Math 6452 - Fall 2021
Homework 4

Work all these problems and talk to me if you have any questions on them, but carefully write up and turn in **only** *problems 4, 5, 7, 8, 11, 12.* **Due: October 13.**

1. (Problem 2 from Section 1.5 in Guillemin and Pollack) Which of the following spaces intersect transversely?
   - The xy-plane and the z-axis in $\mathbb{R}^3$.
   - The xy-plane and the plane spanned by $(3,2,0)$ and $(0,4,-1))$ in $\mathbb{R}^3$.
   - The spaces $\mathbb{R}^k \times \{0\}$ and $\{0\} \times \mathbb{R}^l$ in $\mathbb{R}^n$. (This depends on k, l, and n.)
   - The spaces $\mathbb{R}^k \times \{0\}$ and $\mathbb{R}^l \times \{0\}$ in $\mathbb{R}^n$. (This depends on k, l, and n.)
   - The spaces $V \times \{0\}$ and the diagonal in $V \times V$, where V is a vector space.
   - The symmetric ($A^t = A$) and skew-symmetric ($A^t = -A$) matrices in $M(n)$.
2. For which values of r does the sphere $x^2 + y^2 + z^2 = r$ and $x^2 + y^2 - z^2 = 1$ intersect transversely? Draw the intersection for representative values of r.
3. A space X is called *contractible* if the identity map is homotopic to a constant map (that is there is some point $p \in X$ such that the map $id : X \to X : x \mapsto x$ is homotopic to the map $c : X \to X : x \mapsto p$). Show that if X is contractible then for any space Y any two maps $Y \to X$ are homotopic. Also show that $\mathbb{R}^n$ is contractible for any n.
4. A space X is called *simply connected* if every continuous map from S^1 to X is homotopic to a constant map. Show a contractible space is simply connected. Moreover show that the n-sphere S^n is simply connected if $n > 1$.
   Hint: Given a smooth map $S^1 \to S^n$ use Sard's theorem to say it misses a point and then think about stereographic projection.
5. Show that $S^n \times S^1$ is not simply connected for $n \geq 0$.
   Hint: Consider the submanifold $S = S^n \times \{p\}$ for some $p \in S^1$ and the map $f : S^1 \to S^n \times S^1 : \theta \mapsto (x, \theta)$ for some $x \in S^n$.

   Notice that problems 4 and 5 imply that S^3 and $S^1 \times S^2$, which are both S^1 bundles over S^2, are not diffeomorphic.
6. If M and N are submanifolds of $\mathbb{R}^n$ then show that for almost every $x \in \mathbb{R}^n$ the translate $M + x$ is transverse to N. (Here *almost everywhere* means "off of a set of measure zero" and $M + x = \{y + x : y \in M\}$.)
7. Suppose that $f : M \to N$ is transverse to the submanifold S in N. Show that $T_p f^{-1}(S)$ is give by $(df_p)^{-1}(T_{f(p)}S)$. In particular if S_1 and S_2 are submanifolds of N and they intersect transversely then $T_p(S_1 \cap S_2) = (T_p S_1) \cap (T_p S_2)$.
8. If $f : M \to N$ has p as a regular value and $S = f^{-1}(p)$ show that the normal bundle to S in M is trivial.
9. Let M and N be manifolds of the same dimensions with M compact and N connected. Prove that if $f : M \to N$ has $deg_2(f) \neq 0$ then f is surjective.
10. Let $f : M \to \mathbb{R}$ be a smooth function. A critical point of f is a point $p \in M$ such that $df_p = 0$. We say that p is non-degenerate in the coordinate chart $\phi : U \to V$ if the matrix

$$H = \left(\frac{\partial^2 F}{\partial x^i \partial x^j}(q) \right)$$

is non-singular where $F = f \circ \phi^{-1}$ and $\phi(p) = q$. Show that a critical point is non-degenerate in one coordinate chart if and only it if is non-degenerate in any coordinate chart. Thus it makes sense to talk about non-degenerate critical points independent of coordinate charts.

Note: The matrix H is not well-defined independent of the coordinate chart, but whether it is non-singular or not is.

11. Show that non-degenerate critical points of a function $f : M \to \mathbb{R}$ are isolated (that is each such critical point has a neighborhood containing no other critical points).

    Hint: Work in local coordinate so the function is of the form $f : \mathbb{R}^k \to \mathbb{R}$ and one can then think of df as a function $df : \mathbb{R}^k \to \mathbb{R}^k$. Prove df is a local diffeomorphism near a non-degenerate critical point.

    A function $f : M \to \mathbb{R}$ is called a *Morse function* if all of its critical points are non-degenerate.

12. Show that the function $\mathbb{R}^{n+1} \to \mathbb{R} : (x^1, \ldots, x^{n+1}) \mapsto x^{n+1}$ restricted to S^n is a Morse function with exactly two critical points. (This function is sometimes called the *height function*.)
13. Suppose that M is a submanifold of $\mathbb{R}^{k+1}$. The set of $v \in S^k$ for which the map $f_v : M \to \mathbb{R} : x \mapsto v \cdot x$ is not a Morse function has measure zero. (So every manifold has a lot of Morse functions.)
14. Suppose that M is a submanifold of $\mathbb{R}^{k+1}$. The set of points $p \in \mathbb{R}^{k+1}$ for which the map $f_p : M \to \mathbb{R} : x \mapsto \|x - p\|^2$ is not a Morse function has measure zero.