

## Math 6452 - Fall 2021 Homework 1

Work all these problems and talk to me if you have any questions on them, but carefully write up and turn in **only** problems 2, 4, 5, 7, 8, 9. **Due: In class on September 8.**

1. Lee's book, Problem 1-7
2. Lee's book, Problem 1-9
3. Lee's book, Problem 1-10
4. Let  $M$  be a smooth manifold with or without boundary. Show that  $C^\infty(M)$  is a commutative ring where multiplication and addition are defined point-wise. (Recall  $C^\infty(M)$  is the set of smooth maps from  $M$  to  $\mathbb{R}$ .)
5. Let  $M, X$ , and  $Y$  be smooth manifolds without boundary. Let  $\pi_X : X \times Y \rightarrow X$  and  $\pi_Y : X \times Y \rightarrow Y$  be the projection maps. Show that a function  $f : M \rightarrow X \times Y$  is smooth if and only if  $\pi_X \circ f : M \rightarrow X$  and  $\pi_Y \circ f : M \rightarrow Y$  are smooth.
6. Prove the following maps are smooth.
  - (a)  $p_n : S^1 \rightarrow S^1$ , where  $S^1$  is the unit circle in  $\mathbb{C}$  and  $p_n$  is the map  $z \mapsto z^n$  restricted to  $S^1$ .
  - (b)  $f : S^3 \rightarrow S^2$  given by  $f(w, z) = (z\bar{w} + w\bar{z}, iw\bar{z} - iz\bar{w}, z\bar{z} - w\bar{w})$  where  $S^3$  is the unit sphere in  $\mathbb{C}^2$  and  $S^2$  is the unit sphere in  $\mathbb{R}^3$ .
7. Let  $f : (\mathbb{R}^{n+1} - \{0\}) \rightarrow (\mathbb{R}^{k+1} - \{0\})$  be a smooth homogeneous function of degree  $d \in \mathbb{Z}$ . This means that  $f(cx) = c^d f(x)$  for all  $c \in \mathbb{R}$  and  $x \in (\mathbb{R}^{n+1} - \{0\})$ . Show that the map  $\tilde{f} : \mathbb{R}P^n \rightarrow \mathbb{R}P^k$  defined by  $\tilde{f}([x]) = [f(x)]$  is a well-defined, smooth map.
8. Let  $M$  and  $N$  be smooth manifolds.
  - (a) Show that a continuous function  $f : M \rightarrow N$  defines a linear map  $f^* : C(N) \rightarrow C(M)$  by  $h \in C(N)$  maps to  $h \circ f$ . (Here  $C(M)$  is the set of continuous functions from  $M$  to  $\mathbb{R}$ .)
  - (b) Show that  $f : M \rightarrow N$  is smooth if and only if  $f^*(C^\infty(N)) \subset C^\infty(M)$ .
  - (c) If  $f : M \rightarrow N$  is a homeomorphism, then show it is a diffeomorphism if and only if  $f^*$  restricts to an isomorphism from  $C^\infty(N)$  to  $C^\infty(M)$ .
9. Recall the Grassmann of  $k$ -dimensional subspaces of  $\mathbb{R}^n$  is denoted  $G(k, n)$  (see Lee pages 22-24). Using the standard inner product on  $\mathbb{R}^n$  and denoting the orthogonal complement of a subspace  $V$  of  $\mathbb{R}^n$  by  $V^\perp$ , we can define a map  $f : G(k, n) \rightarrow G(n-k, n)$  by  $f(V) = V^\perp$  for every  $V$  in  $G(k, n)$ . Show that  $f$  is a diffeomorphism.
10. Lee's book, Problem 1-1
11. Lee's book Problem 1-2