III Constructing Spaces

def$^{\underline{n}}$: let X be a topological space
Y a set
$f: X \to Y$ a surjective map

set
$$\mathcal{T}_f = \{ U \subset Y \mid f^{-1}(U) \text{ open in } X \}$$

$\mathcal{T}_f$ is called the quotient topology on Y

(you can check it is a topology)

Th$^{\underline{m}}$ 1: X and Y topological spaces
$f: X \to Y$ surjective map
Then the quotient topology on Y agrees with the given topology on Y iff $\underbrace{(U \text{ open in } Y \Leftrightarrow f^{-1}(U) \text{ open in } X)}_{(*)}$

Proof: $(\Rightarrow)$ U open in $Y \Leftrightarrow U \in \mathcal{T}_Y \Leftrightarrow U \in \mathcal{T}_f = \mathcal{T}_Y \Leftrightarrow f^{-1}(U)$ open in X.

$(\Leftarrow)$ $U \in \mathcal{T}_Y \underset{(*)}{\Leftrightarrow} f^{-1}(U)$ open in $X \Leftrightarrow U \in \mathcal{T}_f$ ✓

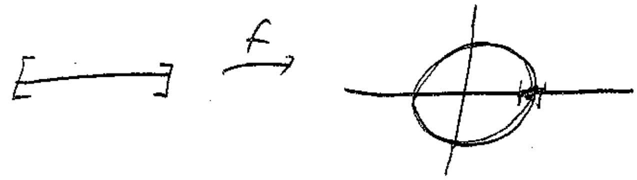
def$^{\underline{n}}$: f surjective and satisfying $(*)$ is called a quotient map (note they are continuous)

lemma 2: $f: X \to Y$ a continuous map onto Y.
f takes closed sets to closed sets $\Leftrightarrow$ f takes open sets to open sets $\Rightarrow$ f is a quotient map

Proof: 1$^{\text{st}}$ $\Leftrightarrow$ should be obvious
$\Rightarrow$ U open in Y, f continuous $\Rightarrow f^{-1}(U)$ open in X
if $f^{-1}(U)$ open in X then $U = f(f^{-1}(U))$ is open in Y
$\therefore (*)$ ✓

example: $X = [0, 2\pi]$ $\quad Y = S^1 = \{ (x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \}$

$$f: X \to Y : \theta \mapsto (\cos\theta, \sin\theta)$$



f is clearly onto Y

f is continuous since we know $\cos\theta$, and $\sin\theta$ are continuous (+ a homework problem)

we show f takes closed sets to closed sets (then f is a quotient map)

let C be a closed set in X

want to show $f(C)$ closed

let p be a limit pt of $f(C)$

since S^1 is 1^{st} countable (since $\mathbb{R}^2$ is) we know there is a sequence of points $p_n \in f(C)$ st. $p_n \to p$ (Th^m I.7)

p_n in $f(C) \Rightarrow \exists\ x_n \in C$ st $f(x_n) = p_n$

X compact & C closed $\Rightarrow C$ compact

$\Rightarrow C$ sequentially cpt (since C & X are 1^{st} countable Th^m II.11 & II.13)

$\Rightarrow \exists$ subsequence x_{n_i} st $x_{n_i} \to x$

$\therefore x$ limit pt of C, C closed $\Rightarrow x \in C$

now $f(x_{n_i}) \to p$ and $f(x_{n_i}) \to f(x)$

since Y is Hausdorff $p = f(x)$ (Th^m I.5)

so $p \in f(C)$

$\therefore f(C)$ closed ✓

note: this example says we can think of S^1 as being a closed interval with the end pts identified. we explore this more later.

