XII Oriented Intersection and Degree

A Oriented Intersection

let U^p and W^q be two oriented subspaces of the vector space V^n with $n = p + q$

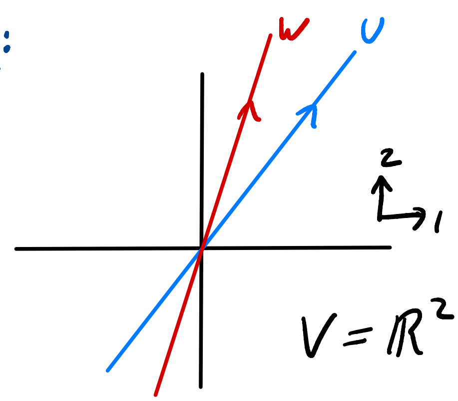
if $U \cap W = \{0\}$ and all the vector spaces are oriented we say $U \cap W = \{0\}$ has orientation $+$

$\iff$

oriented basis for U followed by an oriented basis for W is an oriented basis for V

example:

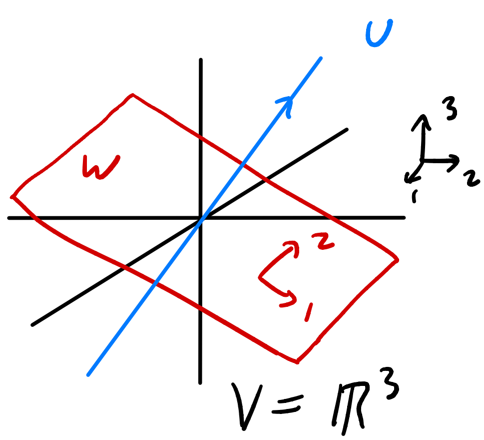
1)



so $U \cap W$ is $+$

$W \cap U$ is $-$

2)



so $U \cap W$ is $+$

$W \cap U$ is $+$

now suppose that M, N are manifolds

- S a submanifold of N
- $\dim S + \dim M = \dim N$
- M and S have no boundary
- S is closed in N and
- M is compact

given any map

$$f: M \to N$$

we can use Th$^{\underline{m}}$ VII.2 to homotop f to f_1 such that $f_1 \pitchfork S$

note: $f_1^{-1}(S) = f_1(M) \cap S = \{\text{finite set of points}\}$

if M, N, S oriented then $f_1^{-1}(S)$ oriented 0-manifold by assigning $\varepsilon(x) = \pm 1$ to $x \in f_1^{-1}(S)$ is $df_x(T_x M) \cap T_{f(x)} S$ is $\pm$ as above

the (oriented) intersection of f with S is

$$I(f, S) = \sum_{x \in f_1^{-1}(S)} \varepsilon(x)$$

to see that this is well-defined we need to consider orientations on preimages

that is, given M, N any oriented manifolds and S an oriented submanifold of N

Suppose $f: M \to N$ is transverse to S

let $\Sigma = f^{-1}(S)$

we show how to orient Σ

let $x \in \Sigma$ any $y = f(x) \in S$

we know

normal bundle

$$T_x M = T_x \Sigma \oplus \nu_x \Sigma$$

and by transversality

$T_y N$ spanned by $T_y S$ and $\mathrm{im}(df_x)$

recall $df_x(T_x \Sigma) \subset T_y S$

(in particular $T_x \Sigma = df_x^{-1}(T_y S)$)

now fiber dim $\nu_x \Sigma$ = codim Σ in M

= codim S in N

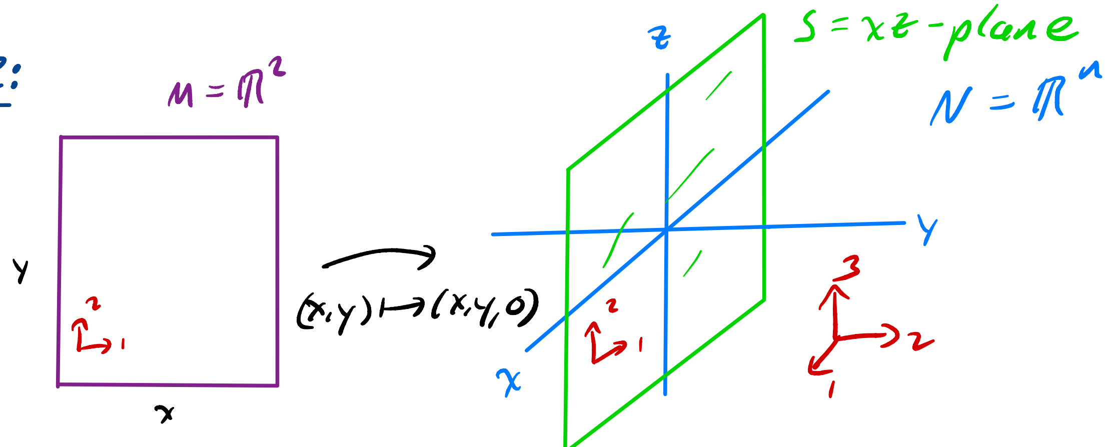
so $T_y N = df_x(\nu_x \Sigma) \oplus T_y S$

