Math 6452 - Fall 2021
Homework 5

Work all these problems and talk to me if you have any questions on them, but carefully write up and turn in **only** *problems 2, 5, 6, 7, 8, 12* **Due: November 5.**

1. For any finite dimensional vector space show that there are canonical isomorphisms $V \otimes \mathbb{R} \cong V \cong \mathbb{R} \otimes V$.
2. For finite dimensional vector spaces V and W show there is a canonical isomorphism $V^* \otimes W \cong Hom(V, W)$.
3. Let $\omega^1, \ldots, \omega^k$ be covectors in V^*. Show they are linearly dependent if and only if $\omega^1 \wedge \cdots \wedge \omega^k = 0$.
4. If $\{\omega^1, \ldots, \omega^k\}$ and $\{\eta^1, \ldots, \eta^n\}$ are linearly independent covectors in V^*, then show they have the same span if and only if $\omega^1 \wedge \cdots \wedge \omega^k = c\eta^1 \wedge \cdots \wedge \eta^k$ for some real number c. Show that $c = \det(A)$ where A is the matrix $(a_{i,j})$ and the $a_{i,j}$ are determined by $\omega_i = \sum_j a_{i,j}\eta_j$.
5. Let M be a smooth manifold and let $\omega \in \Gamma(T^k M)$ be a tensor field. Consider the map

$$\Psi : (\mathcal{X}(M) \times \ldots \times \mathcal{X}(M)) \to C^\infty(M) : (v_1, \ldots, v_k) \mapsto \omega(v_1, \ldots, v_k)$$

   here $\mathcal{X}(M)$ is the set of vector fields. Show that this map is multilinear over $C^\infty(M)$. Moreover show that given any multilinear map over $C^\infty(M)$

$$\Psi : (\mathcal{X}(M) \times \ldots \times \mathcal{X}(M)) \to C^\infty(M)$$

   it is induced from some tensor field by the above construction.
6. Define the 1-form ω on $\mathbb{R}^2 - \{(0,0)\}$ by

$$\omega(x, y) = \left(\frac{-y}{x^2 + y^2}\right) dx + \left(\frac{x}{x^2 + y^2}\right) dy$$

   (a) Compute $\int_C \omega$ where C is a circle of radius r about the origin.
   (b) Is ω the differential of a function on $\mathbb{R}^2 - \{(0,0)\}$? Explain why or why not.
7. Prove that a 1-form α on S^1 is the differential of a function if and only if $\int_{S^1} \alpha = 0$.
8. Prove that the first De Rham cohomology of S^1 is $H^1_{DR}(S^1) \cong \mathbb{R}$.
   Hint: Show that is show that if α is a fixed 1-form on S^1 such that $\int_{S^1} \alpha \neq 0$ then for any other 1-form ω there is a real number c such that $\omega = c\alpha + df$ for some function f.
9. Consider the forms on $\mathbb{R}^3$

$$f \in \omega^0(\mathbb{R}^3), \quad f\,dx + g\,dy + h\,dz \in \omega^1(\mathbb{R}^3), \text{ and}$$

$$f\,dy \wedge dz + g\,dz \wedge dy + h\,dx \wedge dy \in \omega^2(\mathbb{R}^3).$$

   Compute their exterior derivatives. Do they look like anything from vector calculus?
10. Given a vector space V and a vector $v \in V$ define the interior product

$$\iota_v : \Lambda^k(V) \to \Lambda^{k-1}(V)$$

as follows: given $\omega \in \Lambda^k(V)$ define $\iota_v\omega$ to be the $(k-1)$ form:

$$\iota_v\omega(v_1, \ldots, v_{k-1}) = \omega(v, v_1, \ldots, v_{k-1}).$$

If $\omega \in \Lambda^k(V)$ and $\eta \in \Lambda^l(V)$ then show that

$$\iota_v(\omega \wedge \eta) = (\iota_v\omega) \wedge \eta + (-1)^k \omega \wedge (\iota_v\eta).$$

11. On $\mathbb{R}^{2n}$ with coordinates $(x^1, y^1, \ldots, x^n, y^n)$ define the 1-form $\lambda = \frac{1}{2}\sum(x^i\, dy^i - y^i\, dx^i)$. Compute $d\lambda$ and $(d\lambda)^n$ (this means take the wedge product of $d\lambda$ with itself n times, for example $(d\lambda)^3 = (d\lambda) \wedge (d\lambda) \wedge (d\lambda)$). The 2-form $d\lambda$ is called the standard symplectic form on $\mathbb{R}^{2n}$.
12. Suppose V is an n–dimensional vector space. Given a linear map

$$L : V \to V$$

there is an induced linear map

$$L^* : \wedge^n(V) \to \wedge^n(V).$$

Since $\wedge^n(V)$ is 1–dimensional this map is simply multiplication by a constant. We will denote this constant $\det(L)$. Prove the following

(a) If you choose a basis $e_1, \ldots, e_n$ for V then L can be written as a matrix A_L, $\det(L) = \det(A_L)$, where the determinant of a matrix has the usual definition.

(b) $\det(L \circ S) = \det(L)\det(S)$.

(c) If L is the identity map then $\det(L) = 1$.

(d) L is an isomorphism if and only if $\det(L) \neq 0$ and in this case $\det(L^{-1}) = (\det(L))^{-1}$.