Math 6457 - Fall 2008
Homework 3

Work all the problems, but carefully write up and turn in Problems 4, 5, 10 and 11.

1. Let S^3 be the unit sphere in $\mathbb{C}^2$. Fix a point $(z, w) \in S^3$. and define the map
$$\gamma : \mathbb{R} \to S^3 : t \mapsto (e^{it}z, e^{it}w).$$
Express γ in stereographic coordinates and show γ is smooth. (Try to visualize the image of γ for certain choices of (z, w).) Compute $d\gamma_x : T_x\mathbb{R} \to T_{\gamma(x)}S^3$. (Notice that γ induces a map $S^1 \to S^3$ so γ is describing a loop in S^3.)

2. Let S^3 be the unit sphere in $\mathbb{C}^2$ and S^2 be the unit sphere in $\mathbb{R}^3$. Define the map
$$f : S^3 \to S^2$$
by
$$f(z, w) = (z\overline{w} + w\overline{z}, iw\overline{z} - iz\overline{w}, z\overline{z} - w\overline{w}).$$
Express the map using $z = x^1 + iy^1$ and $w = x^2 + iy^2$. Express this map in stereographic coordinates and check that it is smooth. Compute df. (Notice that with γ as above, $f \circ \gamma$ is constant. You might like to show that the inverse image of a point under f is the image of γ for some choice of (z, w).)

3. Show that $\mathbb{C}P^1$ is diffeomorphic to S^2.

4. Let G be a Lie group. Consider the map
$$m : G \times G \to G : (g, h) \mapsto gh.$$
Show
$$dm_{(I,I)}(v, w) = v + w$$
where I is the identity in G. (Hint: compute $dm_{(I,I)}(v, 0)$ and $dm_{(I,I)}(0, w)$ by representing tangent vectors with paths). Compute
$$di_I(v) = -v$$
where
$$i : G \to G : g \mapsto g^{-1}.$$

5. Define the map
$$f : \mathbb{C}P^n \to \mathbb{C}P^m$$
by
$$f([x^0 : \cdots : x^n]) = [x^0 : \cdots : x^n : 0 : \cdots : 0]$$
where $n \leq m$. Show f is smooth and injective. Show df is injective at every point. Thus f is an embedding. (Notice that this shows, for example, that S^2 is a submanifold of $\mathbb{C}P^2$.)

6. With f as in the previous problem show that $\mathbb{C}P^{n+1} - f(\mathbb{C}P^n)$ is diffeomorphic to $\mathbb{C}^{n+1}$. (So for example $\mathbb{C}P^2$ is the union of $\mathbb{C}P^1 \cong S^2$ and $\mathbb{C}^2$. Thus we can think of $\mathbb{C}P^2$ is the compactification of $\mathbb{C}^2$ by an "S^2 at infinity".)

7. A smooth map $f : (\mathbb{C}^{n+1} - \{(0, \ldots, 0)\}) \to (\mathbb{C}^{k+1} - \{(0, \ldots, 0)\})$ is called homogeneous of degree k if $f(\lambda p) = \lambda^k f(p)$ for all $\lambda \neq 0$ and $p \in (\mathbb{C}^{n+1} - \{(0, \ldots, 0)\})$. Show that f induces a map
$$\tilde{f} : \mathbb{C}P^n \to \mathbb{C}P^k.$$
Show this map is smooth.

8. Define the map
$$f : \mathbb{C}P^n \times \mathbb{C}P^m \to \mathbb{C}P^{nm+n+m}$$
by
$$f([x^0 : \cdots : x^n], [y^0 : \cdots : y^m]) = [x^0y^0 : x^0y^1 : \cdots : x^0y^m : x^1y^0 : \cdots : x^ny^m].$$
Show f is a smooth map and that f is one-to-one. Show df is injective at every point. Thus f is an embedding. (Notice that this shows, for example, that $S^2 \times S^2$ is a submanifold of $\mathbb{C}P^3$).

   Of course the last 4 problems could also have been carried out for real projective spaces.

9. For what values of a does $x^2 + y^2 - z^2 = 1$ and $x^2 + y^2 + z^2 = a$ intersect transversely? Determine the intersection for each value of a.

10. Let $f : M \to N$ be a smooth map that is transverse to the submanifold S of N, and set $W = f^{-1}(S)$. Show that T_xW is $df^{-1}(T_{f(x)}S)$.

11. Let M be a smooth submanifold of $\mathbb{R}^n$. Show that there is a linear map $L : \mathbb{R}^n \to R$ such that $L|_M : M \to \mathbb{R}$ is a Morse function.

12. The function
$$f : \mathbb{C}P^n \to \mathbb{R} : [z^0 : \cdots : z^n] \mapsto \frac{\sum_{k=0}^n (k+1)|z_k|^2}{\sum_{k=0}^n |z_k|^2}$$
is well defined on $\mathbb{C}P^n$. Show that it is smooth and a Morse function. Compute its critical points and their index.