CLASS NOTES ON HODGE THEORY

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Abstract. These are notes from an informal class that cover the Hodge theory of real and complex forms. The primary references used for the notes are Warner "Foundations of Differentiable Manifolds and Lie Groups", Wells "Differential Analysis on Complex Manifolds", and Griffiths and Harris "Principles of Algebraic Geometry".

The notes start with a fairly high level overview in the first two sections and then ends with some of the analytic details in the last sections. Towards the end the notes get a little sketching. Hopefully more details will appear in an updated version of these notes. Also, reader beware, these notes have not been proof read so there are almost certainly lots of typos and "mathos". Hopefully these will also be addressed in a future update.

Contents

1. Laplacians and the Hodge Theorem

In this section we prove the Hodge Theorem assuming some a couple analytic theorems which will be established in subsequent sections. To keep the discussion self-contained we briefly recall some facts from manifold theory and Riemannian geometry.

1.1. **Riemannian metrics and the Hodge star operator.** Below we recall a little linear algebra which is then promoted to manifolds.

1.1.1. *Linear theory.* Recall an **inner product** in a vector space V is a map

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$$

(or to $\mathbb{C}$ if V a complex vector space) such that

(1) $\langle v, w\rangle = \langle w, v\rangle$ (or for complex vector spaces $\langle v, w\rangle = \overline{\langle w, v\rangle}$)
(2) $\langle av, w\rangle = a\langle v, w\rangle$ and $\langle v + v', w\rangle = \langle v, w\rangle + \langle v', w\rangle$
(3) $\langle v, v\rangle \geq 0$ and
(4) $\langle v, v\rangle = 0$ if and only if $v = 0$.

Example 1.1. On $V = \mathbb{R}^n$ we have the Euclidean inner product

$$\left\langle \begin{bmatrix} x^1 \\ \vdots \\ x^n \end{bmatrix}, \begin{bmatrix} y^1 \\ \vdots \\ y^n \end{bmatrix} \right\rangle = \sum_{j=1}^{n} x^j y^j$$

If $e_1, \ldots, e_n$ is a basis for V then set

$$g_{ij} = \langle e_i, e_j\rangle.$$

Note that $g_{ij} = g_{ji}$. Now if $v = v^i e_i$ and $w = w^i e_i$ then we have

$$\langle v, w\rangle = v^i w^j g_{ij} = \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix}^t \begin{bmatrix} g_{ij} \end{bmatrix} \begin{bmatrix} w^1 \\ \vdots \\ w^n \end{bmatrix}.$$

So given a basis an inner product is equivalent to a certain type of symmetric matrix (note just any symmetric matrix will not do!).

Given an inner product on V there is a natural inner product on the dual space V^*. Specifically, notice that the non-degeneracy of the inner product says that the map

$$C : V \to V^* : v \mapsto \langle v, \cdot\rangle$$

is an isomorphism. Thus for any two $v^*, w^* \in V^*$ we can define the induced inner product to be

$$\langle v^*, w^*\rangle = \langle C^{-1}(v^*), C^{-1}(w^*)\rangle.$$

It is obvious that this is an inner product on V^*.

Exercise 1.2. We can always find an orthonormal basis for V. (Gram-Schmidt process)

Exercise 1.3. If $e_1, \ldots, e_n$ is an orthonormal basis for V then let $e^1, \ldots, e^n$ be the dual basis for V^*. Show in the induced inner product satisfies

$$\langle e^i, e^j\rangle = \delta^{ij}.$$

That is we could define the inner product on the dual space to be δ^{ij} in a basis dual to an orthonormal basis of V.

Exercise 1.4. If $e_1, \ldots, e_n$ is any basis for V and the inner product in this basis is given by $\begin{bmatrix} g_{ij} \end{bmatrix}$ then show the matrix defining the induced metric on V^* in the dual basis is

$$\begin{bmatrix} g_{ij} \end{bmatrix}^{-1}$$

and we will denote this by $\begin{bmatrix} g^{ij} \end{bmatrix}$.

We can now define an induced inner product on $\wedge^k V^*$ by

$$\langle v_1^* \wedge \ldots \wedge v_k^*, w_1^* \wedge \ldots \wedge w_k^* \rangle = \det\left[\langle v_i^*, w_j^* \rangle\right].$$

Exercise 1.5. This is an inner product on $\wedge^k V^*$.

Exercise 1.6. If $e_1, \ldots, e_n$ is an orthonormal basis for V then show the induced inner product on $\wedge^k V^*$ is determined by saying

$$\{v_{i_1}^* \wedge \ldots \wedge v_{i_k}^* | \text{ for } i_1 < \ldots < i_k\}$$

is an orthonormal basis for $\wedge^k V^*$.

Exercise 1.7. If $e_1, \ldots, e_n$ is any basis for V then show the inner product on $\wedge^n V^* \cong \mathbb{R}$ is given by

$$\mathbb{R} \times \mathbb{R} \to \mathbb{R} : (a, b) \mapsto ab \det\left[g_{ij}\right].$$

Let $e_1, \ldots, e_n$ be an oriented orthonormal basis for V (so $e^1, \ldots, e^n$ is a dual basis for V^*) and define the **Hodge star operator**

$$* : \wedge^k V^* \to \wedge^{n-k} V^*$$

on the basis $e^{i_1} \wedge \ldots \wedge e^{i_k}$ by

$$e^{i_1} \wedge \ldots \wedge e^{i_k} \mapsto e^{j_1} \wedge \ldots \wedge e^{j_{n-k}}$$

where $e_{i_1}, \ldots, e_{i_k}, e_{j_1}, \ldots, e_{j_{n-k}}$ is an oriented basis for V. We note the following

(1) Clearly
$$*1 = e^1 \wedge \ldots \wedge e^n,$$
so
$$\wedge^0 V^* \to \wedge^n V^* : r \mapsto r\, e^1 \wedge \ldots \wedge e^n.$$
(Recall $\wedge_0 V^* \cong \wedge_n V^* \cong \mathbb{R}$.)

(2) If $v_1, \ldots, v_n$ is any basis for V then
$$*1 = \sqrt{\det[\langle v_i, v_j \rangle]}\, v^1 \wedge \ldots \wedge v^n.$$
This is clear since $\sqrt{\det[\langle v_i, v_j \rangle]}\, v^1 \wedge \ldots \wedge v^n = e^1 \wedge \ldots \wedge e^n$.

(3) Similarly
$$*(e^1 \wedge \ldots \wedge e^n) = 1.$$

(4) Also
$$*e^i = (-1)^{i-1} e^1 \wedge \ldots \wedge \widehat{e^i} \wedge \ldots \wedge e^n,$$
where $\widehat{e^i}$ means to leave that term out.

Exercise 1.8. The map

$$** : \wedge^p V^* \to \wedge^p V^*$$

is

$$** = (-1)^{p(n-p)}.$$

Exercise 1.9. For any $v, w \in \wedge^k V^*$ we have

$$\langle v, w \rangle = *(v \wedge *w) = *(w \wedge *v). \tag{1}$$

1.1.2. *Manifold theory.* Let M be an oriented manifold. A **Riemannian metric** on M is a smooth function

$$g : TM \times TM \to \mathbb{R}$$

such that for all $x \in M$

$$g_x : T_xM \times T_xM \to \mathbb{R}$$

is an inner product. We can think of $g \in (TM \otimes TM)^*$. In a coordinate chart $U \subset M, V \subset \mathbb{R}^n, \phi : V \to U$, we have coordinate $x^1, \ldots, x^n$ and $\phi^* g$ has the form

$$\left[g_{ij}(x)\right]$$

in the basis $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$. Given the metric g on TM we also get an induced metric on T^*M. In the basis $dx^1, \ldots, dx^n$ this metric takes the form

$$\left[g^{ij}(x)\right] = \left[g_{ij}(x)\right]^{-1}.$$

We can then induce metrics on all the bundles $\wedge^n(T^*M)$ and get a Hodge star operators

$$* : \wedge^k T^*M \to \wedge^{n-k} T^*M$$

and

$$* : \Omega^k(M) \to \Omega^{n-k}(M)$$

by applying the first map pointwise $(*\alpha)(x) = *(\alpha(x))$. Here $\Omega^k(M)$ is the set of sections of $\wedge^k T^*M$.

On a Riemannian manifold the **volume form** is defined to be

$$dvol = *1$$

where 1 is the constant function on M. In local coordinates we have

$$dvol = \sqrt{\det\left[g_{ij}(x)\right]}\, dx^1 \wedge \ldots \wedge dx^n.$$

Example 1.10. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function and set

$$M = \Gamma_f = \{(x, y, f(x, y))\}.$$

For $v, w \in T_{(x,y,z)}M$ define a Riemannian metric by

$$g(v, w) = v \cdot w \quad \text{(where $\cdot$ is the dot product)}.$$

Local coordinates are given by $V = \mathbb{R}^2, U = M$ and

$$\phi : V \to U : (x, y) \to (x, y, f(x, y)).$$

In these coordinates

$$d\phi_{(x,y)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ f_x(x,y) & f_y(x,y) \end{bmatrix}.$$

So

$$g_{11} = g(d\phi\frac{\partial}{\partial x}, d\phi\frac{\partial}{\partial x}) = \begin{bmatrix} 1 \\ 0 \\ f_x \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ f_x \end{bmatrix} = 1 + f_x^2.$$

and similarly

$$g_{22} = 1 + f_y^2, \quad g_{12} = g_{21} = f_x f_y.$$

Therefor in these coordinates the metric looks like

$$\begin{bmatrix} 1 + f_x^2 & f_x f_y \\ f_x f_y & 1 + f_y^2 \end{bmatrix}$$

and the volume form is

$$dvol_M = \sqrt{1 + f_x^2 + f_y^2}\, dx \wedge dy.$$

One should compare this to the area form on a graph from vector calculus.

Example 1.11. Let S^2 be the unit 2-sphere in $\mathbb{R}^3$. The inverse of stereographic projection from the north pole $N \in S^2$ to the xy-plane give the coordinate chart

$$\phi : \mathbb{R}^2 \to S^2 \subset \mathbb{R}^3 : (x, y) \mapsto \frac{1}{x^2 + y^2 + 1}(2x, 2y, x^2 + y^2 - 1).$$

We have

$$d\phi_{(x,y)} = \frac{1}{(x^2 + y^2 + 1)^2} \begin{bmatrix} -2x^2 + 2y^2 + 2 & -4xy \\ -4xy & 2x^2 - 2y^2 + 2 \\ 4x & 4y \end{bmatrix}.$$

In the basis $e_1 = \frac{\partial}{\partial x}, e_2 = \frac{\partial}{\partial y}$ we have

$$g = \frac{1}{(x^2 + y^2 + 1)^4} \begin{bmatrix} 4((x^2 + y^2) + 1)^2 & 0 \\ 0 & 4((x^2 + y^2) + 1)^2 \end{bmatrix} = \frac{4}{(x^2 + y^2 + 1)^2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus the volume form is

$$dvol_{S^2} = \frac{4}{(x^2 + y^2 + 1)^2}\, dx \wedge dy.$$

Integrating this over $\mathbb{R}^2 = S^2 - N$, we see the volume of the unit sphere is 4π.

Exercise 1.12. Work out the volume form on the unit S^n in $\mathbb{R}^{n+1}$. Also work out the volume form on the flat T^2, that is $\mathbb{R}^2$ module the integer lattice $\mathbb{Z}^2$.

Exercise 1.13. You can also embed T^2 into $\mathbb{R}^3$ as the surface obtained by revolving the unit sphere centered at $(2, 0, 0)$ in the xz-plane about the z-axis. Write down a prameterization of this torus, thus giving a coordinate chart. Work out the volume form. What is the volume of this torus?

1.2. **Harmonic forms and the Hodge theorem.** Let M be an oriented Riemannian manifold with metric g. We have the following inner product on $\Omega^p(M)$

$$\begin{aligned} \langle \alpha, \beta \rangle &= \int_M \langle \alpha(x), \beta(x) \rangle dvol_M \\ &= \int_M (*(\alpha(x) \wedge *\beta(x)) dvol_M \\ &= \int_M \alpha(x) \wedge *\beta(x) \\ &= \int_M \alpha \wedge *\beta. \end{aligned}$$

This is clearly symmetric, bilinear and non-negative. If $\alpha \neq 0$ then there is some open set U where $\langle \alpha(x), \alpha(x) \rangle > 0$ and thus

$$\langle \alpha, \alpha \rangle = \int_M \langle \alpha(x), \alpha(x) \rangle dvol_M \geq \int_U \langle \alpha(x), \alpha(x) \rangle dvol_M > 0$$

and thus we have shown $\langle \alpha, \beta \rangle$ is an inner product. We call this the L^2 **inner product on forms.**

We now define the operator

$$\delta : \Omega^p(M) \to \Omega^{p-1}(M)$$

as the "formal adjoint" of

$$d : \Omega^{p-1}(M) \to \Omega^p(M)$$

with respect to the above inner product. That is for $\beta \in \Omega^p(M)$ we define $\delta\beta$ to be the unique $(p-1)$-form that satisfies

$$\langle \delta\beta, \alpha \rangle = \langle \beta, d\alpha \rangle$$

for all $\alpha \in \Omega^{p-1}(M)$.

Lemma 1.14. *On a closed manifold M the operator $\delta : \Omega^p(M) \to \Omega^{p-1}(M)$ is*

$$\delta\beta = (-1)^{n(p+1)+1} * d * \beta.$$

Proof. If $\alpha \in \Omega^{p-1}(M)$ and $\beta \in \Omega^p(M)$ then

$$\begin{aligned} d(\alpha \wedge *\beta) &= d\alpha \wedge *\beta + (-1)^{p-1}\alpha \wedge d*\beta \\ &= d\alpha \wedge *\beta + (-1)^{p-1}(-1)^{(p-1)(n-p+1)}\alpha \wedge ** d*\beta \\ &= d\alpha \wedge *\beta - (-1)^{n(p+1)+1}\alpha \wedge ** d*\beta. \end{aligned}$$

Thus by Stokes' theorem we have

$$\begin{aligned} 0 = \int_{\partial M} \alpha \wedge *\beta &= \int_M d(\alpha \wedge *\beta) \\ &= \int_M d\alpha \wedge *\beta - (-1)^{n(p+1)+1}\alpha \wedge ** d*\beta \\ &= \langle d\alpha, \beta \rangle - \langle \alpha, (-1)^{n(p+1)+1} * d*\beta \rangle. \end{aligned}$$

□

Example 1.15. Here we notice how d and δ are related to classical differential operators from vector calculus. Through this example we will always assume that we are in dimension 3. Denote the space of vector fields on a Riemannian manifold (M, g) by $\mathcal{V}(M)$. Recall that we can use the metric g to define an isomorphism between vector fields and 1-forms

$$\phi_g : \mathcal{M} \to \Omega^1(M) : v \mapsto \iota_v g,$$

that is $\phi_g(v)$ is the 1-form that evaluates to $g(v, w)$ on the vector field w. Moreover the Hodge star operator provides isomorphisms

$$* : \Omega^2(M) \to \Omega^1(M)$$

and

$$* : \Omega^3(M) \to \Omega^0(M).$$

Recall $\Omega^0(M)$ is the space of functions on M which we also denote by $\mathcal{F}(M)$. Now consider the following diagram

$$\begin{array}{ccccccc} \mathcal{F}(M) & \xrightarrow{D_1} & \mathcal{V}(M) & \xrightarrow{D_2} & \mathcal{V}(M) & \xrightarrow{D_3} & \mathcal{F}(M) \\ \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \phi_g} & & \downarrow{\scriptstyle *\circ\phi_g} & & \downarrow{\scriptstyle id} \\ \Omega^0(M) & \xrightarrow{d} & \Omega^1(M) & \xrightarrow{d} & \Omega^2(M) & \xrightarrow{d} & \Omega^3(M). \end{array}$$

Define the operators D_1, D_2 and D_3 via d and the vertical isomorphisms in the diagram. One may easily check that in the flat metric on $\mathbb{R}^3$ we have $D_1(f)$ is the gradient of the function f, $D_2(v)$ is the curl of the vector field v and $D_3(v)$ is the divergence of the vector field v. In particular δ is conjugate to the divergence operator on vector fields and d is conjugate to the curl operator on vector fields and the gradient operator on functions. Notice that we can use the above diagram to generalize the classical notions of divergence, gradient and curl to a general Riemannian 3-manifold (of course the divergence and gradient can be generalized to any Riemannian manifold of any dimension).

We define the **Laplace-Beltrami operator** (or simply the Laplacian) $\Delta : \Omega^p(M) \to \Omega^p(M)$ to be

$$\Delta = \delta d + d\delta.$$

Exercise 1.16. Show the Laplacian and the Hodge star operator commute:

$$*\Delta = \Delta * .$$

Exercise 1.17. Show that the Laplacian is self-adjoint. That is show that

$$\langle \Delta\alpha, \beta \rangle = \langle \alpha, \Delta\beta \rangle.$$

We call a p-form α **harmonic** if

$$\Delta\alpha = 0.$$

Lemma 1.18. *For any form α then*

$$\Delta\alpha = 0 \Leftrightarrow d\alpha = 0 \text{ and } \delta\alpha = 0.$$

Proof. The implication $\Leftarrow$ is clear. For the other implication assume $\Delta\alpha = 0$. Thus

$$0 = \langle \Delta\alpha, \alpha \rangle = \langle d\delta\alpha, \alpha \rangle + \langle \delta d\alpha, \alpha \rangle = \langle \delta\alpha, \delta\alpha \rangle + \langle d\alpha, d\alpha \rangle$$

and hence $\langle \delta\alpha, \delta\alpha \rangle = 0$ and $\langle d\alpha, d\alpha \rangle = 0$. And so $\delta\alpha = 0 = d\alpha$. □

Example 1.19. The Laplacian

$$\Delta : \Omega^0(M) \to \Omega^0(M)$$

is simply

$$\Delta = \delta d.$$

(Recall $\Omega^0(M)$ is the set of functions on M.) Thus

$$\Delta f = 0 \Leftrightarrow df = 0.$$

So the harmonic 0-forms (*ie* functions) on a closed manifold are exactly the constant functions.

