Math 6452 - Fall 2021
Homework 4

Work all these problems and talk to me if you have any questions on them, but carefully write up and turn in only problems 4, 5, 7, 8, 11, 12. Due: October 13.

1. (Problem 2 from Section 1.5 in Guillemin and Pollack) Which of the following spaces intersect transversely?
   - The $xy$-plane and the $z$-axis in $\mathbb{R}^3$.
   - The $xy$-plane and the plane spanned by $(3, 2, 0)$ and $(0, 4, -1)$ in $\mathbb{R}^3$.
   - The spaces $\mathbb{R}^k \times \{0\}$ and $\{0\} \times \mathbb{R}^l$ in $\mathbb{R}^n$. (This depends on $k, l, \text{and } n$.)
   - The spaces $\mathbb{R}^k \times \{0\}$ and $\mathbb{R}^l \times \{0\}$ in $\mathbb{R}^n$. (This depends on $k, l, \text{and } n$.)
   - The spaces $V \times \{0\}$ and the diagonal in $V \times V$, where $V$ is a vector space.
   - The symmetric $(A^t = A)$ and skew-symmetric $(A^t = -A)$ matrices in $M(n)$.

2. For which values of $r$ does the sphere $x^2 + y^2 + z^2 = r$ and $x^2 + y^2 - z^2 = 1$ intersect transversely? Draw the intersection for representative values of $r$.

3. A space $X$ is called contractible if the identity map is homotopic to a constant map (that is there is some point $p \in X$ such that the map $id : X \to X : x \mapsto x$ is homotopic to the map $c : X \to X : x \mapsto p$). Show that if $X$ is contractible then for any space $Y$ any two maps $Y \to X$ are homotopic. Also show that $\mathbb{R}^n$ is contractible for any $n$.

4. A space $X$ is called simply connected if every continuous map from $S^1$ to $X$ is homotopic to a constant map. Show a contractible space is simply connected. Moreover show that the $n$-sphere $S^n$ is simply connected if $n > 1$.
   Hint: Given a smooth map $S^1 \to S^n$ use Sard’s theorem to say it misses a point and then think about stereographic projection.

5. Show that $S^n \times S^1$ is not simply connected for $n \geq 0$.
   Hint: Consider the submanifold $S = S^n \times \{p\}$ for some $p \in S^1$ and the map $f : S^1 \to S^n \times S^1 : \theta \mapsto (x, \theta)$ for some $x \in S^n$.
   Notice that problems 4 and 5 imply that $S^3$ and $S^1 \times S^2$, which are both $S^1$ bundles over $S^2$, are not diffeomorphic.

6. If $M$ and $N$ are submanifolds of $\mathbb{R}^n$ then show that for almost every $x \in \mathbb{R}^n$ the translate $M + x$ is transverse to $N$. (Here almost everywhere means “off of a set of measure zero” and $M + x = \{y + x : y \in M\}$.)

7. Suppose that $f : M \to N$ is transverse to the submanifold $S$ in $N$. Show that $T_pf^{-1}(S)$ is given by $(df_p)^{-1}(T_{f(p)}S)$. In particular if $S_1$ and $S_2$ are submanifolds of $N$ and they intersect transversely then $T_p(S_1 \cap S_2) = (T_pS_1) \cap (T_pS_2)$.

8. If $f : M \to N$ has $p$ as a regular value and $S = f^{-1}(p)$ show that the normal bundle to $S$ in $M$ is trivial.

9. Let $M$ and $N$ be manifolds of the same dimensions with $M$ compact and $N$ connected. Prove that if $f : M \to N$ has $\deg f \neq 0$ then $f$ is surjective.

10. Let $f : M \to \mathbb{R}$ be a smooth function. A critical point of $f$ is a point $p \in M$ such that $df_p = 0$. We say that $p$ is non-degenerate in the coordinate chart $\phi : U \to V$ if the matrix

$$H = \left( \frac{\partial^2 F}{\partial x^i \partial x^j} \right)$$
is non-singular where $F = f \circ \phi^{-1}$ and $\phi(p) = q$. Show that a critical point is non-degenerate in one coordinate chart if and only if it is non-degenerate in any coordinate chart. Thus it makes sense to talk about non-degenerate critical points independent of coordinate charts.

Note: The matrix $H$ is not well-defined independent of the coordinate chart, but whether it is non-singular or not is.

11. Show that non-degenerate critical points of a function $f : M \to \mathbb{R}$ are isolated (that is each such critical point has a neighborhood containing no other critical points).

Hint: Work in local coordinate so the function is of the form $f : \mathbb{R}^k \to \mathbb{R}$ and one can then think of $df$ as a function $df : \mathbb{R}^k \to \mathbb{R}^k$. Prove $df$ is a local diffeomorphism near a non-degenerate critical point.

A function $f : M \to \mathbb{R}$ is called a Morse function if all of its critical points are non-degenerate.

12. Show that the function $\mathbb{R}^{n+1} \to \mathbb{R} : (x^1, \ldots, x^{n+1}) \mapsto x^{n+1}$ restricted to $S^n$ is a Morse function with exactly two critical points. (This function is sometimes called the height function.)

13. Suppose that $M$ is a submanifold of $\mathbb{R}^{k+1}$. The set of $v \in S^k$ for which the map $f_v : M \to \mathbb{R} : x \mapsto v \cdot x$ is not a Morse function has measure zero. (So every manifold has a lot of Morse functions.)

14. Suppose that $M$ is a submanifold of $\mathbb{R}^{k+1}$. The set of points $p \in \mathbb{R}^{k+1}$ for which the map $f_p : M \to \mathbb{R} : x \mapsto \|x - p\|^2$ is not a Morse function has measure zero.