1. Let $\omega$ be a 1-form on a 3-dimensional manifold $M$. Suppose that $\omega$ is not zero at any point so for each $x \in M$ the kernel $\xi_x$ of $\omega(x)$ is a plane in $T_x M$. We say that $\xi$ is integrable if for any two vector fields $v$ and $w$ with values in $\xi$ (that is $v$ and $w$ are sections of $\xi$) we have that the Lie bracket $[v, w]$ is also a section of $\xi$. For this problem assume that $\omega$ is integrable.

(a) Show that $\omega \wedge d\omega = 0$.
(b) Show there exists a 1-form $\alpha$ such that $d\omega = \omega \wedge \alpha$. (Hint: prove this locally and then use a partition of unity.)
(c) Show that $\omega \wedge d\alpha = 0$.
(d) If $\beta$ is another 1-form such that $d\omega = \omega \wedge \beta$ then there is a function $f$ such that $\beta = \alpha + f\omega$ and $\alpha \wedge d\alpha = \beta \wedge d\beta$.

2. Given an area form $\omega$ on a surface $\Sigma$ (that is a 2–form that is never zero) then one can define the divergence of a vector field $v$ on $\Sigma$ as the unique function $\text{div}_\omega v$ such that

$L_v \omega = (\text{div}_\omega v)\omega$.

(a) Show that if $\omega'$ is another area form (defining the same orientation) then there is a unique positive function $f$ such that $\omega' = f\omega$ and that

$\text{div}_\omega(v) = \text{div}_{\omega'}(v) + d(\ln f)(v)$.

(b) Derive a formula for $\text{div}_{\omega'}(v')$ in terms of $\text{div}_\omega(v)$ if $v' = gv$ for some function $g$.
(c) Show that given a function $f : \Sigma \to \mathbb{R}$ there is a unique vector field $v_f$ that satisfies $\iota_{v_f} \omega = df$.
(d) Show the flow of $v_f$ from the previous item preserves the level sets of $f$ and has zero divergence.

3. Let $a : S^n \to S^n$ be the antipodal map, that is the map $a(x) = -x$ when we think of $S^n$ as the unit sphere in $\mathbb{R}^n$. Show that $a$ is orientation preserving if and only if $n$ is odd.

4. Show that $\mathbb{R}P^n$ is orientable if and only if $n$ is odd.

5. Suppose that $M$ and $N$ are oriented manifolds and $f : M \to N$ is a local diffeomorphism. If $M$ is connected then show that $f$ is either orientation preserving or orientation reversing.

6. On $\mathbb{R}^n - \{0\}$ consider the $(n-1)$-form

$$\omega = \frac{1}{\|x\|^n} \sum_{i=1}^{n} (-1)^{i-1} x^i \, dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^n.$$ 

Compute $d\omega$. 

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7. Let $S^2$ be the unit sphere in $\mathbb{R}^3$ and $\omega$ the 2-form from the previous exercise. If $i : S^2 \to \mathbb{R}^3$ is the inclusion map then compute

$$\int_{S^2} i^* \omega.$$ 

Is there a 1-form $\eta$ on $\mathbb{R}^3 - \{0\}$ such that $d\eta = \omega$? Explain why or why not. Notice that this and the previous exercise imply that $H^2_{DR}(\mathbb{R}^3 - \{0\}) \neq 0$. If you feel like it maybe try to work this problem again for $S^{n-1}$ (this is not required to be turned in).

8. Use Stokes theorem to prove the classical Green’s formula: Give a region $R$ in $\mathbb{R}^2$ with smooth boundary $\partial R = \gamma$ then show

$$\int_{\gamma} f \, dx + g \, dy = \int_{R} \left( \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) \, dxdy.$$ 

9. Given any embedding $f : T^2 \to S^3$ show that for any closed 2-form $\omega$ on $S^3$ we have

$$\int_{T^2} f^* \omega = 0.$$ 

Hint: Show that there is a smooth homotopy $H : T^2 \times [0, 1] \to S^3$ from $f$ to a constant map. Now use Stokes theorem.

10. Show there is some embedding $f : T^2 \to T^3$ and a closed 2-form $\omega$ on $T^3$ such that

$$\int_{T^2} f^* \omega \neq 0.$$ 

Notice that this problem together with the previous one implies that $S^3$ is not diffeomorphic to $T^3$. 

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