Example 1.20. Let $x^1, \ldots, x^n$ be coordinates on $\mathbb{R}^n$ and consider the standard Euclidean (flat) metric $g_{ij} = \delta_{ij}$. Given a function $f : \mathbb{R}^n \to \mathbb{R}$ in $\Omega^0(\mathbb{R}^n)$ we have

$$df = \frac{\partial f}{\partial x^j} dx^j.$$

We now compute $\delta : \Omega^1(\mathbb{R}^n) \to \Omega^0(\mathbb{R}^n)$.

$$\begin{aligned}
\delta(h_j(x)dx^j) &= (-1)^{n(1+1)+1} * d * (h_j(x)dx^j) \\
&= -(-1)^{j-1} * d(h_j(x)\, dx^1 \wedge \ldots \wedge \widehat{dx^j} \wedge \ldots \wedge dx^n) \\
&= -(-1)^{j-1} * \frac{\partial h_j}{\partial x^j}(x)\, dx^j \wedge dx^1 \wedge \ldots \wedge \widehat{dx^j} \wedge \ldots \wedge dx^n \\
&= - * \frac{\partial h_j}{\partial x^j}(x)\, dx^1 \wedge \ldots \wedge dx^n \\
&= -\frac{\partial h_j}{\partial x^j}(x).
\end{aligned}$$

Thus

$$\Delta f = \delta df = \delta(\sum_{j=1}^n \frac{\partial f}{\partial x^j} dx^j) = -\sum_{j=1}^n \left(\frac{\partial}{\partial x^j}\right)^2 f.$$

Exercise 1.21. If we have a metric on $\mathbb{R}^n$ given by g_{ij} compute

$$\begin{aligned}
\Delta f &= -\frac{1}{\sqrt{\det\,[g_{ij}]}} \frac{\partial}{\partial x_j}\left(\sqrt{\det\,[g_{ij}]}\, g_{ij} \frac{\partial f}{\partial x^i}\right) \\
&= g_{ij} \frac{\partial^2 f}{\partial x^i \partial x^j} + \sum c_j(x) \frac{\partial f}{\partial x^j}.
\end{aligned}$$

Here the $c_j(x)$'s are just smooth functions. You can work out a formula for them in terms of the metric, but the point of the second line is that we have identified the highest order term explicitly. This is the term that governs the behavior of Δ as a differential operator as we will see below.

Theorem 1.22 (Hodge Theorem)**.** *Let M be a closed oriented Riemannian n-manifold and set*

$$H^p = \ker(\Delta : \Omega^p(M) \to \Omega^p(M)).$$

Then we have

1. *The space of harmonic forms H^p is finite dimensional.*
2. *The smooth p-forms decompose as*

$$\begin{aligned}
\Omega^p(M) &= H^p \oplus \Delta(\Omega^p(M)) \\
&= H^p \oplus d\delta(\Omega^p(M)) \oplus \delta d(\Omega^p(M)) \\
&= H^0 \oplus d(\Omega^{p-1}(M)) \oplus \delta(\Omega^{p+1}(M)).
\end{aligned}$$

We will prove this theorem in the next section (modulo a great deal of analytic work which is done later), but for now let us explore the consequences of this theorem. But first let us consider the content of the theorem. First the space of solutions to a PDE can easily be an infinitely dimensional space (for example $L = \frac{\partial}{\partial x}$ on $\mathbb{R}^2$ has kernel all smooth functions in y). So the finiteness is quite non-trivial. Secondly with a little thought the second statement would be (relatively) obvious (see the proof below) if we were dealing with L^2-forms (that is the coefficients that define the forms are just measurable functions and not smooth). So the real content hear is that the kernel of Δ extended to the L^2-forms consists of smooth forms and hence is H^p and $(H^p)^\perp = \Delta(\Omega^p(M))$.

From here on in this section M will be a closed oriented n-manifold.

Corollary 1.23. *The equation*

$$\Delta\omega = \alpha$$

has a solution $\omega \in \Omega^p(M)$ *if and only if* α *is* L^2*-orthogonal to the space of harmonic forms. Moreover, if* ω *is a solution to the equation the set of solutions is* $\omega + H^p$.

With a little work one can use this result to prove part (2) of the Hodge Theorem and in fact that is more or less what we do. But we observe the corollary follows from (2) as well. Thus the corollary and (2) are equivalent statements.

Proof. Since part (2) of the Hodge Theorem says $(H^p)^\perp = \Delta(\Omega^p(M))$ the condition for solvability is clear and since $\Delta\omega = \alpha$ is a linear equation it is also clear that $\omega + \ker\Delta$ is the set of solutions if ω is a solution. ☐

Corollary 1.24. *Each deRham cohomology class has a unique harmonic representative. That is the natural inclusion*

$$H^p \to H^p_{dR}(M)$$

is an isomorphism.

Proof. We first establish the following diagram:

$$\begin{tikzcd}
\Omega^{p-1} \arrow[d, "d"'] & \cong & H^{p-1} & \oplus & d\Omega^{p-2} & \oplus & \delta\Omega^p \arrow[lldd, "d"', "\cong"] \\
 & & & & & & \\
\Omega^{p} \arrow[d, "d"'] & \cong & H^{p} & \oplus & d\Omega^{p-1} & \oplus & \delta\Omega^{p+1} \arrow[lld, "d"', "\cong"] \\
\Omega^{p+1} & \cong & H^{p+1} & \oplus & d\Omega^{p} & \oplus & \delta\Omega^{p+2}
\end{tikzcd}$$

To see this notice that if $\alpha \in \delta\Omega^p(M)$ then $d\alpha = 0$ implies that α is harmonic (since clearly $\delta\alpha = 0$) and thus $\alpha = 0$ since it is orthogonal to the harmonic forms. So we see that $\ker d = H^p \oplus d\Omega^{p-1}(M)$ and $d : \delta\Omega^p(M) \to d\Omega^{p-1}(M)$ is an isomorphism as claimed.

Example 1.25. Show that $\delta : d\Omega^{p-1}(M) \to \delta\Omega^p(M)$ is an isomorphism as well. (Note: d and δ are not inverses of one another!)

Now

$$H^p_{dR}(M) = \frac{\ker d}{\operatorname{im} d} = \frac{H^p \oplus d\Omega^{p-1}(M)}{d\Omega^{p-1}(M)} \cong H^p.$$

☐

An immediate corollary of the Hodge theorem and the last corollary is the following.

Corollary 1.26. *The deRham cohomology of a closed manifold is finite dimensional.*

We have an obvious pairing

$$H^p_{dR}(M) \times H^{n-p}_{dR}(M) \to \mathbb{R}$$

given by

$$([\alpha], [\beta]) \mapsto \int_M \alpha \wedge \beta.$$

We denote this pairing by $\langle[\alpha],[\beta]\rangle$. We begin by noticing that his pairing is well-defined. Indeed, if we take another representative of $[\alpha]$ it can be written $\alpha+d\gamma$ for some $(p-1)$-form γ. Thus

$$\begin{aligned}\int_M(\alpha+d\gamma)\wedge\beta &= \int_M \alpha\wedge\beta+\int_M (d\gamma)\wedge\beta\\ &\quad\text{(since } d\beta=0)\\ &= \int_M \alpha\wedge\beta+\int_M d(\gamma\wedge\beta)\\ &= \int_M \alpha\wedge\beta+\int_{\partial M} \gamma\wedge\beta\\ &= \int_M \alpha\wedge\beta.\end{aligned}$$

You can similarly check that the pairing does not depend on $[\beta]$. The pairing is clearly bilinear.

Theorem 1.27 (Poincaré Duality)**.** *The pairing is non-singular. That is it induces an isomorphism*

$$H^{n-p}_{dR}(M)\cong(H^p_{dR}(M))^*$$

and in particular

$$\dim H^{n-p}_{dR}(M)=\dim H^p_{dR}(M)$$

and $H^n_{dR}(M)=\mathbb{R}$.

Proof. Given a class $[\alpha]\in H^p_{dR}(M)$ such that $[\alpha]\neq 0$ we can choose $\alpha\in[\alpha]$ so that $\Delta\alpha=0$. Recall that $*\Delta=\Delta*$ and so $*\alpha$ is also harmonic and represents a class $[*\alpha]\in H^{n-p}_{dR}(M)$. We also can easily see

$$\langle[\alpha],[*\alpha]\rangle=\int_M\alpha\wedge*\alpha=\int_M|\alpha|^2dvol=\|\alpha\|^2\neq 0.$$

Thus the pairing is non-singular. □

We now look at some applications to Riemannian geometry. We first observe the following lemma.

Lemma 1.28. *Let* $\pi:\widetilde{M}\to M$ *be a finite covering space of a closed manifold* M*. Then*

$$b_j(M)\leq b_j(\widetilde{M})$$

for all j*, where* b_j *stands for the* j^{th} *betti number (*ie *dimension of the* j^{th} *deRham cohomology).*

Proof. Pick a metric g on M and let $\widetilde{g}=\pi^*g$ be the pull-back metric on $\widetilde{M}$. Notice that

(1) $\pi^*d=d\pi^*$,
(2) $\pi^*dvol_g=dvol_{\widetilde{g}}$, and
(3) $\pi^*\circ*=*\circ\pi^*$.

Thus

$$\pi^*\circ\delta=\delta\circ\pi^*$$

and

$$\pi^*\circ\Delta=\Delta\circ\pi^*.$$

Also notice that if the fold of the cover is p and U_i is an open cover of M by coordinate charts that are evenly covered by π and ϕ_i is a partition of unit subordinate to the cover U_i then

$$\begin{aligned} p\int_M \omega &= p\int_{U_i} \sum \phi_i\omega = p\sum \int_{U_i} \phi_i\omega \\ &= \sum \int_{\pi^{-1}(U_i)} (\phi_i \circ \pi)\, \pi^*\omega = \int_{\widetilde{M}} \pi^*\omega, \end{aligned}$$

where the first and last inequalities are by the definition of the integral over manifolds.

Now if $\omega_1, \ldots, \omega_n$ is an orthonormal basis for $H^j(M)$ then $\pi^*\omega_1, \ldots, \pi^*\omega_n$ are harmonic forms on $\widetilde{M}$. Moreover, if $j \neq k$ then

$$\langle \pi^*\omega_j, \pi^*\omega_k\rangle = p\langle \omega_j, \omega_k\rangle = 0.$$

Thus all the $\pi^*\omega_j$'s are orthogonal which implies

$$\dim H^j(\widetilde{M}) \geq \dim H^j(M).$$

□

Corollary 1.29. *If M is a closed n-manifold that admits a metric of $1/4$-pinched positive sectional curvature (that is sectional curvatures larger than $\frac{1}{4}K$ and less than or equal to K for some fixed constant K), then $b_j(M) = 0$ for all $j = 1, \ldots, n-1$.*

Proof. By the Sphere Theorem the universal cover of M is S^n and so the result follows from the lemma. □

Corollary 1.30. *If M is a closed n-manifold that admits a metric with zero sectional curvature (*ie *M is flat) then*

$$b_j(M) \leq \binom{n}{j}.$$

(Notice that $\binom{n}{j} = \dim H^i(T^n)$.)

Proof. One of Bieberbach's theorems imply that M is finitely covered by T^n. □

Another application of the Hodge theorem to Riemannian geometry involves Ricci curvature. One has the famous Bochner formula for 1-forms

$$\Delta\alpha = \nabla^*\nabla\alpha + Ric(\alpha)$$

where ∇ is the covariant derivative associated to a metric and ∇^* is its formal adjoint. (The term $\nabla^*\nabla$ is called the covariant Laplacian.) We say $Ric > 0$ if $\langle Ric(\alpha), \alpha\rangle > 0$ for all α. (That is Ric is a positive operator.)

Corollary 1.31. *If M is a closed manifold with a metric having $Ric > 0$ then $H^1(M) = 0$. If instead $Ric \geq 0$ then $\dim H^1(M) \leq n$.*

Proof. Take $\alpha \in H^1(M)$ (that is α is a harmonic 1-form). Then

$$\begin{aligned} 0 = \langle \Delta\alpha, \alpha\rangle &= \langle \nabla^*\nabla\alpha, \alpha\rangle + \langle Ric(\alpha), \alpha\rangle \\ &= \langle \nabla\alpha, \nabla\alpha\rangle + \langle Ric(\alpha), \alpha\rangle \\ &= \|\nabla\alpha\|^2 + \langle Ric(\alpha), \alpha\rangle \geq 0. \end{aligned}$$

Thus there is no such α is $Ric > 0$ and if $Ric = 0$ then $\nabla\alpha = 0$ which means α is covariantly constant. Thus it is determined by its value at one point, proving the last inequality. □

1.3. **Proof of the Hodge theorem (modulo some analytic details).** To prove the Hodge theorem we will need the notion of a **weak solution** to the equation

$$\Delta\omega = \alpha \tag{2}$$

where $\alpha \in \Omega^p(M)$ is given and we are looking for ω. Suppose we have such a solution ω then notice that

$$\langle \alpha, \phi \rangle = \langle \Delta\omega, \phi \rangle = \langle \omega, \Delta\phi \rangle$$

for all $\phi \in \Omega^p(M)$ since Δ is formally self-adjoint. Thus if we define a linear function

$$l : \Omega^p(M) \to \mathbb{R}$$

by $l(\beta) = \langle \omega, \beta \rangle$ then from above we see that

$$l(\Delta\phi) = \langle \alpha, \phi \rangle \tag{3}$$

for all $\phi \in \Omega^p(M)$. We also notice that using the Cauchy-Schwarz inequality

$$|l(\beta)| = |\langle \omega, \beta \rangle| \leq \|\omega\| \|\beta\|.$$

So if $K = \|\omega\|$ we have

$$|l(\beta)| \leq K\|\beta\|, \tag{4}$$

that is l is a bounded linear operator.

Thus we see a solution ω to Equation (2) gives a linear functional $l \in (\Omega^p(M))^*$ satisfying Equations (3) and (4). Such an l is called a **weak solution** to Equation (2). We have seen that solutions give weak solutions, but for the equation we are concerned with we have a converse.

Theorem A. Given $\alpha \in \Omega^p(M)$ and a weak solution $l \in (\Omega^p(M))^*$ to $\Delta\omega = \alpha$ there is an $\omega \in \Omega^p(M)$ such that $l(\beta) = \langle \omega, \beta \rangle$ for all $\beta \in \Omega^p(M)$. In particular $\Delta\omega = \alpha$. □

Notice that the last part of the theorem clearly follows from the first part. In particular

$$\langle \alpha, \beta \rangle = l(\Delta\beta) = \langle \omega, \Delta\beta \rangle = \langle \Delta\omega, \beta \rangle$$

for all β. And hence

$$\langle \Delta\omega - \alpha, \beta \rangle = 0$$

for all β. But his implies

$$\Delta\omega - \alpha = 0.$$

We also have

Theorem B. Let $\{\alpha_n\}$ be a sequence of elements in $\Omega^p(M)$ for which there is some constant c such that

$$\|\alpha_n\| \leq c$$

and

$$\|\Delta\alpha_n\| \leq c$$

then there is a subsequence of $\{\alpha_n\}$ that is Cauchy in the L^2 norm on $\Omega^p(M)$. □

It will take quite a bit of analytic machinery to prove these theorems. We will state the main results that come out of this machinery in the next section and derive Theorems A and B from them. The machinery will be established in the following sections. But for now we show how to prove the Hodge theorem once we know Theorems A and B.

Proof of Hodge Theorem 1.22. We begin with statement (1), that is we show that the dimension of H^p is finite. Suppose the dimension was not finite. Then there would be a sequence $\{\alpha_n\}$ of orthonormal forms in H^p. Thus if $c > 1$ then

$$\|\alpha_n\| = 1 < c$$

and

$$\|\Delta\alpha_n\| = 0 < c.$$

Thus we can use Theorem B to find a subsequence of the α_n's (which we still denote α_n) that is Cauchy. But now consider

$$\|\alpha_n - \alpha_m\|^2 = \langle \alpha_n - \alpha_m, \alpha_n - \alpha_m \rangle = \|\alpha_n\|^2 + \|\alpha_m\|^2 = 2$$

(the second to last equality comes from orthogonality of the α_n's). This clearly contradicts the sequence being Cauchy and hence H^p could not be infinite dimensional.