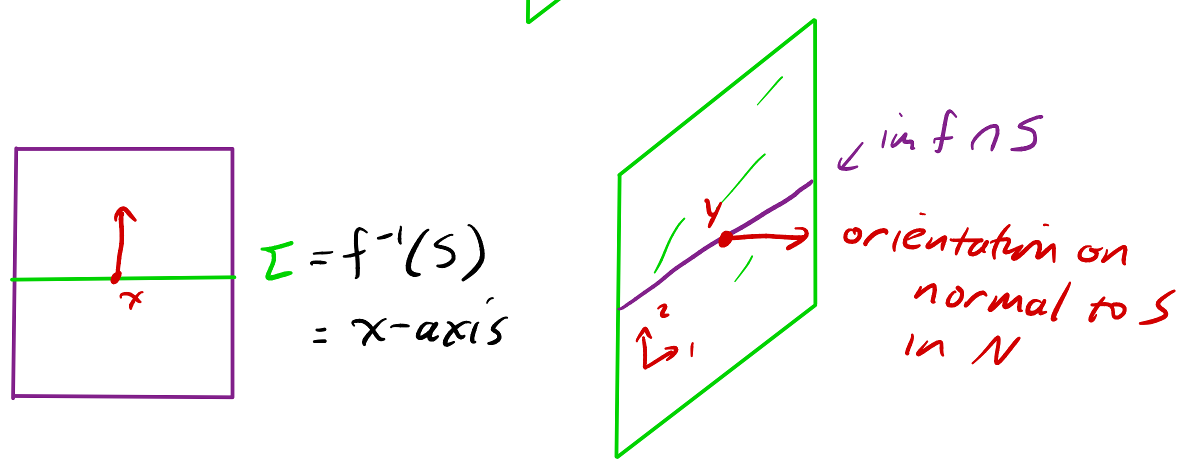
and $df_x|_{\nu_x \Sigma}$ injective

an orientation on $T_y S$ and $T_y N$ give an orientation on $df_x(\nu_x \Sigma)$ so by injectivity an orientation on $\nu_x \Sigma$

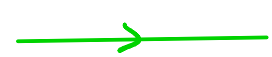
now an orientation on $T_x M$ and $\nu_x \Sigma$ give an orientation on $T_x \Sigma$ i.e. Σ is oriented!

<u>example:</u>



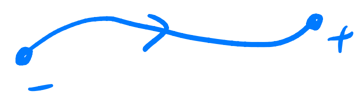


red arrow maps to red arrow by df_x

so orientation on Σ is 

Remarks:

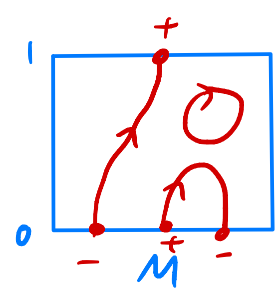
1) $I(f,S)$ is well-defined just like for $I_2(f,S)$ in Th^{m} VII.6 you just need to know the signed count of ∂(1-manifold) is 0



(indeed, f_0 homotopic to f_1 by homotopy

$$F: M \times [0,1] \to N$$

then make $F \pitchfork S$ and $F^{-1}(S)$ is oriented 1-manifold



)

2) $f_1 \simeq f_2 \Rightarrow I(f_1, S) = I(f_2, S)$

just as in Th^{m} VII.6

3) clearly $I_2(f,S) = I(f,S) \bmod 2$

4) Th^{m} VII.7 true for $I(f,S)$ too with same proof

i.e. M, N, S as above $f: M \to N$
if $\exists$ a compact oriented manifold W with $\partial W = M$ and f can be extended to $F: W \to N$ then $I(f, S) = 0$

we can also define the degree of a map:
given M, N closed compact oriented manifolds of same dimension and a function $f: M \to N$

we define
$$\deg(f) = I(f, \{x\})$$
for any $x \in N$

exercise: if f is transverse to x then show
$$\operatorname{def}(f) = \sum_{y \in f^{-1}(x)} \operatorname{sgn}\left(\det(\psi_x \circ f \circ \phi_y^{-1})\right)$$
where $\psi_x : U' \to V'$ oriented coordinate chart about x
and $\phi_y : U \to V$ oriented coordinate chart about y

Remarks:

1) $\deg(f)$ well-defined (just as in Thm VII.7)

2) homotopic maps have same degree (just as in Cor VII.9)

3) Thm VII.11 true for $\deg(f)$

i.e. if $M = \partial W$ with W compact oriented manifold and f extends to $F: W \to N$ then $\deg(f) = 0$

4) Corollary: every non-constant complex polynomial has a root

(same proof as VII.12)

B. Degree and Integration

Thm 1:

If $f: M \to N$ a smooth map between compact, oriented, n-manifolds

and ω is an n-form

then
$$\int_M f^* \omega = \deg(f) \int_N \omega$$

Remark: Could use this to <u>define</u> degree

Proof: let $x \in N$ be a regular value of f

so $$\deg(f) = \sum_{y \in f^{-1}(x)} \varepsilon(y)$$

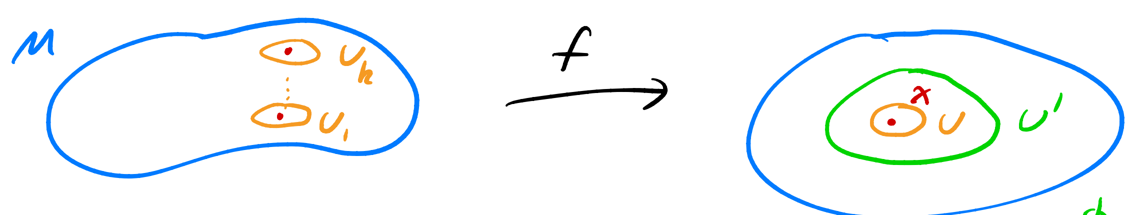
where $$\varepsilon(y) = \begin{cases} +1 & \text{if } df_y \text{ orientation preserving} \\ -1 & \text{if } df_y \text{ '' '' reversing} \end{cases}$$

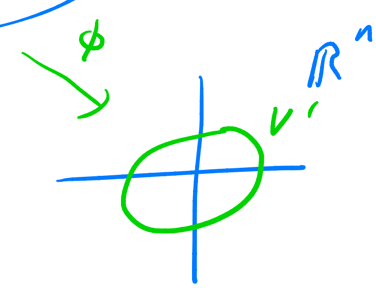
we know from lemma VII.8 that there is a neighborhood U of x such that

$$f^{-1}(U) = U_1 \cup \dots \cup U_k \qquad (\text{let } \{y_i\} = U_i \cap f^{-1}(x))$$

where U_i's are disjoint and $f|_{U_i} : U_i \to U$ is a diffeomorphism

we can assume U is in an oriented coordinate chart $\phi : U' \to V'$





now take an n-form

$$\tilde{\omega} = f\, dx^1 \wedge \dots \wedge dx^n$$

on $\mathbb{R}^n$ with support in $\phi(U)$ such that

$$\int_{\mathbb{R}^n} \tilde{\omega} = \int_{\phi(U)} \tilde{\omega} = c > 0$$

now let $\hat{\omega} = \begin{cases} \phi^* \tilde{\omega} & \text{on } U' \\ 0 & \text{elsewhere} \end{cases}$

so $\int_N \hat{\omega} = \int_{\mathbb{R}^n} \tilde{\omega} = c > 0$

now notice that $f^* \hat{\omega}$ is supported in $U_1, \dots, U_k$ and each U_i is a coordinate chart for M

i.e. $\varphi_i : U_i \longrightarrow \phi(f(U_i))$
$y \longmapsto \phi(f(y))$

is a coordinate chart and it is orientation preserving

$\Longleftrightarrow$

$\varepsilon(y_i) = +1$

so $\int_M f^* \hat{\omega} = \int_{U_1 \cup \dots \cup U_k} f^* \hat{\omega} = \sum_{i=1}^{k} \int_{U_i} f^* \hat{\omega}$

$$= \sum_{i=1}^{k} \varepsilon(y_i) \int_{\phi_i(U_i)} \tilde{\omega}$$

$$= c \sum_{i=1}^{k} \varepsilon(y_i) = c \deg(f)$$

$$= \deg(f) \int_N \hat{\omega}$$

finally recall Th$^{\underline{m}}$ $\underline{\text{XI}}$.14 says

$$H^n_{DR}(N) \cong \mathbb{R}$$

so $d\hat{\omega} = 0 = d\omega$ both give cohomology classes $[\hat{\omega}]$ and $[\omega]$ in $H^n_{DR}(N)$

$\therefore \exists\, a \in \mathbb{R}$ such that $[\omega] = a[\hat{\omega}]$

thus $\exists\, \eta \in \Omega^{n-1}(N)$ such that

$$\omega = k\hat{\omega} + d\eta$$

$$\therefore \int_M f^*\omega = \int_M f^*(a\hat{\omega}) + f^* d\eta$$

$$= a\int_M f^*\hat{\omega} + \int_M d(f^*\eta)$$

$$= a \deg(f) \int_N \hat{\omega} + \cancelto{0}{\int_{\partial M} f^*\eta} \quad \partial M = \emptyset$$

$$= \deg(f)\left(\int_N a\hat{\omega} + \cancelto{0}{\int_{\partial N} d\eta}\right)$$

$$= \deg(f) \int_N (a\hat{\omega} + d\eta) = \deg f \int_N \omega$$

exercise:

Given $M \xrightarrow{f} N \xrightarrow{g} W$ show that

$$\deg(g \circ f) = \deg(g)\deg(f)$$