Th^m 3: $\boxed{\text{given } f: X \to Y \text{ a quotient map then a map } g: Y \to Z \text{ is continuous iff } g \circ f: X \to Z \text{ is continuous}}$

Proof: ($\Rightarrow$) we know composition of continuous maps is continuous

($\Leftarrow$) U open in $Z \Rightarrow (g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$ open in X

$\Rightarrow g^{-1}(U)$ open in Y (q-topology)

$\therefore g$ is continuous ✓

$$\begin{tikzcd} X \arrow[d, "f"'] \arrow[dr, bend left, "f\circ g"] & \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$

note: 1) any continuous map $X \to Z$ that sends all the points $f^{-1}(y)$ to the same pt in Z induces a continuous map $Y \to Z$

2) any continuous map $Y \to Z$ induces a continuous map $g \circ f : X \to Z$

so we can understand continuous maps on Y by just looking at X

example: $[0, 2\pi] \xrightarrow{f} S^1$

think of continuous maps on S^1 as just continuous maps on $[0, 2\pi]$ that take 0 & 2π to the same pt!

def^{n}: X a topological space

a decomposition $\mathcal{D}$ of X is a collection of disjoint subsets of X whose union is all of X

define a topology on $\mathcal{D}$ by saying

$$\mathcal{S} \subset \mathcal{D} \text{ is open} \iff \bigcup_{S \in \mathcal{S}} S \text{ open in } X$$

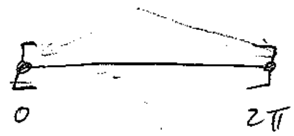
with this topology $\mathcal{D}$ is called a quotient space or decomposition space of X.

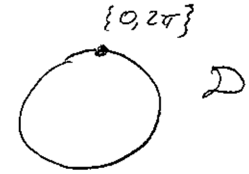
there is a natural surjective map $p : X \to \mathcal{D}$ that takes $x \in X$ to the set $S \in \mathcal{D}$ containing x

lemma 4: the topology on $\mathcal{D}$ is the quotient topology induced on $\mathcal{D}$ via the map $p : X \to \mathcal{D}$

Proof: exercise.

Remark: you should think of the space $\mathcal{D}$ as X with the sets in $\mathcal{D}$ crushed to points *example:* $X = [0, 2\pi]$ $\quad \mathcal{D} = \{ \{x\} \mid x \in (0,2\pi) \} \cup \{0, 2\pi\}$





Th^m 5 $f: X \to Z$ be a surjective continuous map.
$\mathcal{D} = \{ f^{-1}(z) \mid z \in Z \}$
give $\mathcal{D}$ the quotient topology, $p: X \to \mathcal{D}$ the natural map
f induces a continuous bijection $g: \mathcal{D} \to Z$
g is a homeomorphism iff f is a quotient map.

$$\begin{tikzcd} & X \arrow[dl, "p"'] \arrow[d, "f"] \\ \mathcal{D} \arrow[r, "g"'] & Z \end{tikzcd}$$

Proof: clearly g is a bijection
$g \circ p = f$ is continuous so Th^m 3 $\Rightarrow$ g is continuous
if f a quotient map then let $\mathcal{S} \subset \mathcal{D}$ be open set
so $p^{-1}(\mathcal{S})$ open
$p^{-1}(\mathcal{S}) = f^{-1}(g(\mathcal{S}))$
$\therefore g(\mathcal{S})$ open since f q-map
$\therefore g$ a homeo.
if g a homeo then U open in $Z \Leftrightarrow g^{-1}(U)$ open in $\mathcal{D}$
$\Leftrightarrow \underset{f^{-1}(U)}{\underset{\shortparallel}{p^{-1}(g^{-1}(U))}}$ open in X
$\therefore f$ a q-map

Cor 5: $f: X \to Z$ and $\mathcal{D}$ as above. if X cpt & Z Hausdorff then $g: \mathcal{D} \to Z$ is homeo

example: $f: [0, 2\pi] \to S^1$
$\theta \longmapsto (\cos\theta, \sin\theta)$

$\mathcal{D}$ in example above is $\{ f^{-1}(p) \mid p \in S^1 \}$
we know f a q-map so $\mathcal{D}$ is homeo to S^1

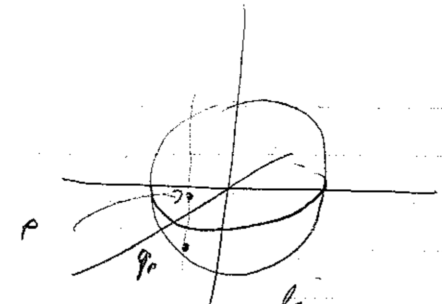
this makes precise the idea S^1 is $[0, 2\pi]$ with endpts identified.

more examples:
1) $X = D_2^2 = \{ (x,y) \in \mathbb{R}^2 \mid \sqrt{x^2+y^2} \le 2 \}$