Now for part (2) of the Hodge Theorem. That is we need to show

$$\begin{aligned} \Omega^p(M) &= H^p \oplus \Delta(\Omega^p(M)) \\ &= H^p \oplus d\delta(\Omega^p(M)) \oplus \delta d(\Omega^p(M)) \\ &= H^0 \oplus d(\Omega^{p-1}(M)) \oplus \delta(\Omega^{p+1}(M)). \end{aligned}$$

Notice the second equality is obvious and the third equality was established in the proof of Corollary 1.24 above. Thus we are left to see

$$\Omega^p(M) = H^p \oplus \Delta(\Omega^p(M))$$

To this end let $\omega_1, \dots, \omega_N$ be an orthonormal basis for H^p. For $\alpha \in \Omega^p(M)$ set

$$h(\alpha) = \alpha - \sum_{j=1}^{N} \langle \alpha, \omega_j \rangle \omega_j.$$

Clearly

$$\langle h(\alpha), \omega_j \rangle = 0$$

for all j. So $h(\alpha) \in (H^p)^\perp$ and we see that

$$\Omega^p(M) = H^p \oplus (H^p)^\perp.$$

Therefore to establish the theorem we are left to show that

$$(H^p)^\perp = \Delta\Omega^p(M).$$

One inclusion is easy. That is suppose $\gamma \in \Delta\Omega^p(M)$ so there is some $\widetilde{\gamma}$ such that $\gamma = \Delta\widetilde{\gamma}$. Thus for all $\omega \in H^p$ we have

$$\langle \gamma, \omega \rangle = \langle \Delta\widetilde{\gamma}, \omega \rangle = \langle \widetilde{\gamma}, \Delta\omega \rangle = \langle \widetilde{\gamma}, 0 \rangle = 0.$$

So $\gamma \in (H^p)^\perp$.

To complete the proof we need to show that for any $\alpha \in (H^p)^\perp$ there is some ω such that $\Delta\omega = \alpha$. To this end we will construct a weak solution and then apply Theorem A. Recall a weak solution is among other things in element $l \in (\Omega^p(M))^*$. To construct this element we will follow a common strategy. That is we will define the linear functional on a subset of $\Omega^p(M)$ where the definition is "obvious" and then use the Hahn-Banach theorem to extend it to all of $\Omega^p(M)$. Thus we begin by recalling the well known theorem.

Hahn-Banach Theorem. Suppose X is a linear space and $\rho : X \to \mathbb{R}$ is a function satisfying
$$\rho(x+y) \leq \rho(x) + \rho(y)$$
and
$$\rho(\lambda x) = \lambda\, \rho(x).$$
Now given a subspace Y of X and a linear function $f : Y \to \mathbb{R}$ satisfying
$$f(x) \leq \rho(x)$$
for all $x \in Y$, then there is a linear function $F : X \to \mathbb{R}$ such that
$$F(x) = f(x)$$
for all $x \in Y$ and
$$F(x) \leq \rho(x)$$
for all $x \in X$. □

We will not prove this theorem here, but it can be found in any functional analysis book.

We begin to construct our weak solution for a given $\alpha \in (H^p)^\perp$ by defining a linear map $l : \Delta\Omega^p(M) \to \mathbb{R}$ as follows: given $\beta \in \Delta\Omega^p(M)$ there is a $\gamma \in \Omega^p(M)$ such that $\Delta\gamma = \beta$, define
$$l(\beta) = \langle \alpha, \gamma \rangle.$$
Notice that l is well-defined since if $\beta = \Delta\gamma'$ too then $\Delta(\gamma - \gamma') = 0$ so $(\gamma - \gamma') \in H^p$ and thus
$$\langle \alpha, \gamma - \gamma' \rangle = 0$$
since $\alpha \in (H^p)^\perp$. So we have
$$\langle \alpha, \gamma \rangle = \langle \alpha, \gamma' \rangle$$
and l is well-defined. The function l is clearly linear. To apply the Hahn-Banach theorem we need the following lemma.

Lemma 1.32. *There is a constant c such that for all $\beta \in (H^p)^\perp$*
$$\|\beta\| \leq c\|\Delta\beta\|.$$

Assuming the lemma is true for the moment notice that if $\beta = \Delta\gamma$ then
$$\begin{aligned}
|l(\beta)| &= |l(\Delta\gamma)| = |l(\Delta(h(\gamma)))| \\
&\qquad \text{(recall } h(\gamma) = (\gamma - \textstyle\sum \langle \gamma, \omega_j \rangle \omega_j) \in (H^p)^\perp) \\
&= |\langle \alpha, h(\gamma) \rangle| \\
&\qquad \text{(by Cauchy-Schwarz inequality)} \\
&\leq \|\alpha\| \|h(\gamma)\| \\
&\qquad \text{(by the last lemma)} \\
&\leq c\|\alpha\| \|\Delta h(\gamma)\| \\
&\qquad \text{(since the } \omega_j \text{ are harmonic)} \\
&= c\|\alpha\| \|\Delta\gamma\| = c\|\alpha\| \|\beta\|.
\end{aligned}$$
Thus if we set $K = c\|\alpha\|$ we see that
$$|l(\beta)| \leq K\|\beta\|$$

and if we set $\rho : \Omega^p(M) \to \mathbb{R} : x \mapsto K\|x\|$ then we can apply the Hahn-Banach theorem to l and ρ to extend l to all of $\Omega^p(M)$ such that it satisfies

$$|l(\beta)| \leq K\|\beta\|$$

for all $\beta \in \Omega^p(M)$ and

$$l(\Delta\gamma) = \langle \alpha, \gamma \rangle$$

by construction. So l is a weak solution to $\Delta\omega = \alpha$ and hence Theorem A says there is some $\omega \in \Omega^p(M)$ such that $l(\beta) = \langle \omega, \beta \rangle$ for all β and hence $\alpha \in \Delta\Omega^p(M)$. Thus establishing $(H^p)^\perp = \Delta\Omega^p(M)$ and completing the proof of the theorem. □

We are now left to prove Lemma 1.32.

Proof of Lemma 1.32. Suppose the bound in the lemma does not exist, then we can fine a sequence $\beta'_n \in (H^p)^\perp$ such that

$$\|\beta'_n\| > n\|\Delta\beta'_n\|$$

for all n. Now set $\beta_n = \frac{1}{\|\beta'_n\|}\beta'_n$, so we have

$$\|\beta_n\| = 1$$

and since $\Delta\beta_n = \frac{1}{\|\beta'_n\|}\Delta\beta'_n$ we have that

$$\|\Delta\beta_n\| = \frac{1}{\|\beta'_n\|}\|\Delta\beta'_n\| \leq \frac{1}{n}.$$

We can now apply Theorem B to $\{\beta_n\}$ to get a Cauchy subsequence.

Exercise 1.33. Suppose this Cauchy sequence converged to β. Show that $\|\beta\| = 1, \beta \in (H^p)^\perp$ and $\beta \in H^p$. This is of course a contradiction and hence our initial assumption the the lemma was not true is false. We have not finished the proof since $\Omega^p(M)$ is not a complete space and hence there is no guarantee that a Cauchy sequence converges.

To get around the lack of completeness of $\Omega^p(M)$ we will use Theorem A again. In particular, notice that for all $\gamma \in \Omega^p(M)$ we have

$$|\langle \beta_n, \gamma \rangle - \langle \beta_m, \gamma \rangle| = |\langle \beta_n - \beta_m, \gamma \rangle| \leq \|\beta_n - \beta_m\|\|\gamma\|.$$

Thus since the β_n are Cauchy we see $\{\langle \beta_n, \gamma \rangle\}$ is a Cauchy sequence in $\mathbb{R}$ which is complete. Thus $\{\langle \beta_n, \gamma \rangle\}$ converges to some number which we denote

$$l(\gamma) = \lim_{n\to\infty} \langle \beta_n, \gamma \rangle.$$

It is clear that l is linear and moreover

$$|l(\gamma)| = |\lim_{n\to\infty} \langle \beta_n, \gamma \rangle| \leq \lim_{n\to\infty} \|\beta_n\|\|\gamma\| = \|\gamma\|.$$

So l is a bounded linear functional on $\Omega^p(M)$. Notice that

$$l(\Delta\gamma) = \lim_{n\to\infty} \langle \beta_n, \Delta\gamma \rangle = \lim_{n\to\infty} \langle \Delta\beta_n, \gamma \rangle \leq \lim_{n\to\infty} \|\Delta\beta_n\|\|\gamma\| = 0$$

and so l is a weak solution to $\Delta\omega = 0$. Thus Theorem A implies there is an ω such that $l(\beta) = \langle \omega, \beta \rangle$ for all β. This ω should be the limit of the β_n's. We show it behaves as if it is. In particular for all $\omega' \in H^p$ notice that

$$\langle \omega, \omega' \rangle = l(\omega') = \lim_{n\to\infty} \langle \beta_n, \omega' \rangle = 0$$

since the β_n''s are in $(H^p)^\perp$. Thus $\omega \in (H^p)^\perp$. But of course $\Delta\omega = 0$ by construction so $\omega \in H^p$ too. Thus $\omega = 0$. We will have our contradiction, and hence establish the lemma, once we see that $\|\omega\| = 1$. To see this notice that

$$\begin{aligned}
\|\omega\|^2 &= \langle \omega, \omega \rangle = l(\omega) \\
&= \lim_{n\to\infty} \langle \beta_n, \omega \rangle = \lim_{n\to\infty} l(\beta_n) \\
&= \lim_{n\to\infty} (\lim_{m\to\infty} \langle \beta_n, \beta_m \rangle) \\
&= \lim_{n\to\infty} (\lim_{m\to\infty} \langle \beta_n - \beta_m + \beta_m, \beta_m \rangle) \\
&= \lim_{n\to\infty} (\lim_{m\to\infty} \langle \beta_n - \beta_m, \beta_m \rangle + \langle \beta_m, \beta_m \rangle) \\
&= \lim_{n\to\infty} (\lim_{m\to\infty} \langle \beta_n - \beta_m, \beta_m \rangle + 1) \\
&= 1 + \lim_{n\to\infty} (\lim_{m\to\infty} \langle \beta_n - \beta_m, \beta_m \rangle).
\end{aligned}$$

Now notice that since $\{\beta_n\}$ is Cauchy we have for any $\epsilon > 0$ an N such that $n, m > N$ implies

$$|\langle \beta_n - \beta_m, \beta_m \rangle| \leq \|\beta_n - \beta_m\| \|\beta_m\| = \|\beta_n - \beta_m\| 1 \leq \epsilon.$$

Thus $|\lim_{n\to\infty}(\lim_{m\to\infty}\langle \beta_n - \beta_m, \beta_m \rangle)| = 0$ and hence $\|\omega\| = 1$. $\square$

1.4. **Formal analytic details.** In this section we show how to prove Theorems A and B given certain analytic theorems.

In the Section 3 we will show how to associate Hilbert spaces $H_s(\Omega^p(M))$ with inner products $\langle \cdot, \cdot \rangle_s$ and associated norms $\| \cdot \|_s$. These will be called Sobolev spaces of forms and they essentially are forms whose first s derivatives are L^2-bounded. The precise definition is not relevant here. For now we just need the following properties.

Theorem 1.34. *We have the following:*

(1) $\Omega^p(M)$ *is a dense subset of* $H_s(\Omega^p(M))$ *for all* s. *(So all these new spaces are just completions of smooth p-forms in some norm.)*
(2) *The inner product* $\langle \cdot, \cdot \rangle_0$ *is the* L^2*-inner product on* $\Omega^p(M)$. *(So all these Sobolev norms are generalizations of the* L^2*-norm.)*
(3) *If* $\alpha \in H_s(\Omega^p(M))$ *and*
$$\langle \alpha, \beta \rangle_s = 0$$
for all $\beta \in \Omega^p(M)$ *then* $\alpha = 0$ *in* $H_s(\Omega^p(M))$.
(4) *For any smooth form* $\alpha \in \Omega^p(M)$ *and number* $s \leq t$ *we have*
$$\|\alpha\|_s \leq \|\alpha\|_t.$$
Thus the identity map on $\Omega^p(M)$ *induces a bounded inclusion*
$$H_t(\Omega^p(M)) \subset H_s(\Omega^p(M)).$$
(5) *(Sobolev inequality) If* $s > \frac{n}{2} + k$ *then there is a constant* c *such that for any smooth form* $\alpha \in \Omega^p(M)$ *we have*
$$\|\alpha\|_{C^k} \leq c\|\alpha\|_s.$$
Thus the identity map on $\Omega^p(M)$ *induces a bounded inclusion*
$$H_s(\Omega^p(M)) \subset C^k(\Omega^p(M)),$$
that is any element in $H_s(\Omega^p(M))$ *has* k *continuous derivatives.*

(6) *(Rellich's lemma) if $s < t$ then any sequence $\{\alpha_j\}$ in $H_t(\Omega^p(M))$ for which $\|\alpha_j\|_t \leq K$ for some fixed K, there is a subsequence that is Cauchy (and hence convergent) in $H_s(\Omega^p(M))$. That is the bounded linear inclusion map*

$$H_t(\Omega^p(M)) \subset H_s(\Omega^p(M))$$

is a compact operator.

(7) *The map*

$$H_0(\Omega^p(M)) \to (H_0(\Omega^p(M)))^*$$

given by

$$\alpha \mapsto \langle \alpha, \cdot \rangle_0$$

is an isomorphism.

Parts (1) and (2) of this theorem follow from the definition of the Sobolev spaces in Section 3.5, while part (3) is an easy consequence of (1). Parts (4)–(6) are a restatement of parts of Theorem 3.17 and part (7) is just the fact that an inner product induces an isomorphism form a space to its dual space. We have the immediate corollary.

Corollary 1.35. *We have*

$$\cap_s H_s(\Omega^p(M)) = \Omega^p(M).$$

It turns out that the Laplacian is what is called an **elliptic operator** and while we do not define this term until Section 4 we note the following results.

Theorem 1.36. *The Laplacian on $\Omega^p(M)$ extends to a bounded linear operator*

$$\Delta : H_s(\Omega^p(M)) \to H_{s-2}(\Omega^p(M)).$$

Theorem 1.37 (Elliptic estimate for the Laplacian)**.** *There is some constant c such that for all $\alpha \in H_{s+2}(\Omega^p(M))$ we have*

$$\|\alpha\|_{s+2} \leq c(\|\Delta\alpha\|_s + \|\alpha\|_s).$$

Theorem 1.38 (Elliptic regularity for the Laplacian)**.** *If $\omega \in H_0(\Omega^p(M))$ and $\alpha \in H_t(\Omega^p(M))$ for some $t > 0$ such that*

$$\Delta\omega = \alpha$$

then $\omega \in H_{t+2}(\Omega^p(M))$.

The first theorem is the statement in part (2) of Theorem 3.17. The second two theorems are direct consequences of Theorems 4.11 and 4.12

Proof of Theorem B. Suppose $\{\alpha_n\}$ is a sequence of elements in $\Omega^p(M)$ for which there is some constant c such that

$$\|\alpha_n\| \leq c'$$

and

$$\|\Delta\alpha_n\| \leq c'$$

then the Elliptic estimate implies that

$$\|\alpha_n\|_2 \leq c(\|\Delta\alpha\|_0 + \|\alpha\|_0) = 2cc'$$

and thus the Sobolev 2-norm is uniformly bounded for all n. Now Rellich's lemma implies that a subsequence of the α_n's is Cauchy in the Sobolev 0-norm, that is in the L^2-norm on $\Omega^p(M)$. □

Proof of Theorem A. Given $\alpha \in \Omega^p(M)$, suppose $l \in (\Omega^p(M))^*$ is a weak solution to $\Delta\omega = \alpha$. That is,

$$l(\Delta\gamma) = \langle \alpha, \gamma \rangle$$

and there is some c such that

$$|l(\gamma)| \leq c\|\gamma\|$$

for all γ.

Since $l : \Omega^p(M) \to \mathbb{R}$ is a bounded linear operator and $\Omega^p(M)$ is dense in $H_0(\Omega^p(M))$ we can extend l to a bounded linear operator on $H_0(\Omega^p(M))$. That is $l \in (H_0(\Omega^p(M)))^*$. So by Theorem 1.34 there is an element $\omega \in H_0(\Omega^p(M))$ such that

$$l(\beta) = \langle \omega, \beta \rangle$$

for all $\beta \in H_0(\Omega^p(M))$. If ω were smooth then we would be done. Notice that

$$\langle \Delta\omega, \beta \rangle = \langle \omega, \Delta\beta \rangle = l(\Delta\beta) = \langle \alpha, \beta \rangle$$

for all $\beta \in \Omega^p(M)$. Thus again by Theorem 1.34 we have

$$\Delta\omega = \alpha$$

in $H_{-2}(\Omega^p(M))$. Since $\alpha \in \Omega^p(M)$ it is in $H_s(\Omega^p(M))$ for all s. Thus Elliptic regularity implies that $\omega \in H_{s+2}(\Omega^p(M))$ for all s. Therefore $\omega \in \cap_s H_s(\Omega^p(M))$ and hence in $\Omega^p(M)$. $\square$

2. Complex manifolds and the Hodge theorem

Here we present another version of the Hodge theorem for complex manifolds (this is what many people mean when that say "Hodge theorem"). But for those interesting in just seeing the details of the proofs of the theorems above feel free to skip this section and go strait to Section 3. In this section we do not introduce anything important to the analysis we will develop later.

