Math 6452 - Fall 2021
Homework 6

Work all these problems and talk to me if you have any questions on them, but carefully write up and turn in **only** *problems 2, 5, 6, 7, 9, and 10.* **Due: November 22**

1. Let ω be a 1-form on a 3-dimensional manifold M. Suppose that ω is not zero at any point so for each $x \in M$ the kernel ξ_x of $\omega(x)$ is a plane in T_xM. We say that ξ is integrable if for any two vector fields v and w with values in ξ (that is v and w are sections of ξ) we have that the Lie bracket $[v, w]$ is also a section of ξ. For this problem assume that ω is integrable.
   (a) Show that $\omega \wedge d\omega = 0$.
   (b) Show there exists a 1-form α such that $d\omega = \omega \wedge \alpha$. (Hint: prove this locally and then use a partition of unity.)
   (c) Show that $\omega \wedge d\alpha = 0$.
   (d) If β is another 1-form such that $d\omega = \omega \wedge \beta$ then there is a function f such that $\beta = \alpha + f\omega$ and $\alpha \wedge d\alpha = \beta \wedge d\beta$.
2. Given an area form ω on a surface Σ (that is a 2–form that is never zero) then one can define the divergence of a vector field v on Σ as the unique function $\mathrm{div}_\omega v$ such that
$$L_v\omega = (\mathrm{div}_\omega v)\omega.$$
   (a) Show that if ω' is another area form (defining the same orientation) then there is a unique positive function f such that $\omega' = f\omega$ and that
$$\mathrm{div}_\omega(v) = \mathrm{div}_{\omega'}(v) + d(\ln f)(v).$$
   (b) Derive a formula for $\mathrm{div}_\omega(v')$ in terms of $\mathrm{div}_\omega(v)$ if $v' = gv$ for some function g.
   (c) Show that given a function $f : \Sigma \to \mathbb{R}$ there is a unique vector field v_f that satisfies $\iota_{v_f}\omega = df$.
   (d) Show the flow of v_f from the previous item preserves the level sets of f and has zero divergence.
3. Let $a : S^n \to S^n$ be the antipodal map, that is the map $a(x) = -x$ when we think of S^n as the unit sphere in $\mathbb{R}^n$. Show that a is orientation preserving if and only if n is odd.
4. Show that $\mathbb{R}P^n$ is orientable if and only if n is odd.
5. Suppose that M and N are oriented manifolds and $f : M \to N$ is a local diffeomorphism. If M is connected then show that f is either orientation preserving or orientation reversing.
6. On $\mathbb{R}^n - \{0\}$ consider the $(n-1)$-form
$$\omega = \frac{1}{\|x\|^n}\sum_{i=1}^{n}(-1)^{i-1}x^i\, dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^n.$$
   Compute $d\omega$.

7. Let S^2 be the unit sphere in $\mathbb{R}^3$ and ω the 2-form from the the previous exercise. If $i : S^2 \to \mathbb{R}^3$ is the inclusion map then compute

$$\int_{S^2} i^*\omega.$$

Is there and 1-form η on $\mathbb{R}^3 - \{0\}$ such that $d\eta = \omega$? Explain why or why not. Notice that this and the previous exercise imply that $H^2_{DR}(\mathbb{R}^3 - \{0\}) \neq 0$.
If you feel like it maybe try to work this problem again for S^{n-1} (this is not required to be turned in).

8. Use Stokes theorem to prove the classical Green's formula: Give a region R in $\mathbb{R}^2$ with smooth boundary $\partial R = \gamma$ then show

$$\int_\gamma f\,dx + g\,dy = \int_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dxdy.$$

9. Given any embedding $f : T^2 \to S^3$ show that for any closed 2-form ω on S^3 we have

$$\int_{T^2} f^*\omega = 0.$$

Hint: Show that there is a smooth homotopy $H : T^2 \times [0,1] \to S^3$ from f to a constant map. Now use Stokes theorem.

10. Show there is some embedding $f : T^2 \to T^3$ and a closed 2-form ω on T^3 such that

$$\int_{T^2} f^*\omega \neq 0.$$

Notice that this problem together with the previous one implies that S^3 is not diffeomorphic to T^3.