$Y = S^2 = \{ (x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1 \}$

we define $f : X \to Y$

for pts $p \in D_1^2$

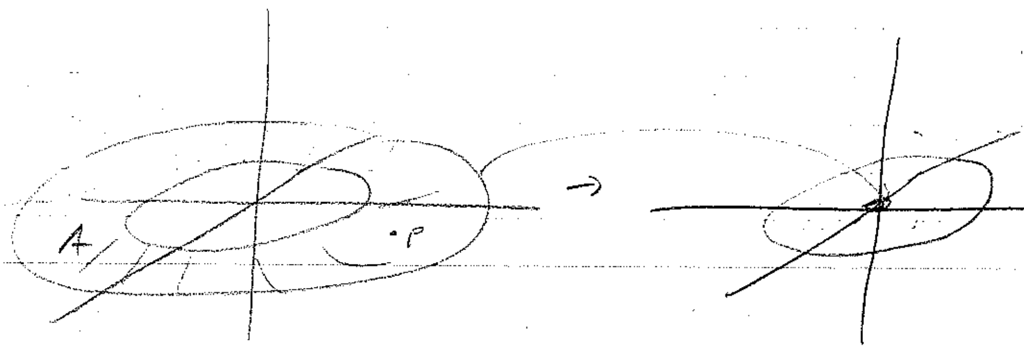


look at vertical line l_p through p
l_p intersects the southern hemisphere in one pt q_p

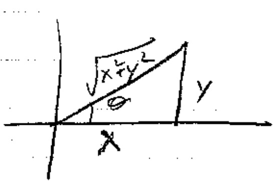
define $f(p) = q_p$

formula $f(x,y) = (x, y, -\sqrt{1-x^2-y^2})$ $\quad (x,y) \in D_1^2$

for pts $p \in A_{1,2} = \{ (x,y) \mid 1 \le \sqrt{x^2+y^2} \le 2 \}$



then some map as above



$(x,y) \mapsto (r,\theta) \mapsto (2-r, \theta) \mapsto ((2-r)\cos\theta, (2-r)\sin\theta)$

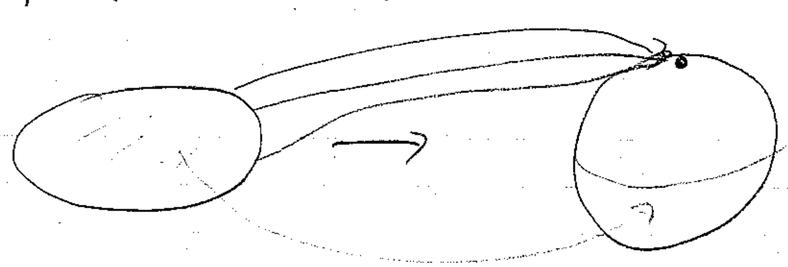
$= \left((2-\sqrt{x^2+y^2}) \cos\left(\tan^{-1}\frac{y}{x}\right), (2-\sqrt{x^2+y^2}) \sin\left(\tan^{-1}\frac{y}{x}\right) \right)$

$f(x,y) = \left((2-\sqrt{x^2+y^2}) \frac{x}{\sqrt{x^2+y^2}}, (2-\sqrt{x^2+y^2}) \frac{y}{\sqrt{x^2+y^2}}, \sqrt{1-4-x^2-y^2+4\sqrt{\ \ }} \right)$

$\frac{x^2(4+(x^2+y^2)-4\sqrt{\ \ }) + y^2(\ \ )}{x^2+y^2} = 4 - 4\sqrt{\ \ } + x^2+y^2$

$1 - (4 - 4\sqrt{\ \ } + x^2+y^2)$

note: $\mathcal{D}_f = \{ \{x\} \mid x \in B_2(0) \} \cup \{ (x,y) \mid x^2+y^2 = 4 \}$



"so we crush ∂ of disk to north pole."

f is clearly continuous and onto

f is closed map (X cpt so if C closed subset it is also cpt $\therefore f(C)$ cpt in Y, Y Hausdorff so $f(C)$ closed)

$\therefore f$ a q-map

$\therefore$ induced map $g: \mathcal{D} \to Y$ is <u>homeomorphism</u>

so we can think of S^2 as the disk with ∂ crushed to a pt.