2.1. **Complex manifolds.**

2.1.1. *Linear theory.* Let V be a $\mathbb{R}$ vector space. A **complex structure on** V is a linear automorphism

$$J : V \to V$$

such that

$$J^2 = -Id_V.$$

Notice that a complex structure J on V gives V the structure of a $\mathbb{C}$ vector space:

$$(a + ib)v = av + b(Jv)$$

and

$$\dim_{\mathbb{C}} V = \frac{1}{2} \dim_{\mathbb{R}} V.$$

Thus $\dim_{\mathbb{R}} V$ is always even if V has a complex structure.

Exercise 2.1. Prove the following statements.

- If $e_1, \ldots, e_n$ is a $\mathbb{C}$-basis for V and $f_j = J(e_j)$ then $e_1, f_1, \ldots, e_n, f_n$ is a $\mathbb{R}$-basis for V.
- In this basis
$$J = \begin{bmatrix} 0 & -1 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & -1 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$
- A complex structure induces an orientation on V.

Notice that if V is a $\mathbb{C}$ vector space then the map

$$J : V \to V : v \mapsto iv$$

is a complex structure on the underlying $\mathbb{R}$ vector space. Thus we see that a complex structure on a $\mathbb{R}$ vector space is essentially equivalent to endowing the vector space with the structure of a $\mathbb{C}$ vector space. It might seem strange to introduce this new idea that is equivalent to an old idea, but we will see it is useful when moving to the manifold level. But first we further explore a complex structure J on a $\mathbb{R}$-vector space V.

Exercise 2.2. If V and V' are $\mathbb{R}$-vector spaces with complex structures J and J', respectively, then an $\mathbb{R}$-linear map $\phi : V \to V'$ is $\mathbb{C}$-linear (for the $\mathbb{C}$ vector spaces structures induced by the complex structures J and J') if and only if

$$\phi \circ J = J' \circ \phi.$$

Exercise 2.3. If ϕ is a complex linear isomorphism then ϕ preserves the orientations induced from J and J'.

The eigenvalues of J are $\pm$ the square root of the eigenvalues $-Id_V$ so they are $\pm i$. Thus we cannot diagonalize J over $\mathbb{R}$. To diagonalize J we must complexify V that is consider

$$V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}.$$

Clearly J induces a map $J : V_{\mathbb{C}} \to V_{\mathbb{C}}$. Set

$$V^{(1,0)} = \ i\text{-eigenspace of } J$$

and

$$V^{(0,1)} = \ -i\text{-eigenspace of } J.$$

Then clearly

$$V_{\mathbb{C}} = V^{(1,0)} \oplus V^{(0,1)}.$$

Exercise 2.4. Show that

$$V^{(1,0)} = \{\frac{1}{2}(v - iJv) | v \in V\}$$

and

$$V^{(1,0)} = \{\frac{1}{2}(v + iJv) | v \in V\}$$

Thus each of these spaces is $\mathbb{R}$-isomorphic to V.
A word of caution: we should technically write $v \otimes 1 + (Jv) \otimes i$ for $v + iJv$ and similarly for the other expression. There is concern for confusion since we earlier said that J gives V the structure of a complex vector space by defining $iv = Jv$. But here we are thinking of V as a real vector space J an operator on it and $V \otimes \mathbb{C}$ the complexification of V.
You might be wondering about the $\frac{1}{2}$ here. This will be clear below when we talk about the dual picture.

Exercise 2.5. Define "complex conjugation" to be

$$C : V_{\mathbb{C}} \to V_{\mathbb{C}} : v + iw \mapsto v - iw.$$

Show that C induces an isomorphism from $V^{(1,0)}$ to $V^{(0,1)}$. Thus we identify V with a subset of $V_{\mathbb{C}}$ by $v \mapsto \frac{1}{2}(v - iJv)$ then

$$V_{\mathbb{C}} = V \oplus \overline{V}$$

where $\overline{V} = C(V)$.

The complex structure J on V induces a complex structure J^* on the dual space V^*. In particular if $w^* \in V^*$ then we define $J^*(w^*)$ to be the unique element in V^* defined by

$$J^*(w^*)(v) = w^*(J(v)).$$

That is J^* is the adjoint of J under the non-degenerate pairing

$$V \times V^* \to \mathbb{R} : (v, w^*) \mapsto w^*(v).$$

Now as above we have

$$V^*_{\mathbb{C}} = V^* \otimes_{\mathbb{R}} \mathbb{C} \cong (V^*)^{(1,0)} \oplus (V^*)^{(0,1)}.$$

Exercise 2.6. Show that

$$\bigwedge^k V^*_{\mathbb{C}} = \bigoplus_{p+q=k} \bigwedge^{(p,q)} V$$

where

$$\bigwedge^{(p,q)} V = \bigwedge^p (V^*)^{(1,0)} \wedge \bigwedge^q (V^*)^{(0,1)}.$$

Elements of $\bigwedge^{(p,q)} V$ are called forms of type (p, q).

If $e_1, \dots, e_n$ is a $\mathbb{C}$-basis for V with complex structure J, then $e_1, f_1, \dots, e_n, f_n$ is a $\mathbb{R}$-basis for V, where $f_j = J(e_j)$. Moreover the dual basis for V^* is $e^1, f^1, \dots, e^n, f^n$. Now we have that

$$V^{(1,0)} \text{ is spanned by } g_j = \frac{1}{2}(e_j - if_j), j = 1, \dots n,$$

and

$$V^{(0,1)} \text{ is spanned by } \overline{g}_j = \frac{1}{2}(e_j + if_j), j = 1, \dots, n.$$

Similarly we have

$$(V^*)^{(1,0)} \text{ is spanned by } g^j = e^j + if^j, j = 1, \dots, n$$

and

$$(V^*)^{(0,1)} \text{ is spanned by } \overline{g}^j = e^j - if^j, j = 1, \dots n.$$

Exercise 2.7. Show that $g^1, \dots, g^n$ is the dual basis to $g_1, \dots, g_n$. (This explains why there needs to be a $\frac{1}{2}$ somewhere, the reason we put it in the definition of g_j will be clear later.)

Finally if $\omega \in \bigwedge^{(p,q)} V$ then

$$\omega = \omega_{\alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_q} g^{\alpha_1} \wedge \dots \wedge g^{\alpha_p} \wedge \overline{g}^{\beta_1} \wedge \dots \wedge \overline{g}^{\beta_q},$$

where we as usual use the summation convention.

2.1.2. *Hermitian Structure.* A **Hermitian inner product** on a vector space V with a complex structure J is an $\mathbb{R}$-linear map

$$h : V \times V \to \mathbb{C}$$

such that

(1) $h(v, w) = \overline{h(w, v)}$
(2) $h(v, v) > 0$ for $v \neq 0$ (note that $h(v, v)$ is real by (1))
(3) $h(Jv, w) = ih(v, w)$

Notice that the first and last properties imply

$$h(v, Jw) = -ih(v, w),$$

thus we have

$$h(Jv, w) = -h(v, Jw),$$

that is J is skew-adjoint with respect to h, and

$$h(Jv, Jw) = h(v, w),$$

that is h is J invariant.

Exercise 2.8. Notice that we can extend $h : V \times V \to \mathbb{C}$ to the complexified vector space $V_{\mathbb{C}}$

$$h : V_{\mathbb{C}} \times V_{\mathbb{C}} \to \mathbb{C}$$

and it satisfies

(1') $h(\overline{v}, \overline{w}) = \overline{h(v, w)}$,
(2') $h(v, \overline{v}) > 0$ for all $v \neq 0$, and
(3') $h(v, w) = 0$ for all $v \in V^{(1,0)}$ and $w \in V^{(0,1)}$.

Given an Hermitian inner product $h : V \times V \to \mathbb{C}$ set

$$g(v, w) = \text{Real part of } h(v, w).$$

Exercise 2.9. Show $g : V \times V \to \mathbb{R}$ is an inner product and satisfies

$$g(Jv, Jw) = g(v, w),$$

that is, g is J invariant. In addition show that if g' is any J invariant inner product on a vector space V then

$$g(v, w) + ig(v, Jw)$$

is a Hermitian inner product.

Now set

$$\omega(v, w) = -\frac{1}{2} \text{Imaginary part of } h(v, w).$$

We easily see that

(1) $\omega \in \wedge^2 V$:

$$\omega(v, w) = -\frac{1}{2}\text{Im } h(v, w) = \frac{1}{2}\text{Im } h(w, v) = -\omega(w, v),$$

(2) ω is non-degenerate
(3) $\omega(v, Jv) = -\frac{1}{2}\text{Im } h(v, Jv) = \frac{1}{2}\text{Im } ih(v, v) = g(v, v) > 0$ for $v \neq 0$, and
(4) $\omega(Jv, Jw) = \omega(v, w)$.

Exercise 2.10. Show that given and $\omega \in \wedge^2 V$ that satisfies (2)–(4) then

$$2(\omega(v, Jw) - i\omega(v, w))$$

is a Hermitian inner product.

Given a Hermitian inner product $h : V \times V \to \mathbb{C}$ let g and ω be the associated inner product and 2-form. Choose a complex basis $e_1, \ldots, e_n$ for V that is orthonormal with respect to h. Let $f_\alpha = Je_\alpha$. So $e_1, f_1, \ldots e_n, f_n$, is a real basis for V. Notice that

$$g(f_\alpha, f_\beta) = g(Je_\alpha, Je_\beta) = g(e_\alpha, e_\beta) = \delta_{\alpha\beta},$$

where $\delta_{\alpha\beta}$ is the Kroneker delta. Also

$$g(f_\alpha, e_\beta) = \operatorname{Re} h(Je_\alpha, e_\beta) = \operatorname{Re} ih(e_\alpha, e_\beta) = -2\omega(e_\alpha, e_\beta),$$

so $g(f^\alpha, e^\beta) = 0$ if for all α and β. That is $e_1, f_1, \ldots, e_n, f_n$ is a real orthonormal basis for V with respect to the metric g. Let $e^1, f^2, \ldots, e^n, f^n$, be the dual basis. We clearly have

$$g = \sum_{\alpha=1}^{n} (e^\alpha \otimes e^\alpha + f^\alpha \otimes f^\alpha).$$

In addition, from above we see that $\omega(v, w) = -\frac{1}{2} g(v, Jw)$, so

$$\omega = -\frac{1}{2} \sum_{\alpha=1}^{n} -e^\alpha \otimes f^\alpha + f^\alpha \otimes e^\alpha) = \sum_{\alpha=1}^{n} e^\alpha \wedge f^\alpha.$$

Set $g^\alpha = e^\alpha + if^\alpha$ in $V_{\mathbb{C}}$ and notice that $g^1, \overline{g}^1, \ldots, g^n, \overline{g}^n$ is a complex basis for $V_{\mathbb{C}}$. From above one may check that

$$h = \sum_{\alpha=1}^{n} g^\alpha \otimes \overline{g}^\alpha.$$

It is also amusing to notice that

$$\omega = \frac{i}{2}\sum_{\alpha=1}^{n} g^\alpha \wedge \overline{g}^\alpha.$$

2.1.3. *Manifold theory.* A **complex structure** on s vector bundle $p : E \to M$ is a bundle automorphism

$$\begin{tikzcd} E \arrow[rr, "J"] \arrow[dr, "p"'] & {} \arrow[d, phantom, "\circ"] & E \arrow[dl, "p"] \\ & M & \end{tikzcd}$$

such that for all $x \in M$ we have

$$E_x \xrightarrow{J_x} E_x$$

is a complex structure where $E^x = p^{-1}(x)$ and J_x is J restricted to E_x. An **almost complex structure** on a manifold M is a complex structure on the tangent bundle TM of M. We will see below why the word "almost" is used here, basically it is because "complex structure" already has a meaning for manifolds.

Notice that if a manifold admits an almost complex structure then it is even dimensional and orientable.

Example 2.11. Let $M = \mathbb{C}^n$, so for every $z \in \mathbb{C}^n$ we have $T_z\mathbb{C}^n = \mathbb{C}^n$. Thus we have

$$J_z : T_z\mathbb{C}^n \to T_z\mathbb{C}^n : v \mapsto iv$$

is an almost complex structure on $M = \mathbb{C}^n$.

Example 2.12. Let M be a **complex manifold**. That is M has an atlas of coordinate charts into $\mathbb{C}^n$ and the transition maps are **holomorphic** maps.

A function $f : \mathbb{C}^n \to \mathbb{C}^n : (z^1, \dots, z^n) \mapsto (f^1(z^1, \dots, z^n), \dots, f^n(z^1, \dots, z^n))$ where $f^j = u^j + iv^j$ is **holomorphic** if it satisfies the Cauchy-Riemann equations

$$\frac{\partial u^j}{\partial x^k} = \frac{\partial v^j}{\partial y_k}$$
$$\frac{\partial u^j}{\partial y_k} = -\frac{\partial v^j}{\partial x_k}$$

for all $k, j = 1, \dots, n$. This is equivalent to

$$\frac{\partial f^j}{\partial \overline{z}^k} = 0$$

where of course

$$\frac{\partial}{\partial \overline{z}^k} = \frac{1}{2}\left(\frac{\partial}{\partial x^k} + i\frac{\partial}{\partial y^k}\right).$$

Exercise 2.13. Show that a map f is holomorphic if and only if df is invariant under the almost complex structure J defined in the previous example.

Since the fixed almost complex structure on $\mathbb{C}^n$ is preserved by all the transition maps in an atlas of coordinate charts for M we get an induced almost complex structure on M. That is at a point $x \in M$ let $f : V \to U$ be a coordinate chart where $U \subset M$ and $V \subset \mathbb{C}^n$. define the complex structure J' at x to be

$$J'_{f(x)} = df_x \circ J_x \circ (df_x)^{-1}.$$

One may easily check that this does not depend on the choice of coordinate chart.

Given two complex manifolds M and M' with associated almost complex structures J and J' we say a map $f : M \to M'$ is a **holomorphic map** if

$$df \circ J = J' \circ df.$$

Notice that this is equivalent to saying that in local coordinates on M and M', f satisfies the Caucy-Riemann equations.

Suppose M has an almost complex structure J. Then just as in the linear case we can consider the complexified tangent bundle (basically just complexify pointwise) and split it into the eigenspaces of J :

$$TM_{\mathbb{C}} = TM \otimes \mathbb{C} = (TM)^{(1,0)} \oplus (TM)^{(0,1)}.$$

The first summand is called the holomorphic tangent bundle and the second is called the anti-holomorphic tangent bundle. The composition of the inclusion of TM into $TM_{\mathbb{C}}$ followed by (the correct) projection to $(TM)^{(1,0)}$ is given by

$$TM \to (TM)^{(1,0)} : v \mapsto \frac{1}{2}(v + iJv)$$

and is an $\mathbb{R}$-linear map. We similarly have

$$T^*M_{\mathbb{C}} = T^*M \otimes \mathbb{C} = (T^*M)^{(1,0)} \oplus (T^*M)^{(0,1)},$$

and

$$\bigwedge^k T^*M_{\mathbb{C}} = \bigwedge^k (T^*M \otimes \mathbb{C}) = \bigoplus_{p+q=k} \bigwedge^{(p,q)} T^*M.$$

We also have the sections of these bundles

$$\Omega^{p,q}(M) = \Gamma(\bigwedge^{(p,q)} T^*M).$$

From our study of the linear situation we can locally choose vector fields $v_1, \dots v_n$ such that

$$v_1, Jv_1, \dots, v_n, Jv_n$$

span TM. Then of course we have

$$\begin{aligned}
w_j &= \frac{1}{2}(v_j - iJv_j) \qquad & j = 1, \dots n \text{ span } (TM)^{(1,0)},\\
w^j &= v^j + iJv^j & j = 1, \dots n \text{ span } (T^*M)^{(1,0)},\\
\overline{w}_j &= \frac{1}{2}(v_j + iJv_j) & j = 1, \dots n \text{ span } (TM)^{(0,1)},\\
\overline{w}^j &= v^j - iJv^j & j = 1, \dots n \text{ span } (T^*M)^{(0,1)},
\end{aligned}$$

where the v^j are the duals to v_j. For $\eta \in \Omega^{p,q}(M)$ we have local functions η_{AB} for each set of non-negative integers $A = \{a_1, \dots, a_n\}$ and $B = \{b_1, \dots, b_n\}$, such that $|A| = p$ and $|B| = q$, (recall $|A| = a_1 + \cdots + a_n$ and similarly for $|B|$) so that we can write

$$\eta = \sum_{|A|=p, |B|=q} \eta_{AB}\, w^A \wedge \overline{w}^B$$

where, as usual, $w^A = (w^1)^{a_1} \wedge \ldots \wedge (w^n)^{a_n}$.

