \omega + d\beta$

thus $\langle \cdot , \cdot \rangle$ is well-defined if $\partial M = \emptyset$

Th^{m} 19 (Poincaré Duality):

If M is oriented and has a finite good cover, then $\langle \cdot, \cdot \rangle$ is non-degenerate

in particular, $H^k_{DR}(M) \cong (H^{n-k}_c(M))^*$

Cor 20:

If M is compact and oriented, then

$$H^k_{DR}(M) \cong (H^{n-k}_{DR}(M))^* \cong H^{n-k}_{DR}(M)$$

Canonical non canonical

Proof M compact $\Rightarrow H^k_c(M) = H^k_{DR}(M)$

and M has finite good cover

for proof of Poincaré duality need a lemma

lemma 21 (the 5 lemma):

If the rows of the following commutative diagram are exact

$$\begin{tikzcd}
\cdots \arrow[r] & A \arrow[r, "f_1"] \arrow[d, "\alpha"'] & B \arrow[r, "f_2"] \arrow[d, "\beta"'] & C \arrow[r, "f_3"] \arrow[d, "\gamma"'] & D \arrow[r, "f_4"] \arrow[d, "\delta"'] & E \arrow[r] \arrow[d, "\varepsilon"'] & \cdots \\
\cdots \arrow[r] & A' \arrow[r, "f_1'"'] & B' \arrow[r, "f_2'"'] & C' \arrow[r, "f_3'"'] & D' \arrow[r, "f_4'"'] & E' \arrow[r] & \cdots
\end{tikzcd}$$

and $\alpha, \beta, \delta, \varepsilon$ isomorphisms then so is γ

Proof: γ injective:

$c \in C$ if $\gamma(c) = 0$ then $0 = f_3' \circ \gamma(c) = \delta \circ f_3(c)$

so $f_3(c) = 0$ (δ an isomorphism)

by exactness $\exists\, b \in B$ st $f_2(b) = c$

but $f_2'(\beta(b)) = \gamma(f_2(b)) = \gamma(c) = 0$

thus exactness gives $a' \in A$ such that

$$f_1'(a') = \beta(b)$$

α is an isomorphism so let $a = \alpha^{-1}(a')$

now $\beta(f_1(a)) = f_1'(\alpha(a)) = \beta(b)$

β an isomorphism $\Rightarrow f_1(a) = b$

finally $c = f_2(b) = f_2(f_1(a)) = 0$

by exactness

exercise: Show γ surjective

Proof of Th$^{\underline{m}}$ 19:

we first observe that the two Mayer-Vietoris sequences fit together with integration

$$
(\circledast)\quad
\begin{tikzcd}
H^k_{DR}(M) \arrow[r, "k^*\oplus l^*"] & H^k_{DR}(U)\oplus H^k_{DR}(V) \arrow[r, "i^*-j^*"] & H^k_{DR}(U\cap V) \arrow[r, "\delta"] & H^{k+1}_{DR}(M) \\
\times & \times & \times & \times \\
H_c^{n-k}(M) \arrow[d, "\int_M"] & H_c^{n-k}(U)\oplus H_c^{n-k}(V) \arrow[l, "k_*+l_*"'] \arrow[d, "\int_U+\int_V"] & H^{n-k}(U\cap V) \arrow[l, "i\oplus j"'] \arrow[d, "\int_{U\cap V}"] & H_c^{n-k-1}(M) \arrow[l, "\delta_c"'] \arrow[d, "\int_M"] \\
\mathbb{R} \arrow[r, equal] & \mathbb{R} \arrow[r, equal] & \mathbb{R} \arrow[r, equal] & \mathbb{R}
\end{tikzcd}
$$

diagram commutes upto sign

e.g. $\int_U + \int_V \big((i^*\oplus j^*)\eta\big)\wedge(\omega_1\oplus\omega_2)$

$$= \int_U (i^*\eta)\wedge\omega_1 + \int_V (j^*\eta)\wedge\omega_2$$

ω_1 cpt supp in U $\quad$ ω_2 cpt supp in V

$$= \int_M (i^*\eta)\wedge\omega_1 + \int_M j^*\eta\wedge\omega_2$$

$$= \int_M \eta\wedge\omega_1 + \int_M \eta\wedge\omega_2$$

$$= \int_M \eta\wedge i_*\omega_1 + \int \eta\wedge j_*\omega_2$$

$$= \int_M \eta\wedge(i_*+j_*)(\omega_1,\omega_2)$$

similarly for 2^{nd} square

for 3 square we need to see

$$\int_{U\cap V} \omega \wedge \delta_c \tau = \pm \int_{U\cup V} \delta\omega \wedge \tau$$