2) In general $D^n = \{(x_1, \dots x_n) \in \mathbb{R}^n \mid x_1^2 + \dots x_n^2 \leq 1\}$

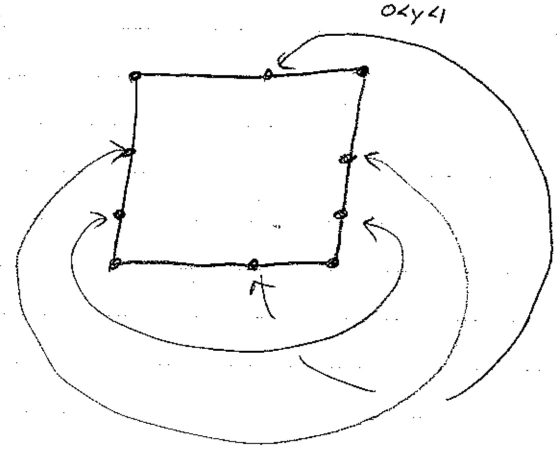
$S^n = \{(x_1 \dots x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \dots x_{n+1}^2 = 1\}$

$\mathcal{D} = \{(x_1 \dots x_n)\}_{x_1^2 + \dots x_n^2 < 1} \cup \{(x_1 \dots x_n) \mid x_1^2 + \dots x_n^2 = 1\}$.

then $\mathcal{D}$ homeo to S^n.

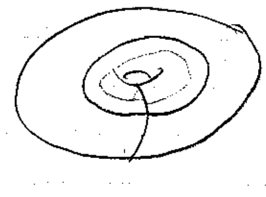
3) $X = [0,1] \times [0,1]$

$\mathcal{D} = \{(x,y)\}_{\substack{0<x<1 \\ 0<y<1}} \cup \{\{(1,y),(0,y)\}\}_{0<y<1} \cup \{\{(x,1),(x,0)\}\}_{0<x<1} \cup \{(0,0),(1,1),(0,1),(1,0)\}$

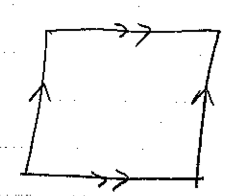


<u>Claim</u>: $\mathcal{D}$ homeo to $S^1 \times S^1$

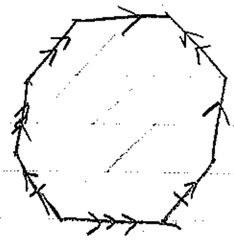
<u>torus</u>

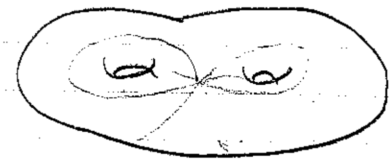


could have said



identify edges with same labels

4) $X =$ 

Claim: $\mathcal{D} \cong$ 

5)

$X = S^3 \subset \mathbb{R}^4 = \mathbb{C}^2$

$\uparrow$ unit sphere

$S^3 = \{(z_1, z_2) \mid |z_1|^2 + |z_2|^2 = 1\}$

$S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$

$(r_1 e^{i\theta_1}, r_2 e^{i\theta_2})$
r_1, r_2

say $\underbrace{\text{2 pts in } S^3}_{(z_1, z_2) \text{ and } (\tilde{z}_1, \tilde{z}_2)}$ are equivalent if there is a $z \in S^1$

such that $(z_1, z_2) = (z\tilde{z}_1, z\tilde{z}_2)$

let $\mathcal{D}$ = equivalence classes (some times this is denoted $X/\sim$)

Claim: $\mathcal{D}$ is homeomorphic to S^2

map $S^3 \to S^2$ is called the Hopf map or Hopf fibration

def$^{\underline{n}}$: let $S^{2n+1} \subset \mathbb{C}^{n+1}$ be unit sphere
let $\mathbb{C}P^n = S^{2n+1}/\sim$ where $\sim$ is as above
$\mathbb{C}P^n$ is called complex projective space