Example 2.14. Of M is a complex manifold with local coordinate $z^1, \ldots, z^n$ (and $z^j = x^j + iy^j$) then

$$\begin{aligned}
\frac{\partial}{\partial z_j} &= \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right) && j = 1, \ldots n \text{ span } (TM)^{(1,0)},\\
dz^j &= dx^j + idy^j && j = 1, \ldots n \text{ span } (T^*M)^{(1,0)},\\
\frac{\partial}{\partial \overline{z}_j} &= \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right) && j = 1, \ldots n \text{ span } (TM)^{(0,1)},\\
d\overline{z}^j &= dx^j - idy^j && j = 1, \ldots n \text{ span } (T^*M)^{(0,1)}.
\end{aligned}$$

Back to a general almost complex manifold M. We can extend the DeRham differential $d : \Omega^p M \to \Omega^{p+1} M$ to the complexified p forms. So if η is a (p, q)-form represented as above then

$$d\eta = \sum_{|A|=p, |B|=q} d\eta_{AB} \wedge w^A \wedge w^B + \sum_{|A|=p, |B|=q} \eta_{AB}\, dw^A \wedge w^B + \sum_{|A|=p, |B|=q} \eta_{AB}\, w^A \wedge dw^B.$$

In the first term in this expression we see $d\eta_{AB}$ which is a 1-form and can have $(1, 0)$ and $(0, 1)$ components. Thus all the terms in the first sum are $(p + 1, q)$ and $(p, q + 1)$ forms. Consider dw^A. This can be for example

$$(dw^1) \wedge ((p - 1, 0)\text{-form}).$$

since dw^1 is a 2-form it can have $(2, 0), (1, 1)$ and $(0, 2)$ components. Thus the second sum consists of terms that are $(p+1, q), (p, q+1)$ and $(p-1, q+2)$ forms. Similarly the last sum consists of terms that are $(p + 2, q - 1), (p + 1, q)$ and $(p, q + 1)$ forms. Thus we see that

$$d : \Omega^{p,q}(M) \to \Omega^{p+2,q-1}(M) \oplus \Omega^{p+1,q}(M) \oplus \Omega^{p,q+1}(M) \oplus \Omega^{p-1,q+2}(M).$$

However, if J comes from a complex structure in M then we can use as a basis for $T^*M_{\mathbb{C}}$, the dz^j and $d\overline{z}^j$. In this case we see that

$$d(dz^j) = 0 = d(d\overline{z}^j)$$

since $d^2 = 0$. Thus on a complex manifold we have

$$d : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M) \oplus \Omega^{p.q+1}(M).$$

Actually this characterizes when an almost complex structure comes from a complex structure

Theorem 2.15. *Let J be an almost complex structure on M. Then there is a complex structure on M inducing J if and only if*

$$d(\Omega^{p,q}(M)) \subset \Omega^{p+1,q}(M) \oplus \Omega^{p.q+1}(M).$$

□

We will not prove this theorem here. In this generality it is a fairly involved PDE problem. An almost complex structure coming from an underlying complex structure is called **integrable**.

Now denote the projection to $\Omega^{p,q}(M)$ by

$$\pi^{p,q} : \Omega^*(M) \to \Omega^{p,q}(M).$$

Define

$$\partial : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M)$$

by $\partial = \pi^{p+1,q} \circ d$ and

$$\overline{\partial} : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M)$$

by $\overline{\partial} = \pi^{p,q+1} \circ d$. We also define

$$N^* = \pi^{0,2} \circ d : \Omega^{1,0}(M) \to \Omega^{0,2}(M).$$

Exercise 2.16. Let N be the conjugate dual operator

$$N : \Gamma(T^{2,0}(M)) \to \Gamma(T^{0,1}(M))$$

The map N is called the **Nijenhaus tensor**. Show the following:

(1) $N(v, w) = -8\mathrm{Re}([v^{1,0}, w^{1,0}]^{0,1})$.
(2) N is a tensor.
(3) Interperating $N \in \Gamma(TM \otimes T^*M \otimes T^*M)$ we can write

$$N(v, w) = [Jv, Jw] - [v, w] - J[v, Jw] - J[Jv, w],$$

where v, w are vector fields.
(4) $N = 0$ if and only if $T^{1,0}M$ is closed under the Lie bracket.
(5) $N = 0$ if and only if $\overline{\partial}^2 = 0$ on functions.
(6) $N = 0$ if and only if $d(\Omega^{p,q}(M)) \subset \Omega^{p+1,q}(M) \oplus \Omega^{p,q+1}(M)$. (Hence if and only if J is integrable!)

On a complex manifolds we clearly have

$$d = \partial + \overline{\partial}$$

so

$$0 = d^2 = \partial^2 + \overline{\partial}^2 + (\partial\overline{\partial} + \overline{\partial}\partial).$$

When this equation is applied to a (p, q) form we see that the first term on the right is a $(p + 2, q)$ form, then next term is a $(p, q + 2)$ form and the last term is a $(p + 1, q + 1)$ form. Since the decomposition of forms into type is a direct sum decomposition each of these three terms must each be zero. Thus

$$\partial^2 = 0, \quad \overline{\partial}^2 = 0, \quad \text{and} \quad \partial\overline{\partial} + \overline{\partial}\partial = 0.$$

Hence we can consider the chain complex

$$\Omega^{p,0}(M) \xrightarrow{\overline{\partial}} \Omega^{p,1}(M) \xrightarrow{\overline{\partial}} \Omega^{p,2}(M) \xrightarrow{\overline{\partial}} \ldots.$$

We define th **Dolbeault cohomology of** M to be the cohomology of this complex

$$H^{p,q}_{\overline{\partial}}(M) = \frac{\ker(\overline{\partial} : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M))}{\mathrm{im}(\overline{\partial} : \Omega^{p,q-1}(M) \to \Omega^{p,q}(M))}.$$

Exercise 2.17. Let $f : M \to N$ be a holomorphic function between two complex manifolds. Then $f^* : \Omega^k(N) \to \Omega^k(M)$ induces a map

$$f^* : \Omega^{p,q}(N) \to \Omega^{p,q}(M)$$

and

$$f^* \circ \overline{\partial} = \overline{\partial} \circ f^*.$$

So we get a map

$$f^* : H^{p,q}_{\overline{\partial}}(N) \to H^{p,q}_{\overline{\partial}}(M).$$

2.2. **Complex Laplacian and the Hodge theorem.** Now consider a Hermitian structure on the almost complex manifold (M, J). That is

$$h : TM \otimes TM \to \mathbb{R}$$

such that h is a Hermitian inner product when restricted to $T_xM \otimes T_xM$ for each $x \in M$. Note we get a Riemannian metric by

$$g = \text{Real } h$$

and hence a volume from on M which we denote ω_h. We also get

$$\omega = -\frac{1}{2}\text{Im } h$$

which is a non-degenerate 2-form. Actually $\omega \in \Omega^{1,1}(M)$.

Example 2.18. The simplest example of a Hermitian structure is on $M = \mathbb{C}^n$. Here we let

$$\begin{aligned}
h &= \sum_{j=1}^{n} dz^j \otimes d\overline{z}^j \\
&= \sum_{j=1}^{n} (dx^j + i\, dy^j) \otimes (dx^j - i\, dy^j) \\
&= \sum_{j=1}^{n} \left((dx^j \otimes dx^j + dy^j \otimes dy^j) - i(dx^j \otimes dy^j - dy^j \otimes dx^j) \right) \\
&= \sum_{j=1}^{n} \left((dx^j \otimes dx^j + dy^j \otimes dy^j) - i2\, dx^j \wedge dy^j \right).
\end{aligned}$$

Thus g is the standard flat metric on $\mathbb{R}^{2n} = \mathbb{C}^n$ and $\omega = \sum_{j=1}^{n} dx^j \wedge dy^j$.

Exercise 2.19. Show that ω in the above example can also be expressed as

$$\omega = i \sum_{j=1}^{n} dz^j \wedge d\overline{z}^j.$$

Exercise 2.20.

(1) If $f : M \to N$ is a holomorphic function such that

$$df : (T_z(M))^{(1,0)} \to (T_{f(z)}N)^{(1,0)}$$

for all $z \in M$ then for any Hermitian structure h on N the pull-back f^*h is a Hermitian structure on M. Moreover

$$f^*\omega_h = \omega_{f^*h}.$$

(2) If h is a Hermitian metric on a complex n-manifold M (here n is the complex dimension) then

$$\text{Volume } (M) = \frac{1}{n!} \int_m \omega^n.$$

In other words $n! \int_M \omega_h = \int_M \omega^n$.

We get an L^2-inner product from the metric g. Specifically given two functions $f(x)$ and $g(x)$ on M we have

$$\langle f, g\rangle = \int_M f(x)g(x)\frac{1}{n!}\,\omega^n$$

and we can similarly define inner products on vector fields and forms. Using this L^2-inner product on forms we can define the L^2-adjoint of $\overline{\partial}$

$$\overline{\partial}^* : \Omega^{p,q}(M) \to \Omega^{p,q-1}(M).$$

Specifically, $\overline{\partial}^*\psi$ is the unique element in $\Omega^{p,q-1}(M)$ such that

$$\langle \overline{\partial}^*\psi, \eta\rangle = \langle \psi, \overline{\partial}\eta\rangle$$

for all $\eta \in \Omega^{p,q-1}(M)$.

Let $*$ denote the Hodge star operator induced by the Riemannian metric g.

Exercise 2.21. Check that $*$ of a (p,q)-form is a $(n-p, n-q)$-form:

$$* : \Omega^{p,q}(M) \to \Omega^{n-p,n-q}(M).$$

Of course applied to (p,q)-forms we have that

$$\langle \psi, \phi\rangle\,\frac{1}{n!}\omega^n = \psi \wedge *\phi$$

and

$$* * \psi = (-1)^{p+q}\psi.$$

Exercise 2.22. Show the following formula for the L^2-adjoint of $\overline{\partial}$:

$$\overline{\partial}^* = - * \overline{\partial} * .$$

Notice that this of course implies that $\overline{\partial}^* \circ \overline{\partial}^* = 0$.

We can now define the complex Laplace operator to be

$$\Delta_{\overline{\partial}} = \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\,\overline{\partial}.$$

Exercise 2.23. Check that for a function $f : \mathbb{C}^n \to \mathbb{R}$ we have

$$\Delta_{\overline{\partial}} f = \overline{\partial}^*\,\overline{\partial} f = -2\sum \frac{\partial^2}{\partial z^j\partial \overline{z}^j} f.$$

Also show that

$$\frac{\partial^2}{\partial z^j\partial \overline{z}^j} = \frac{1}{4}(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}),$$

so that the complex Laplacian is just a constant multiple of the ordinary Laplacian when applied to functions on $\mathbb{C}^n$.

A form $\alpha \in \Omega^{p,q}(M)$ is called **harmonic** if

$$\Delta_{\overline{\partial}}\,\alpha = 0.$$

Define the set of harmonic (p,q)-forms

$$H^{p,q}(M) = \ker(\overline{\partial} : \Omega^{p,q}(M) \to \Omega^{p,q}(M)).$$

Theorem 2.24 (Hodge Theorem)**.** *Let M be a compact complex n-manifold with Hermitian structure h. Then*

(1) *$H^{p,q}(M)$ is finite dimensional.*

(2) *We have*

$$\begin{aligned}\Omega^{p,q}(M) &= H^{p,q}(M) \oplus \Delta_{\overline{\partial}}(\Omega^{p,q}(M) \\ &= H^{p,q}(M) \oplus \overline{\partial}(\Omega^{p,q-1}(M)) \oplus \overline{\partial}^*(\Omega^{p,q+1}(M)).\end{aligned}$$

Just as for our first Hodge theorem we have the immediate corollaries:

Corollary 2.25. *Given $\alpha \in \Omega^{p,q}(M)$ the equation $\Delta_{\overline{\partial}}\omega = \alpha$ has a solution $\omega \in \Omega^{p,q}(M)$ if and only if α is L^2-orthogonal to $H^{p,q}(M)$ in $\Omega^{p,q}(M)$. Moreover, if there is a solution ω to the equation then the set of all solutions to the equation is $\omega + H^{p,q}(M)$.*

Corollary 2.26. *The Dolbeault cohomology*

$$H^{p,q}_{\overline{\partial}}(M) \cong H^{p,q}(M)$$

is finite dimensional.

Noting that $*$ takes (p,q)-forms to $(n-p, n-q)$-forms we also have the following form of duality.

Theorem 2.27. *For a compact complex n-manifold M with Hermitian structure h we have that*

$$H^{n,n}(M) \cong \mathbb{C}$$

and the pairing

$$H^{p,q}(M) \times H^{n-p,n-q}(M) \to \mathbb{C} : (\alpha, \beta) \mapsto \int_M \alpha \wedge \beta$$

is non-degenerate and hence

$$H^{n-p,n-q}(M) \cong (H^{p,q}(M))^*.$$

We define the **Hodge numbers** to be

$$h^{p,q} = \dim H^{p,q}(M).$$

The previous results tell us that

$$h^{p,q} < \infty,$$
$$h^{n,n} = 1,$$

and

$$h^{p,q} = h^{n-p,n-q}.$$

This is about all we can say for a general complex manifold. In particular, the relation between the Dolbeault cohomology and the topology of the manifolds is unclear. To understand this connection we need another condition on the Hermitian structure. We discuss this in the next subsection.

We note the proofs of all these theorems is essentially identical to the proofs of the corresponding theorems for the DeRham differential once we know the following result.

Lemma 2.28. *The complex Laplacian $\Delta_{\overline{\partial}}$ on a closed complex manifold is an elliptic operator.*

Exercise 2.29. When you have finished reading the proof in the DeRham case, prove this lemma and then derive all the above results from it.

2.3. **Kähler manifolds.** Let M be a complex manifold with induced almost complex structure J and Hermitian metric h. Let $g = \text{Re } h$ be the associated Riemannian metric and $\omega = -\frac{1}{2}\text{Im } h$ the associated non-degenerate $(1,1)$-form. We call (M, J, h) **Kähler** if

$$d\omega = 0.$$

As a side note we mention a closed non-degenerate 2-form on a manifold is called a **symplectic form**. So a Kähler manifold is a complex manifold with a Hermitian inner product such that its associated 2-form is a symplectic form. The study of symplectic forms is quite interesting in its own right, but we will not go into this right now.

To better understand the Kähler condition better suppose we have a Kähler metric h on a complex manifold (M, J). In a coordinate chart U choose a complex orthonormal basis $e^1, \ldots, e^n$ for T^*U. Set $f^j = Je^j$. As worked out above we see that $\{\phi^j = e^j + if^j\}_{j=1}^n$ is an $\mathbb{R}$-basis for $(T^*U)^{1,0}$ and $\{\overline{\phi}^j\}_{j=1}^n$ is a $\mathbb{R}$-basis for $(T^*U)^{0,1}$. Moreover

$$\omega = \frac{i}{2}\sum_{j=1}^n \phi^j \wedge \overline{\phi}^j.$$

We notice that $d\phi^j \in \Omega^{(2,0)}(U) \oplus \Omega^{(1,1)}(U)$. Let τ^j be the $(2,0)$ part of $d\phi^j$. We call the τ^j the **torsion** of the metric. The form $d\phi^j - \tau^j$ is in $\Omega^{(1,1)}(U)$ which has a basis $\{\phi^\alpha \wedge \overline{\phi}^\beta\}$. Thus there are functions $c^j_{\alpha\beta}$ such that

$$d\phi^j - \tau^j = \sum c^j_{\alpha\beta}\overline{\phi}^\beta \wedge \phi^\alpha.$$

Set $\psi^j_\alpha = \sum c^j_{\alpha\beta}\overline{\phi}^\beta$. The ψ^j_α are called the **connection matrix** of the metric. We clearly have

$$d\phi^j = \tau^j + \sum \psi^j_\alpha \wedge \phi^\alpha.$$

Exercise 2.30. Show that $\overline{\psi^j_k} + \psi^k_j = 0$.

We say that h has the same k **jet as the Euclidean metric** if for all $z_0 \in M$ there is a holomorphic coordinate chart about z_0 in which

$$h = \sum_{j=1}^n (\delta_{jk} + g_{jk})\, dz^j \otimes d\overline{z}^j$$

where δ_{jk} is the Kronecker delta function and g_{jk} are functions that vanish to order k at z_0.

Lemma 2.31. *For a complex manifold M with Hermitian metric h the following are equivalent*

1. *h is Kähler,*
2. *$\tau^j = 0$ for all j, and*
3. *h has the same 1 jet as the Euclidean metric at each point of M.*

Proof. We begin by computing

$$\begin{aligned}
\frac{2}{i} d\omega &= \sum (d\phi^j \wedge \overline{\phi}^j - \phi^j \wedge d\overline{\phi}^j) \\
&= \sum \left(\psi^j_\alpha \wedge \phi^\alpha \wedge \overline{\phi}^j - \phi^j \wedge \overline{\psi}^j_\alpha \wedge \overline{\phi}^\alpha + \tau^j \wedge \overline{\phi}^j - \phi^j \wedge \overline{\tau}^j \right) \\
&\quad \text{(using Exercise 2.30)} \\
&= \sum \left(\psi^j_\alpha \wedge \phi^\alpha \wedge \overline{\phi}^j - \psi^\alpha_j \wedge \phi^j \wedge \overline{\phi}^\alpha + \tau^j \wedge \overline{\phi}^j - \phi^j \wedge \overline{\tau}^j \right) \\
&= \sum \left(\tau^j \wedge \overline{\phi}^j - \phi^j \wedge \overline{\tau}^j \right)
\end{aligned}$$

Noting that τ^j has type $(2,0)$ and similarly $\phi^j, \overline{\phi}^j$, and $\overline{\tau}^j$ have type $(1,0), (0,1)$ and $(0,2)$, respectively, we see that $d\omega = 0$ if and only if all the $\tau^j = 0$, thus establishing the equivalence of (1) and (2).

Now assume that (3) is true. So we have

$$\omega = \frac{i}{2} \sum_{j=1}^{n} (\delta_{jk} + g_{jk})\, dz^j \wedge d\overline{z}^j.$$

Since $g_{jk}(z_0) = dg_{jk}(z_0) = 0$, we see $(d\omega)_{z_0} = 0$ for any z_0. Thus (3) implies (1).