for this we need to recall the def^{n} of δ

looking back at the proof of Th^{m} 10 if $\omega \in \Omega^k(U\cap V)$

then take a partition of unity $\{\varphi_U, \varphi_V\}$ for $\{U, V\}$

$$\varphi_V\, \omega \in \Omega^k(U)$$

$$\varphi_U \omega \in \Omega^k(V)$$

and $d(\varphi_V \omega) + d(\varphi_U \omega) = d((\varphi_V + \varphi_U)\omega) = d\omega = 0$

on $U \cap V$

$\therefore\ i^*(d\varphi_V \omega) - j^*(d\varphi_U \omega) = 0$

and $\exists\ \eta \in \Omega^{k+1}(M)$ st.

$$k^*\eta = \eta|_U = d\varphi_V \omega = d\varphi_V \wedge \omega$$

$$l^*\eta = \eta|_V = d\varphi_U \omega = d\varphi_U \wedge \omega$$

define $\delta\omega = \eta$

similarly if $\tau \in \Omega_c^{n-k-1}(M)$ then $\delta_c \tau$ is

an $(n-k)$ form such that

$(\text{extension by } 0 \text{ of } \delta_c\tau \text{ to } U, \text{extension by } 0 \text{ of } \delta_c \tau \text{ to } V)$

$$= (d(\varphi_U \tau), d(\varphi_V \tau))$$

$$= (d\varphi_U \wedge \tau, d\varphi_V \wedge \tau)$$

so $\int_{U\cap V} \omega \wedge \delta_c \tau = \int_{U\cap V} \omega \wedge (d\psi_V \wedge \tau)$

$$= (-1)^{|\omega|} \int_{U\cap V} (d\psi_V \wedge \omega) \wedge \tau$$

$$= (-1)^{|\omega|} \int_{U\cup V} (k^* \eta) \wedge \tau$$

$d\psi_V \wedge \omega$ supported in $U \cap V$

$k^*\eta = \eta$ since $k^*\eta = d\psi_V \wedge \omega$ is supported in $U \cap V$

$$= (-1)^{|\omega|} \int_M \eta \wedge \tau = (-1)^{|\omega|} \int_M \delta\omega \wedge \tau$$

<u>exercise</u>: (*) is equivalent to

$$\begin{array}{ccccccc}
\cdots \to & H^k_{DR}(M) & \to & H^k(U) \oplus H^k(V) & \longrightarrow & H^k(U\cap V) & \to \cdots \\
& \downarrow \langle\cdot,\cdot\rangle & & \downarrow \langle\cdot,\cdot\rangle & & \downarrow \langle\cdot,\cdot\rangle & \\
\cdots \leftarrow & (H^{n-k}_c(M))^* & \leftarrow & (H^{n-k}(U) \oplus H^{n-k}(V))^* & \leftarrow & (H^{n-k}(U\cap V))^* & \leftarrow \cdots \\
& & & \cong & & & \\
& & & (H^{n-k}(U))^* \oplus (H^{n-k}(V))^* & & &
\end{array}$$

is a commutative diagram (upto sign)

note lemma 21 says if Th$^{\text{m}}$ 19 true for U, V, and $U\cap V$, then true for $U \cup V$

now let M be a manifold with a finite good cover

we prove Thm by induction on the length of the cover

<u>Base Case:</u> one element $M \cong \mathbb{R}^n$

note after Cor 17 we observed

$$H^k_c(\mathbb{R}^n) \cong H^{n-k}_{DR}(\mathbb{R}^n)$$

so only nontrivial case to check is

$$\begin{array}{ccccc} H^0_{DR}(\mathbb{R}) & \times & H^n_c(\mathbb{R}^n) & \to & \mathbb{R} \\ \wr\Vert & & \wr\Vert & & \\ \mathbb{R} & & \mathbb{R} & & \end{array}$$

we just need to see generators pair nontrivially, easy exercise

<u>Inductive step:</u> exactly like proof of Thm 14