6) given
- Y, Z topological spaces
- $A \subset Y$
- $f : A \to Z$

let $\mathcal{D}$ be a decomp of $Y \amalg Z$ where only non trivial elts are

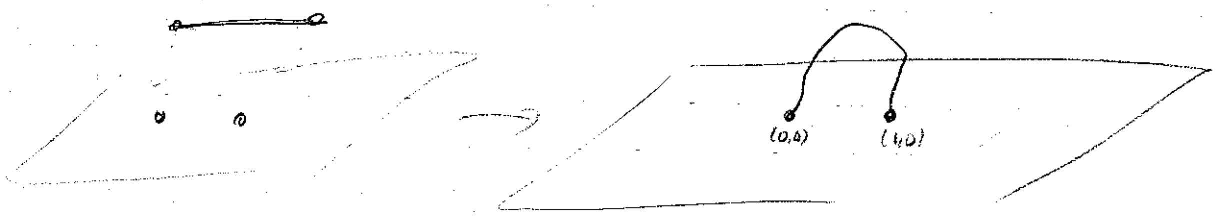
$\{\{a, f(a)\}\}_{a \in A}$

$\mathcal{D}$ is the space obtained by "gluing Y to Z along A with f"

denote $Y \cup_A Z$ or $Y \cup_f Z$

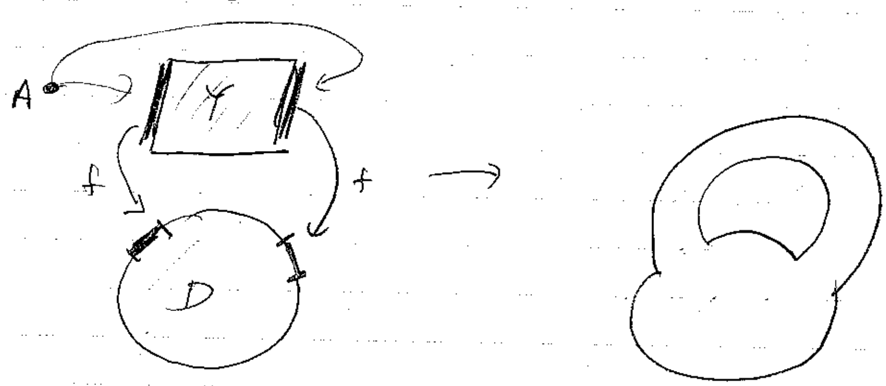
<u>eg</u>: a) $Y = [0,1]$ $\quad Z = \mathbb{R}^2$

$A = \{0,1\}$ $\quad f: \{0,1\} \to \mathbb{R}^2 : \begin{matrix} 0 \mapsto (0,0) \\ 1 \mapsto (1,0) \end{matrix}$



b) $Y = [0,1] \times [0,1]$ $\quad A = \{0,1\} \times [0,1]$

$Z = D^2$



c) $Y = B^4 = \{ (z_1, z_2) \in \mathbb{C}^2 \,/\, |z_1|^2 + |z_2|^2 \leq 1 \}$

$A = \{ (z_1, z_2) \in \mathbb{C}^2 \,/\, |z_1|^2 + |z_2|^2 = 1 \}$

$Z = S^2$ & $f: A \to S^2$ the Hopf map

<u>Claim</u>: $Y \cup_f S^2$ homeo $\mathbb{C}P^2$ (<u>Hard</u>)

7) $X = \mathbb{R}$ $\quad$ say x & y are equivalent if $x - y = n$ an <u>integer</u>

Claim $X/\sim$ homeo to S^1

<u>define</u>: $f: \mathbb{R} \to S^1 \subset \mathbb{R}^2$

$x \mapsto (\cos 2\pi x, \sin 2\pi x)$

<u>note</u>: f is continuous, surjective and induces a map $g: X/\sim \to S^1$

(since $f(x) = f(y)$

$\iff \cos 2\pi x = \cos 2\pi y$, $\sin 2\pi x = \sin 2\pi y$

$\iff x = y + n$)

$$\begin{tikzcd} X \arrow[d, "p"'] \arrow[dr, "f"] & \\ X/\sim \arrow[r, "g"'] & S^1 \end{tikzcd}$$

so g is continuous, surjective and one-to-one

g will be a homeo iff f is a q-map.
in particular if f is an open map
(you can check f sends small open intervals
to open sets in S^1 the result then follows
by taking unions)

think of S^1 or $\mathbb{R}^1$ curled up like a garden hose