Now assume that (1) is true. Can always choose coordinates around a point z_0 so that

$$\omega = \frac{i}{2} \sum_{j=1}^{n} \left(\delta_{jk} + a_{jkl}\, z^l + a_{jk\bar{l}}\, \overline{z}^l + \text{(higher order terms)} \right) dz^j \wedge d\overline{z}^j.$$

Exercise 2.32. Use our assumption to show that $a_{jk\bar{l}} = \overline{a_{kjl}}$ and $a_{jkl} = a_{lkj}$.

Exercise 2.33. Make the coordinate change

$$z^k = w^k + \frac{1}{2} \sum b_{klm} w^l w^m$$

and compute ω in these new coordinates.

Exercise 2.34. Show that if $b_{klj} = -a_{jkl}$ then in these new coordinates h has the same 1 jet as the Euclidean metric at z_0.

This completes the proof of the lemma. □

Example 2.35. Any metric on a compact Riemann surface Σ is Kähler since $d\omega$ is a 3-form and hence zero.

Example 2.36. The standard metric on $\mathbb{C}^n$ is Kähler:

$$h = \sum_{j=1}^{n} dz^j \otimes d\overline{z}^j$$

and

$$\omega = \sum_{j=1}^{n} dx^j \wedge dy^J$$

which is clearly closed.

Example 2.37. Let $T^{2n} = \mathbb{C}^n / \Lambda$ where Λ is a lattice in $\mathbb{C}^n$. Since the standard metric on $\mathbb{C}^n$ is invariant under translations, it descends to a Kähler form on T^{2n}.

Example 2.38. Consider complex projective space $\mathbb{C}P^n$. Recall $\mathbb{C}P^n$ is the space of complex planes in $\mathbb{C}^{n+1}$, that is

$$\mathbb{C}P^n = (\mathbb{C}^{n+1} - \{(0,0,\ldots,0)\})/(\mathbb{C} - \{0\}).$$

Denote the quotient map

$$\pi : (\mathbb{C}^{n+1} - \{(0,0,\ldots,0)\}) \to \mathbb{C}P^n$$

and let $z^0, \ldots, z^n$ be coordiantes on $\mathbb{C}^{n+1}$. Let U be a coordinate chart for $\mathbb{C}P^n$ so that $\pi^{-1}(U) = U \times (\mathbb{C} - \{0\})$. Choose a holomorphic function

$$z : U \to (\mathbb{C} - \{0\}) \subset \mathbb{C}.$$

Now set

$$\omega = \frac{i}{2\pi}\partial\overline{\partial}(\log \|z\|^2).$$

We claim that ω is independent of z. Indeed given another holomorphic function z' there is a non-zero holomorphic function f such that $z' = fz$. Thus

$$\begin{aligned}
\frac{i}{2\pi}\partial\overline{\partial}\log \|z'\|^2 &= \frac{i}{2\pi}\partial\overline{\partial}(\log \|z\|^2 f\overline{f} + \log f + \log \overline{f}) \\
&= \frac{i}{2\pi}\partial\overline{\partial}(\log \|z\|^2 + \log f + \log \overline{f}) \\
&= \omega + \frac{i}{2\pi}\partial\overline{\partial}(\log f + \log \overline{f}).
\end{aligned}$$

Moreover $\overline{\partial}\log f = 0$ since $\log f$ is holomorphic and similarly $\partial\overline{\partial}\log \overline{f} = 0$, so ω is independent of z. Convince yourself that this implies that we can define ω on all of $\mathbb{C}P^n$.

Since $d = \partial + \overline{\partial}$ we clearly have that $d\omega = 0$.

Exercise 2.39. Let $U = \pi(\{z^0 \neq 0\})$. On this set we have "homogeneous coordinates" $w^j = \frac{z^j}{z^0}$. In these coordinates show

$$\omega = \frac{i}{2\pi}\sum_{j=1}^{n} dw^j \wedge d\overline{w}^j.$$

Thus ω is a non-degenerate $(1,1)$-form and is J invariant.

Thus we can use ω to define a Hermitian metric on $\mathbb{C}P^n$ that is Kähler. This metric is called the Fubini-Study metric.

Exercise 2.40. Show that the product of Kähler manifolds is Kähler.

Exercise 2.41. Show that if $S \subset M$ is a complex submanifold of a Kähler manifold, then the induced metric on S is a Kähler metric.

Example 2.42. Note the previous exercises and examples show that any compact complex submanifold of $\mathbb{C}^n$ or $\mathbb{C}P^n$ is a Kähler manifold (*ie* any smooth projective variety is Kähler).

We have the immediate connections between the Dolbeault cohomology of a Kähler manifold and its algebraic topology.

Theorem 2.43. *For a closed Kähler manifold M we have*

(1) $b_{2k}(M) > 0$,
(2) $H^{q,0}_{\overline{\partial}}(M)$ *injects into* $H^q_{dR}(M)$, *and*

(3) *for any complex submanifold (or variety) $V \subset M$ of complex dimension k we have $[V] \neq 0$ in $H_{2k}(M)$.*

Proof. For (1) we notice that $d\omega^k = 0$ so it represents some element of $H^{2k}_{dR}(M)$. To see that it is not the zero element note that if it were not then $\omega^k = d\phi$ and hence $\omega^n = d\phi \wedge \omega^{n-k} = cd(\phi \wedge \omega^{n-k})$, for some constant c. Thus Stokes' theorem implies

$$0 = \int_{\partial M} c\phi \wedge \omega^{n-k} = \int_M cd(\phi \wedge \omega^{n-k}) = \int_M \omega^n.$$

This contradicts the fact observed above that ω^n is a non-zero multiple of the volume form on M which of course has non-zero integral.

For item (2) suppose that $[\eta] \in H^{q,0}(M)$. So we know $\overline{\partial}\eta = 0$. We want to show that $d\eta = 0$ and that if $\eta = d\phi$ then $[\eta] = 0$ in $H^{q,0}(M)$. To this end let $\phi^1, \ldots, \phi^n$ be an orthonormal frame for $\Omega^{1,0}(M)$, so we can write

$$\eta = \sum \eta_I \phi^I$$

for some functions η_I. So we have $\eta \wedge \overline{\eta} = \sum \eta_I \overline{\eta}_I \phi^I \wedge \overline{\phi}^I$.

Exercise 2.44. Show that

$$\omega^{n-q} = c_q \sum_{|K|=n-q} \phi^K \wedge \overline{\phi}^K$$

where c_q is a constant depending only on q.

Notice that

$$\eta \wedge \overline{\eta} \wedge \omega^{n-q} = c_q \sum_{|I|=q} \eta_I \overline{\eta}_I \phi^I \wedge \overline{\phi}^I \wedge \phi^K \wedge \overline{\phi}^K$$

where $K = 1, \ldots, n - I$ in the sum. Thus

$$\eta \wedge \overline{\eta} \wedge \omega^{n-q} = \frac{c_q}{c_n} (\sum_{|I|=q} |\eta_I|^2) dvol_g$$

So we see that a $(q, 0)$-form η is non-zero if and only if $\int_M \eta \wedge \overline{\eta} \wedge \omega^{n-q} \neq 0$.

Now assuming η represents a class in $H^{q,0}(M)$ we have $\overline{\partial}\eta = 0$. So $d\eta = (\partial + \overline{\partial})\eta = \partial\eta$. Of course $\partial\eta$ is in $\Omega^{q+1,0}(M)$. Thus $\partial\eta = 0$ if and only if

$$\int_M d\eta \wedge \overline{d\eta} \wedge \omega^{n-q-1} = 0.$$

But $d(\eta \wedge \bar{(}d\eta) \wedge \omega^{n-q-1}) = d\eta \wedge \overline{d\eta} \wedge \omega^{n-q}$. Thus Stokes' theorem implies $\int_M d\eta \wedge \overline{d\eta} \wedge \omega^{n-q-1} = 0$ and hence $\partial\eta = 0$.

Now suppose that $\eta = d\phi$. Then

$$\int_M \eta \wedge \overline{\eta} \wedge \omega^{n-q} = \int_M d(\phi \wedge \overline{\eta} \wedge \omega^{n-q}) = 0$$

and we see that $\eta = 0$ and hence $[\eta] = 0$ in $H^{p,0}(M)$.

For statement (3) we observe that if V is a complex submanifold of M of complex dimension k then the Kähler form ω pulls back to a Kähler form ω_V on V. Thus integrating ω^k over V gives a non-zero number. This implies that V the fundamental class of V in M cannot be null-homologous. $\square$

We are now ready to see a more subtle connection between the Dolbeault cohomology of a Kähler manifold and its algebraic topology. The key is to relate the ordinary Laplacian to the complex Laplacian. To that end we define the following operators. First two projections:

$$\pi^{p,q} : \Omega^*(M) \to \Omega^{p,q}(M)$$

and

$$\pi^r : \Omega^*(M) \to \Omega^r(M)$$

where clearly $\pi^r = \sum_{p+q=r} \pi^{p,q}$. From the operators $\partial, \overline{\partial}$ and $d = \partial + \overline{\partial}$ we have

$$d^c = \frac{i}{4\pi}(\overline{\partial} - \partial)$$

and the various Laplacians

$$\Delta_d = dd^* + d^*d, \quad \Delta_{\overline{\partial}} = \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\,\overline{\partial} \text{ and } \quad \Delta_\partial = \partial\partial^* + \partial^*\partial.$$

Finally using the $(1,1)$-form ω coming from the Hermitian metric we have

$$L : \Omega^{p,q}(M) \to \Omega^{p+1,q+1}(M) : \eta \to \eta \wedge \omega$$

and its adjoint

$$\Lambda : \Omega^{p,q}(M) \to \Omega^{p-1.q-1}(M).$$

Theorem 2.45. *Let M a Kähler manifold. Then*

(1) $[\Lambda, d] = -4\pi(d^c)^*$,
(2) $[L, d^*] = 4\pi d^c$,
(3) $[L, \Delta_d] = 0 = [\Lambda, \Delta_d]$,
(4) $[\Delta_d, \pi^{p,q}] = 0$, *and*
(5) $\Delta_d = 2\Delta_{\overline{\partial}} = 2\Delta_\partial$.

Before proving this theorem notice we now have

Theorem 2.46 (Hodge decomposition theorem)**.** *For a closed compact Káhler manifold M we have*

$$H^r_{dR}(M) = \bigoplus_{p+q=r} H^{p,q}_{\overline{\partial}}(M)$$

and

$$H^{p,q}_{\overline{\partial}}(M) = \overline{H^{q,p}_{\overline{\partial}}(M)}.$$

Thus we see that the deRahm cohomology, which we know is related to the ordinary cohomology, is actually the sum of the appropriate Dolbeault cohomology on a Kähler manifold. So, in some sense, we can think of the Dolbeault cohomology of a Kähler manifold as a refinement of the ordinary cohomology (think about in what sense this is true and in what sense it is not). Recall we know that $h^{k,k} > 0$ and $h^{n,n} = 1 = b^{2n}$. Moreover, from the last theorem we know that $h^{p,q} = h^{q,p}$. We have the "Hodge diamond":

$$
\begin{array}{ccccccccccc}
 & & & & & h^{n,n} & & & & & \\
 & & & \iddots & & \vdots & & \ddots & & & \\
 & & & & & h^{k,k} & & & & & \\
h^{n,0} & & & & & \vdots & & & & & h^{0,n} \\
 & \ddots & & & & \vdots & & & & \iddots & \\
 & & h^{3,0} & & h^{2,1} & \vdots & h^{1,2} & & h^{0,3} & & \\
 & & & h^{2,0} & & h^{1,1} & & h^{0,2} & & & \\
 & & & & h^{1,0} & & h^{0,1} & & & & \\
 & & & & & h^{0,0} & & & & &
\end{array}
$$

This figure is symmetric about the horizontal middle line. Of course an immediate corollary of the Hodge decomposition theorem is the following.

Theorem 2.47. *The odd Betti numbers of a Kähler manifold are even, that is*

$$\dim H^{2k+1}_{dR}(M) \text{ is even.}$$

Proof of Theorem 2.46. Set

$$
\begin{aligned}
H^{p,q}_d &= \{\eta \in \Omega^{p,q}(M) | \Delta_d \eta = 0\} \\
&= \{\eta \in \Omega^{p,q}(M) | \Delta_{\overline{\partial}} \eta = 0\} = H^{p,q}_{\overline{\partial}}(M).
\end{aligned}
$$

Now since

$$H^r = \{\eta \in \Omega^r(M) | \Delta_d \eta = 0\} \cong H^r_{dR}(M)$$

and

$$[\Delta_d, \pi^{p,q}] = 0$$

we clearly have

$$H^r_{dR}(M) = \bigoplus_{p+q=r} H^{p,q}_{\overline{\partial}}(M).$$

Since Δ_d is a real operator for any $\eta \in \Omega^{p,q}(M)$ we clearly have

$$\Delta_d \eta = 0 \quad \text{if and only if} \quad \Delta_d \overline{\eta} = 0$$

and thus we have

$$H^{p,q}_{\overline{\partial}}(M) = \overline{H^{q,p}_{\overline{\partial}}(M)}.$$

□

Now for the proof of the technical theorem.

Proof of Theorem 2.45. We begin by showing that equation (1) in the theorem implies the other equations. Equation (1) says

$$\Lambda d - d\Lambda = -4\pi(d^c)^* = -i(\overline{\partial}^* - \partial^*).$$

The adjoint equation is

$$d^*L - Ld^* = -i(\overline{\partial} - \partial) = -4\pi d^c.$$

That is the adjoint is just Equation (2). To verify Equation (3) notice that

$$d(\omega \wedge \eta) = \omega \wedge d\eta$$

and hence $[d, L] = 0 = [L, d]$ and $[d^*, \Lambda] = 0 = [\Lambda, d^*]$. Moreover

$$\begin{aligned} d^c d &= \frac{i}{4\pi}(\overline{\partial} - \partial)(\partial + \overline{\partial}) = \frac{i}{4\pi}(-\partial\overline{\partial} + \overline{\partial}\partial) \\ &= -\frac{i}{4\pi}(\partial + \overline{\partial})(\overline{\partial} - \partial) = -dd^c. \end{aligned}$$

Taking adjoints we also have $(d^c)^*d^* = -d^*(d^c)^*$. Thus we see

$$\begin{aligned} L(dd^* + d^*d) &= dLd^* + (d^*L + 4\pi d^c)d \\ &= dd^*L + d(4\pi d^c) + d^*dL + 4\pi d^c d \\ &= (dd^* + d^*d)L \end{aligned}$$

so $L\Delta_d = \Delta_d L$ and taking adjoints gives $\Delta_d \Lambda = \Lambda \Delta_d$, thus establishing Equation (3).

Let us work on Equation (5) now. To this end notice that

$$\begin{aligned} \Lambda d - d\Lambda = -4\pi(d^c)^* &\Leftrightarrow \Lambda(\partial + \overline{\partial}) - (\partial + \overline{\partial})\Lambda = -i(\overline{\partial}^* - \partial^*) \\ &\qquad \text{(decomposing by type)} \\ &\Leftrightarrow [\Lambda, \partial] = -i\overline{\partial}^* \text{ and } [\Lambda, \overline{\partial}] = i\partial^*. \end{aligned}$$

So we have

$$\begin{aligned} \partial\overline{\partial}^* + \overline{\partial}^*\partial &= -i(\partial(\Lambda\partial - \partial\Lambda) + (\Lambda\partial - \partial\Lambda)\partial) \\ &= -i(\partial\Lambda\partial - \partial\Lambda\partial) = 0. \end{aligned}$$

From which we can deduce

$$\begin{aligned} \Delta_d &= (\partial + \overline{\partial})(\partial^* + \overline{\partial}^*) + (\partial^* + \overline{\partial}^*)(\partial + \overline{\partial}) \\ &= (\partial\partial^* + \partial^*\partial) + (\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}) + (\partial\overline{\partial}^* + \overline{\partial}\partial^* + \partial^*\overline{\partial} + \overline{\partial}^*\partial) \\ &\qquad \text{(from the last equation)} \\ &= (\partial\partial^* + \partial^*\partial) + (\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}) = \Delta_\partial + \Delta_{\overline{\partial}}. \end{aligned}$$

In addition we have

$$\begin{aligned} \Delta_\partial &= \partial\partial^* + \partial^*\partial \\ &= -i(\partial(\Lambda\overline{\partial} - \overline{\partial}\Lambda) + (\Lambda\overline{\partial} - \overline{\partial}\Lambda)\partial) \\ &= -i(\partial\Lambda\overline{\partial} + \Lambda\overline{\partial}\partial - \partial\overline{\partial}\Lambda - \overline{\partial}\Lambda\partial) \end{aligned}$$

and similarly

$$\begin{aligned}\Delta_{\overline{\partial}} &= i(\overline{\partial}\Lambda\partial - \overline{\partial}\partial\Lambda + \Lambda\partial\overline{\partial} - \partial\Lambda\overline{\partial}) \\ &= i(\overline{\partial}\Lambda\partial + \partial\overline{\partial}\Lambda - \Lambda\overline{\partial}\partial - \partial\Lambda\overline{\partial}) \\ &= \Delta_{\partial}\end{aligned}$$

These last three computations yield equation (5) in the theorem. As it is clear that Δ_∂ commutes with $\pi^{p,q}$ we also see that Δ_d commutes using (5). Thus (4) also holds.

We are left to establish (1).

Exercise 2.48. Show that if (1) is true in $\mathbb{C}^n$ with the standard Hermitian structure then it is true for any Kähler manifold using the the fact that a Kähler metric has the same 1 jet as the standard Hermitian structure on $\mathbb{C}^n$ by Lemma 2.31.

We now consider the standard Hermitian $\mathbb{C}^n$. Let $z^1, \dots, z^n$ be the standard coordinates on $\mathbb{C}^n$. Define the wedge product maps

$$e_k : \Omega^{p,q}(\mathbb{C}^n) \to \Omega^{p+1,q}(\mathbb{C}^n) : \phi \mapsto dz^k \wedge \phi$$

and

$$\overline{e}_k : \Omega^{p,q}(\mathbb{C}^n) \to \Omega^{p,q+1}(\mathbb{C}^n) : \phi \mapsto d\overline{z}^k \wedge \phi.$$

Let i_k and $\overline{i}_k$ be their formal adjoints. We claim

$$i_k \circ e_k(dz^J \wedge d\overline{z}^K) = \begin{cases} 0 & \text{if } k \in K \\ 2dz^J \wedge d\overline{z}^K & \text{if } k \notin K. \end{cases}$$

To see this notice that the first link is obvious since $dz^k \wedge dz^k = 0$. The second line follows by observing that

$$\begin{aligned}\langle i_k \circ e_k(dz^J \wedge d\overline{z}^K), dz^L \wedge d\overline{z}^M\rangle &= \langle dz^k \wedge dz^J \wedge d\overline{z}^K, dz^k \wedge dz^L \wedge d\overline{z}^M\rangle \\ &= \langle dz^k, dz^k\rangle\langle dz^J \wedge d\overline{z}^K, dz^L \wedge d\overline{z}^M\rangle \\ &= 2\langle dz^J \wedge d\overline{z}^K, dz^L \wedge d\overline{z}^M\rangle.\end{aligned}$$

Similarly we have

$$e_k \circ i_k(dz^J \wedge d\overline{z}^K) = \begin{cases} 0 & \text{if } k \notin K \\ 2dz^J \wedge d\overline{z}^K & \text{if } k \in K. \end{cases}$$

From this we have $e_k \circ i_k + i_k \circ e_k = 2$ and $\overline{e}_k \circ \overline{i}_k + \overline{i}_k \circ \overline{e}_k = 2$. If $k \neq l$ then one may easily check that $e_k \circ i_l + i_l \circ e_k = 0$ and $\overline{e}_k \circ \overline{i}_l + \overline{i}_l \circ \overline{e}_k = 0$.

Now notice that

$$L = \frac{i}{2}\sum e_k \circ \overline{e}_k$$

and

$$\Lambda = -\frac{i}{2}\sum \overline{i}_k \circ i_k.$$

If we define $\partial_k \sum \phi_{JK} dz^J \wedge d\overline{z}^K = \sum \left(\frac{\partial \phi_{JK}}{\partial z^k}\right) dz^J \wedge d\overline{z}^K$ and similarly for $\overline{\partial}_k$ then we have

$$\partial = \sum \partial_k \circ e_k = \sum e_k \circ \partial_k$$

and

$$\overline{\partial} = \sum \overline{\partial}_k \circ \overline{e}_k = \sum \overline{e}_k \circ \overline{\partial}_k.$$

Exercise 2.49. Show that the formal adjoint of ∂_k is $-\partial_k$ and the formal adjoint of $\overline{\partial}_k$ is $-\overline{\partial}_k$. Also show that ∂_k and $\overline{\partial}_k$ and their adjoints commute with $e_l, \overline{e}_l, i_l$, and $\overline{i}_l$.

From the exercise we see that
$$\partial^* = -\sum \partial_k \circ i_k$$
and
$$\overline{\partial}^* = -\sum \overline{\partial}_k \circ \overline{i}_k.$$
Now
$$\begin{aligned}
\Lambda\partial &= -\frac{i}{2}\sum \overline{i}_k i_k \partial_l e_l = \sum \partial_l \overline{i}_k i_k e_l\\
&= -\frac{i}{2}\left(\sum_k \partial_k \overline{i}_k i_k e_k + \sum_{k\neq l} \partial_l \overline{i}_k i_k e_l\right)\\
&= \frac{i}{2}\sum_k \partial_k \overline{i}_k e_k i_k - \frac{2i}{2}\sum_k \partial_k \overline{i}_k + \frac{i}{2}\sum_{l\neq k} \partial_l \overline{i}_k e_l i_k\\
&= \frac{i}{2}\sum_{l,k} \partial_l \overline{i}_k e_l i_k - i\sum_k \partial_k \overline{i}_k\\
&= -\frac{i}{2}\sum_{l,k} \partial_l e_l \overline{i}_k i_k - i\overline{\partial}^* = \partial\Lambda - i\overline{\partial}^*.
\end{aligned}$$
Similarly
$$\Lambda\overline{\partial} = \overline{\partial}\Lambda - i\partial^*.$$
Thus
$$\Lambda d = d\Lambda - i(\overline{\partial}^* - \partial^*).$$
Completing the proof of (1). □

We end this section with the Hard Lefschetz Theorem. We do not provide a proof, but refer to Griffiths and Harris "Principles of Algebraic Geometry" for that as well as a discussion of the theorem.

Theorem 2.50 (Hard Lefschetz Theorem)**.** *Let M be a closed Kähler manifold. Then*

(1) *The map*
$$L^k : H^{n-k}_{dR}(M) \to H^{n+k}_{dR}(M),$$
coming from wedging with the k^{th} power of the Kähler form ω, is an isomorphism.

(2) *If*
$$P^{n-k}(M) = \ker(L^{k+1} : H^{n-k}_{dR}(M) \to H^{n+k+2}_{dR}(M)) = \ker(\Lambda) \cap H^{n-k}_{dR}(M)$$
then
$$H^m_{dR}(M) = \sum_k L^k(P^{m-2k}(M)).$$

3. Function spaces

3.1. **Sobolev spaces on T^n.** Let $T^n = \mathbb{R}^n/(2\pi\mathbb{Z}^n)$ with coordinates $(x_1, \ldots, x_n)$.

Let $C^l(T^n, \mathbb{C}^m) = \{f : T^n \to \mathbb{C}^k | f \text{ is } l \text{ times differentiable}\}$

The L^2 inner product on $C^\infty(T^n, \mathbb{C}^m)$ is

$$\langle f, g\rangle = \frac{1}{(2\pi)^n}\int_{T^n} f(x)\cdot g(x) dvol_x$$

and norm is

$$\|f\| = \langle f, f\rangle^{\frac{1}{2}}.$$

The sup norm on C^0 is

$$\|f\|_\infty = \sup_{x\in T^n}\{|f(x)|\}.$$

The Sovolev k-inner product on C^∞ keeps track of the L^2 inner product of k derivatives. If $\alpha = (\alpha_1, \ldots, \alpha_n)$ and

$$D^\alpha = (-i)^{|\alpha|}\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\cdots\left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Now define

$$\langle f, g\rangle_s = \sum_{|\alpha|\le s}\langle D^\alpha f, D^\alpha g\rangle$$

and

$$\|f\|_s = \langle f, f\rangle_s^{\frac{1}{2}}.$$

Notice that the inner product and norm would be the same with or without the factor of i in the definition of D^α. It is included to simplify various formulas below.

Lemma 3.1. *All the above define inner products on C^∞.*

Define $H_s(T^n)$ to be the completion of $C^\infty(T^n, \mathbb{C}^m)$ in the norm $\|\|_s$.

In general two norms $\|\|$ and $\|\|'$ on a vector space V are called **equivalent** if there are constants C and C' such that

$$C\|v\| \le \|v\|' \le C'\|v\|$$

for all $v \in V$.

Lemma 3.2. *Equivalent norms on a vector space V induce equivalent topologies on the vector space. A sequence is Cauchy in one norm if and only if it is Cauchy in the other norm. Thus the completions of V in both norms are isomorphic vector spaces.*

Let

$$\langle f, g\rangle_s' = \langle f, g\rangle + \sum_{|\alpha|=s}\langle D^\alpha f, D^\alpha g\rangle$$

and

$$\|f\|_s' = (\langle f, f\rangle_s')^{\frac{1}{2}}.$$

Lemma 3.3. *The norms $\|\|_s$ and $\|\|_s'$ are equivalent norms.*

This will be easiest to prove after the next section.

Recall def of completion. Completions only depend on norm up to equivalence. bounded linear maps on a dense subset extend to bounded maps on the completions (if image space is complete). If normed linear space is a subset of another complete linear space then its closure is the completion.

Maybe Def of H_s as subset of L^2 functions (so need to define weak derivative).

3.2. **Alternate, Fourier, definition of Sobolev spaces on T^n.** Given a function $f \in C^\infty(T^n, \mathbb{C}^n)$ and an n-tuple of integers $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{Z}^n$ we set

$$f_\xi = \frac{1}{(2\pi)^n} \int_{T^n} f(x) e^{-i\,x\cdot\xi} dvol_x$$

and call this the ξ^{th} Fourier coefficient of f. The Fourier series of f is

$$\sum_{\xi \in \mathbb{Z}^n} f_\xi e^{i\,x\cdot\xi}.$$

Say some stuff about $l^2(\mathbb{Z}^n)$ the l^2 norm and weighted l^2-norms.

Theorem 3.4. *Facts about Fourier series:*

(1) *This series converges uniformly to $f(x)$.*
(2) *(Parseval's identity) The L^2-norm of f can be computed by*

$$\|f\|^2 = \sum_{\xi \in \mathbb{Z}^n} |f_\xi|^2. \tag{5}$$

(3) *Something about $L^2(T^n, \mathbb{C}^m)$ and $l^2(\mathbb{Z}^n, \mathbb{C}^m)$.*

Note

$$\begin{aligned}
(\frac{\partial}{\partial x_j} f)_\xi &= \frac{1}{(2\pi)^n} \int_{T^n} \big(\frac{\partial}{\partial x_j} f(x)\big) e^{-i\,x\cdot\xi} dvol_x \\
&\qquad \text{(integrate by parts and notice no boundary terms)} \\
&= -\frac{1}{(2\pi)^n} \int_{T^n} f(x) \big(\frac{\partial}{\partial x_j} e^{-i\,x\cdot\xi}\big) dvol_x \\
&= -\frac{1}{(2\pi)^n} \int_{T^n} f(x) \big(-i\xi_j e^{-i\,x\cdot\xi}\big) dvol_x \\
&= i\xi_j \frac{1}{(2\pi)^n} \int_{T^n} f(x) e^{-i\,x\cdot\xi} dvol_x \\
&= i\xi_j f_\xi.
\end{aligned} \tag{6}$$

Or better:

$$(-i\frac{\partial}{\partial x_j} f)_\xi = \xi_j f_\xi.$$

In general we clearly have

$$(D^\alpha f)_\xi = \xi^\alpha f_\xi. \tag{7}$$

Using Parseval's identity, Equation (5), we can compute the L^2-norm of $D^\alpha f$:

$$\|D^\alpha f\|^2 = \sum_{\xi} \xi^{2\alpha} |f_\xi|^2. \tag{8}$$

We can use this observation to rewrite the Sobolev s-norm on $C^\infty(T^n, \mathbb{C}^m)$

$$\|f\|_s^2 = \sum_{|\alpha|\le s} \|D^\alpha f\|^2 = \sum_{|\alpha|\le s,\xi} \xi^{2\alpha}|f_\xi|^2.$$

Now consider the norm

$$\|f\|_s'' = \left(\sum_\xi (1+|\xi|^2)^s \|f_\xi\|^2\right)^{\frac{1}{2}}, \tag{9}$$

where s is any number. Say something about this giving norm on $l^2(\mathbb{Z}^n, \mathbb{C}^m)$...

Lemma 3.5. *For any non-negative integer s the norms $\|\|_s$ and $\|\|_s''$ are equivalent. That is there are constants c, c' such that*

$$c\|f\|_s \le \|f\|_s'' \le c'\|f\|_s.$$

Thus we can think of this Fourier defined Sobolev norm as generalizing the derivative defined norm. Below when s is a non-negative integer we will switch between these two norms without warning, using the one most convenient at the time. Discuss H_s as completion of image of C^∞ functions in l^2.

Proof. We begin by claiming that for each t there are constants c_t and c_t' such that for all ξ

$$c_t \sum_{|\alpha|=t} \xi^{2\alpha} \le |\xi|^{2t} \le c_t' \sum_{|\alpha|=t} \xi^{2\alpha} \tag{10}$$

We will prove this below, but given this inequality note that if we set

$$c = \min_{t=0\ldots s, t'=0\ldots s} \{\frac{1}{\binom{s}{t'} c_t'}\}$$

then

$$\begin{aligned} c(1+|\xi|^2)^s &= c(1+\binom{s}{2}|\xi|^2 + \ldots + \binom{s}{s-1}|\xi|^{2(s-1)} + |\xi|^{2s}) \\ &\le 1 + \frac{1}{c_1'}|\xi|^2 + \ldots + \frac{1}{c_{s-1}'}|\xi|^{2(s-1)} + \frac{1}{c_s'}|\xi|^{2s} \\ &\le 1 + \sum_{|\alpha|=1} \xi^{2\alpha} + \ldots + \sum_{|\alpha|=s-1} \xi^{2\alpha} + \sum_{|\alpha|=s} \xi^{2\alpha} \\ &\le \sum_{|\alpha|\le s} \xi^{2\alpha}. \end{aligned} \tag{11}$$

One can similarly define c' so that

$$\sum_{|\alpha|\le s} \xi^{2\alpha} \le c'(1+|\xi|^2)^s.$$

Thus we have

$$c(1+|\xi|^2)^s = \sum_{|\alpha|\le s} \xi^{2\alpha} \le c'(1+|\xi|^2)^s.$$

And hence

$$c(1+|\xi|^2)^s|f_\xi|^2 = \sum_{|\alpha|\le s} \xi^{2\alpha}|f_\xi|^2 \le c'(1+|\xi|^2)^s|f_\xi|^2.$$

Now if we sum over ξ notice that

$$
\begin{aligned}
c^{\frac{1}{2}}\|f\|''_s &= \left(c\sum_{\xi}(1+|\xi|^2)^s|f_\xi|^2\right)^{\frac{1}{2}} \\
&\leq \left(\sum_{\xi,|\alpha|\leq s}\xi^{2\alpha}|f_\xi|^2\right)^{\frac{1}{2}} \\
&= \left(\sum_{|\alpha|\leq s}\|D^\alpha f\|^2\right)^{\frac{1}{2}} \\
&= \|f\|_s.
\end{aligned} \tag{12}
$$

Similarly

$$\|f\|_s \leq (c')^{\frac{1}{2}}\|f\|''_s.$$

We are left to prove Equation (10). To this end consider the function

$$p(\xi) = \frac{|\xi|^{2t}}{\sum_{|\alpha|=t}\xi^{2\alpha}}.$$

We think of p as a function $\mathbb{R}^n \to \mathbb{R}$. Notice that

$$p(\lambda\xi) = p(\xi)$$

for all non-zero λ. Since Equation (10) is clear for $\xi = 0$ we just prove it for $\xi \neq 0$. The unit sphere in $\mathbb{R}^n$ is compact and p is positive on it so it has a positive minimum c_t and maximum c'_t. Thus we see that

$$p(\xi) = p(\xi/|\xi|) \leq c'_t$$

so

$$|\xi|^{2t} \leq c'_t \sum_{|\alpha|=t}\xi^{2\alpha}.$$

Similarly

$$c_t \sum_{|\alpha|=t}\xi^{2\alpha} \leq |\xi|^{2t}.$$

□

3.3. **Operators on Sobolev spaces.** We begin by defining a **differential operator** from $C^\infty(T^n,\mathbb{C}^m)$ to $C^\infty(T^n,\mathbb{C}^{m'})$. This is just an $m \times m'$ matrix with entries for the form $\sum_\alpha \omega_\alpha(x)D^\alpha$. The **order** of the operator is the highest order derivative taken in the matrix.

Example 3.6. The operator

$$L = \sum_{j=1}^n \left(\frac{\partial}{\partial x_i}\right)^2$$

is a second order operator from $C^\infty(T^n,\mathbb{C})$ to itself.

Example 3.7. The operator

$$\begin{bmatrix} L = \cos x_1 \frac{\partial}{\partial x_1} & e^{x_1x_2}\frac{\partial}{\partial x_2} & 5\frac{\partial^2}{\partial x_1\partial x_2} \\ (\frac{\partial}{\partial x_2})^2 & 9\sin x_2 \frac{\partial}{\partial x_2} & 9 \end{bmatrix}$$

is a second order operator from $C^\infty(T^n,\mathbb{C}^3)$ to $C^\infty(T^n,\mathbb{C}^2)$.

Theorem 3.8. (1) *For all s the derivative operator $D^\alpha : C^\infty(T^n, \mathbb{C}^m) \to C^\infty(T^n, \mathbb{C}^m)$ satisfies*

$$\|D^\alpha f\|_s \le \|f\|_{s+|\alpha|} \tag{13}$$

and thus extends to a bounded linear operator

$$D^\alpha : H_{s+|\alpha|}(T^n) \to H_s(T^n). \tag{14}$$

(2) *For all s and all functions $\omega(x) \in C^\infty(T^n, \mathbb{C})$ there are constants c and c' such that the multiplicative operator $M_\omega : C^\infty(T^n, \mathbb{C}^m) \to C^\infty(T^n, \mathbb{C}^m)$ defined by $M_\omega(f(x)) = \omega(x)f(x)$ satisfies*

$$\|M_\omega f\|_s \le c\|\omega\|_\infty \|f\|_s + c'\|f\|_{s-1} \tag{15}$$

and in particular (since $\|f\|_{s-1} \le \|f\|_s$) there is a constant c'' such that $\|M_\omega f\|_s \le c''\|f\|_s$ and thus the operator extends to a bounded linear operator

$$M_\omega : H_s(T^n) \to H_s(T^n). \tag{16}$$

(3) *For all s and linear differential operator of order k, $L : C^\infty(T^n, \mathbb{C}^m) \to C^\infty(T^n, \mathbb{C}^m)$ there is a constant c such that*

$$\|Lf\|_s \le c\|f\|_{s+k} \tag{17}$$

and thus extends to a bounded linear operator

$$L : H_{s+k}(T^n) \to H_s(T^n). \tag{18}$$

Proof. Notice that

$$\xi^{2\alpha} = \xi_1^{2\alpha_1} \cdots \xi_n^{2\alpha_n} \le (1 + |\xi|^2)^{|\alpha|}$$

since $\xi_1^{2\alpha_1} \cdots \xi_n^{2\alpha_n}$ will occur as one of the terms in $(1 + |\xi|^2)^{|\alpha|} = (1 + \xi_1^2 + \ldots + \xi_n^2)^{|\alpha|}$ with coefficient greater than one. Thus

$$\begin{aligned}
\|D^\alpha f\|_s &= \sum_\xi (1 + |\xi|^2)^s \xi^{2\alpha} |f_\xi|^2 \\
&\le \sum_\xi (1 + |\xi|^2)^s (1 + |\xi|^2)^{|\alpha|} |f_\xi|^2 \\
&= \|f\|_{s+|\alpha|}.
\end{aligned}$$

Thus establishing Inequality (13) for any s.

We first notice that for $s = 0$ we have

$$\|\omega f\|_s = \|\omega f\| \le \|\omega\|_\infty \|f\|. \tag{19}$$

Using the derivative definition of the Sobolev norm we have

$$\begin{aligned}
\|\omega f\|_s^2 &= \sum_{|\alpha|\leq s} \|D^\alpha(\omega f)\|^2 \\
&\quad \text{(by the triangle inequality and Equation 24)} \\
&\leq \sum_{|\alpha|\leq s} \|\omega D^\alpha f\|^2 + \sum_{|\alpha|\leq s} \|D^\alpha(\omega f) - \omega D^\alpha f\|^2 \\
&\quad \text{(since the second term only has derivatives of } f \\
&\quad \text{of order less than } s \text{ times functions with bounded} \\
&\quad \text{sup norm we can use Equation (19))} \\
&\leq \sum_{|\alpha|\leq s} \|\omega\|_\infty^2 \|D^\alpha f\|^2 + c \sum_{|\alpha|\leq s-1} \|D^\alpha f\|^2 \\
&= \|\omega\|_\infty^2 \|f\|_s^2 + c\|f\|_{s-1}^2 \\
&= (\|\omega\|_\infty \|f\|_s + c'\|f\|_{s-1})^2
\end{aligned}$$

Thus establishing Inequality (15) for non-negative integer values of s.
Add proof for all s.

Exercise 3.9. Prove Inequality (26).
Hint: this is essentially obvious from the other inequalities.

□

3.4. Properties of Sobolev spaces on T^n.

Theorem 3.10. (1) *For any smooth function f and number $s \leq t$ we have*

$$\|f\|_s \leq \|f\|_t.$$

Thus the identity map on $C^\infty(T^n, \mathbb{C}^m)$ induces a bounded inclusion

$$H_t(T^n) \subset H_s(T^n).$$

(2) *(Sobolev inequality) If $s > \frac{n}{2} + k$ then there is a constant c such that for any smooth function f we have*

$$\|f\|_{C^k} \leq c\|f\|_s.$$

Thus the identity map on $C^\infty(T^n, \mathbb{C}^m)$ induces a bounded inclusion

$$H_s(T^n) \subset C^k(T^n, \mathbb{C}^m),$$

that is any element in $H_s(T^n)$ has k continuous derivatives.

(3) *(Rellich's lemma) if $s < t$ then any sequence f_j in $H_t(T^n)$ for which $\|f_j\|_t \leq K$ for some fixed K, there is a subsequence that is Cauchy (and hence convergent) in $H_s(T^n)$. That is the bounded linear inclusion map*

$$H_t(T^n) \subset H_s(T^n)$$

is a compact operator.

Corollary 3.11.

$$\cap_s H_s(T^n) = C^\infty(T^n, \mathbb{C}^m).$$

This corollary is clear given that the smooth functions are in all Sobolev spaces.

Proof of Theorem 3.10. The first statement in the theorem is obvious.

For the Sobolev inequality we assume $\frac{n}{2} < s$ and note that since for a smooth function the Fourier series converges absolutely we have

$$\begin{aligned}
|f(x)|^2 &= |\sum_\xi f_\xi e^{ix\cdot\xi}|^2 \leq \sum_\xi |f_\xi|^2 \\
&= \sum_\xi (1+|\xi|^2)^{-s}(1+|\xi|^2)^s |f_\xi|^2 \\
&= \lim_{N\to\infty} \sum_{|\xi|\leq N} (1+|\xi|^2)^{-s}(1+|\xi|^2)^s |f_\xi|^2 \\
&\quad \text{(by Cauchy Schwarz inequality)} \\
&\leq \lim_{N\to\infty} \left(\sum_{|\xi|\leq N} (1+|\xi|^2)^{-s} \right) \left(\sum_{|\xi|\leq N} (1+|\xi|^2)^{s} |f_\xi|^2 \right) \\
&\quad \text{(since, if } s > \frac{n}{s}\text{, the two terms on the next line converge,} \\
&\quad \text{the second since } f \text{ is in } H_s \text{ and the second is checked below,} \\
&\quad \text{we have)} \\
&\leq \left(\lim_{N\to\infty} \sum_{|\xi|\leq N} (1+|\xi|^2)^{-s} \right) \left(\lim_{N\to\infty} \sum_{|\xi|\leq N} (1+|\xi|^2)^{s} |f_\xi|^2 \right) \\
&\quad \text{(setting } c = \sum_\xi (1+|\xi|^2)^{-s}\text{))} \\
&= c\|f\|_s.
\end{aligned}$$

Thus

$$\|f\|_{C^0} \leq c\|f\|_s$$

and we have an embedding of $C^0(T^n)$ into $H_s(T^n)$, if $s > \frac{n}{2}$. Now if $s > \frac{n}{2} + k$ then for $|\alpha| \leq k$ we have $D^\alpha f \in H_{s-|\alpha|}(T^n)$ and $s - |\alpha| \geq s - k > \frac{n}{2}$ and thus Inequality (13) gives

$$\|D^\alpha f\|_{C^0} \leq \|D^\alpha f\|_{s-|\alpha|} \leq \|f\|_s.$$

We can hence conclude that there is a c such that

$$\|f\|_{C^k} \leq c\|f\|_s.$$

We are left to check our claim that $\sum_\xi (1+|\xi|^2)^{-s}$ is uniformly convergent if $s > \frac{n}{2}$. To this end set

$$S_j = \{\xi = (\xi_1, \ldots, \xi_n) : \max |\xi_l| \leq j\}.$$

Since for any $\xi \in S_j$ there is one ξ_l equal to $\pm j$ and the other $n-1$ terms are between $-j$ and j we see

$$|S_j| \leq 2n(2j+1)^{n-1}.$$

In addition $\xi \in S_j$ implies $|\xi|^2 \geq j^2$. Thus we have

$$\begin{aligned} s_j &= \sum_{\xi \in S_j} \left(\frac{1}{1+|\xi|^2} \right)^2 \\ &\leq \frac{2n(2j+1)^{n-1}}{(1+j^2)^s} \leq \frac{2n(2j+1)^{n-1}}{j^{2s}} \\ &\leq \frac{2n(3j)^{n-1}}{j^{2s}} \leq cj^{n-1-2s}. \end{aligned}$$

So we have

$$\sum_{\xi} \frac{1}{(1+|\xi|^2)^s} = 1 + \sum_{j=1}^{\infty} s_j \leq 1 + c \sum_{j=1}^{\infty} j^{n-1-2s}.$$

The last sequence converges if $1 + 2s - n > 1$.

Now for Rellich's lemma. Let $\{f^j\}$ be a sequence in $H_s(T^n)$ such that

$$\|f^j\|_s \leq C.$$

For each fixed ξ we let

$$u^j_\xi = (1+|\xi|^2)^{\frac{t}{2}} f^i_\xi$$

and notice that $|u^j_\xi|^2 = u^j_\xi \cdot \overline{u}^j_\xi = (1+|\xi|^2)^t |f^j_\xi|^2 \leq \|f\|^2_t \leq \|f\|^2_s < C^2$. Thus for a fixed ξ we have a bounded sequence $\{u^j_\xi\}$ in C^m which of course must have a convergent subsequence. Ordering the ξ's we can use the Cantor diagonalization argument to find a subsequence of the f^j's, which we still denote f^j, so that the corresponding u^j_ξ's converge for each fixed ξ. We now claim that $\{f^j\}$ is a Cauchy sequence in $H_t(T^n)$. To see this notice that

$$\begin{aligned} \|f^j - f^k\|^2_s = &\sum_{|\xi|<N} (1+|\xi|^2)^{s-t} (1+|\xi|^2)^t |f^j_\xi - f^k_\xi|^2 \\ &+ \sum_{|\xi|\geq N} (1+|\xi|^2)^{s-t} (1+|\xi|^2)^t |f^j_\xi - f^k_\xi|^2 \end{aligned}$$

We call the first term on the left I and the second term II. Since $s - t < 0$ we have the following bound on II

$$\begin{aligned} II &\leq \sum_{|\xi|\geq N} (1+N^2)^{s-t} (1+|\xi|^2)^t |f^j_\xi - f^k_\xi|^2 \\ &= (1+N^2)^{s-t} \sum_{\xi} (1+|\xi|^2)^t |f^j_\xi - f^k_\xi|^2 \\ &= (1+N^2)^{s-t} \|f^j - f^k\|^2_t \\ &\qquad \text{(by the triangle inequality and Equation 24)} \\ &\leq c(1+N^2)^{s-t} (\|f^j\|^2_t + \|f^k\|^2_t) \leq (1+N^2)^{s-t} cC^2. \end{aligned}$$

Thus we can choose N sufficiently large so that $II \leq \frac{\epsilon}{2}$ for any pre assigned ϵ. To bound I notice that

$$(1+|\xi|^2)^{s-t} \leq 1$$

and thus

$$I \leq \sum_{|\xi|<N} (1+|\xi|^2)^t |f^j_\xi - f^k_\xi|^2.$$

Now since for each fixed ξ the sequence $\{u^j_\xi = (1+|\xi|^2)^{\frac{t}{2}} f^i_\xi\}$ is Cauchy, and since there are only finitely many ξ with $|\xi| < n$ we see there is some N' such that for all $j,k > N'$ we have

$$|u^j_\xi - u^k_\xi|^2 < \frac{\epsilon}{2l}$$

where l is the number of ξ with $|\xi| < n$. We now have

$$I \leq \sum_{|\xi|<N} (1+|\xi|^2)^t |f^j_\xi - f^k_\xi|^2 = \sum_{|\xi|<N} |u^j_\xi - u^k_\xi|^2 \leq \frac{\epsilon}{2l} l = \frac{\epsilon}{2}.$$

Combining this with the above estimate shows that given ϵ there is an N' such that for $j,k > N'$ we have

$$\|f^j - f^k\|_s \leq \epsilon$$

and thus $\{f^j\}$ is Cauchy in the s-norm. □

Here are several useful inequalities.

Lemma 3.12. (1) *(Peter-Paul inequality) Let $t' < t < t''$ then for any $\epsilon > 0$ there is a constant c (depending on ϵ, t, t' and t'') such that*

$$\|f\|_t \leq \epsilon \|f\|_{t''} + c\|f\|_{t'}, \tag{20}$$

for all $f \in H_{t''}(T^n)$.

(2) *For any function $\omega : T^n \to \mathbb{C}$ and functions $f, g \in H_s(T^n)$ we have*

$$|\langle \omega f, g\rangle_s - \langle f, \omega g\rangle| \leq c(\|f\|_s \|g\|_{s-1} + \|f\|_{s-1}\|g\|_s). \tag{21}$$

(3) *There is a constant c (which happens to be $\frac{1}{2}$) such that for any numbers a, b we have*

$$|ab| \leq c(|a|^2 + |b|^2). \tag{22}$$

Moreover, given an ϵ there is a constant c such that

$$|ab| \leq \epsilon |a|^2 + c|b|^2. \tag{23}$$

(4) *There is a constant c such that*

$$c(\sum_{j=1}^N |a_j|)^2 \leq \sum_{j=1}^N |a_j|^2 \leq (\sum_{j=1}^N |a_j|)^2 \tag{24}$$

Proof. To prove the Peter-Paul inequality notice that if $y \neq 0$ then y or $\frac{1}{y} > 1$. Thus since $t'' - t$ and $t - t'$ are greater than 0 we have

$$1 \leq (y)^{t''-t} + \left(\frac{1}{y}\right)^{t-t'}.$$

If we set $y = \epsilon^{\frac{1}{t''-t}}(1+|\xi|^2)$ then

$$1 \leq \epsilon(1+|\xi|^2)^{t''-t} + \frac{1}{\epsilon^{\frac{t-t'}{t''-t}}(1-|\xi|^2)^{t-t'}}$$

and thus

$$(1+|\xi|^2)^t \leq \epsilon(1+|\xi|^2)^{t''} + \epsilon^{\frac{t''-t}{t-t'}}(1+|\xi|^2)^{t'}.$$

Multiplying both sides by $|f_\xi|^2$ and summing over ξ gives the desired inequality with $c = \epsilon^{\frac{t''-t}{t-t'}}$.

For Inequality (21) we first notice that if $s = 0$ then we clearly have

$$\langle \omega f, g\rangle - \langle f, \overline{\omega} g\rangle = \frac{1}{(2\pi)^n}\int_{T^n} (\omega f \cdot g - f \cdot (\overline{\omega} g)) = 0.$$

Now for $s = 1$ we have

$$\begin{aligned}
|\langle \omega f, g\rangle_1 - \langle f, \overline{\omega} g\rangle_1| &= \frac{1}{(s\pi)^n}\int_{T^n} \\
&= \frac{1}{(2\pi)^n}\Big|\Big(\int_{T^n}\omega f\cdot g + \sum_j \int_{T^n}(D_j\omega f)\cdot D_j g \\
&\qquad - \int_{T^n}\omega f\cdot g - \sum_j\int_{T^n} D_j f\cdot D_j(\overline{\omega} g)\Big)\Big| \\
&\quad \text{(the second and third terms cancel)} \\
&= \frac{1}{(2\pi)^n}\Big|\Big(\sum_j\Big[\int_{T^n}\omega D_j f\cdot D_j g - \int_{T^n} D_j f\cdot \overline{\omega} D_j g \\
&\qquad + \int_{T^n}(D_j\omega) f\cdot D_j g + \int_{T^n} D_j f\cdot (D_j\overline{\omega}) g\Big]\Big)\Big| \\
&\quad \text{(the first and second terms cancel)} \\
&\le \frac{1}{(2\pi)^n}\Big(\Big|\int_{T^n}(D_j\omega) f\cdot D_j g\Big| + \Big|\int_{T^n} D_j f\cdot (D_j\overline{\omega}) g\Big|\Big) \\
&\quad \text{(the Cauchy-Schwarz inequality gives)} \\
&\le \sum_j\Big(\|(D_j\omega) f\|\|D_j g\| + \|D_j f\|\|(D_j\overline{\omega}) g\|\Big) \\
&\quad \text{(Inequality (15) gives)} \\
&\le \sum_j\Big(c\|f\|\|D_j g\| + c'\|D_j f\|\|g\|\Big) \\
&= c\|f\|\sum_j\|D_j g\| + c'\|g\|\sum_j\|D_j f\| \\
&\le c\|f\|\|g\|_1 + c'\|f\|_1\|g\| \le c''(\|f\|\|g\|_1 + \|f\|_1\|g\|).
\end{aligned}$$

This establishes the inequality for $s = 1$.

Exercise 3.13. Prove the inequality for general s.

(In Section **??** we will discuss a "trick" for proving this inequality, but hopefully it is clear that one should expect the inequality to be true from the above computation.)

Inequality 24 follows from Inequality (22) which in turn clearly follows form Inequality (23). For This inequality notice that for a real number $c \neq 0$ we have

$$0 \le (c|a| - \frac{1}{2c}|b|)^2 = c^2|a|^2 - |ab| + \frac{1}{4c^2}|b|^2$$

so

$$|ab| \le c^2|a|^2 + \frac{1}{4c^2}|b|^2.$$

Thus the inequality follows if we set $c = \sqrt{\epsilon}$. $\square$

3.5. **Sobolev spaces on a compact manifold** M**.** We now define the Sobolev spaces of sections of a bundle over a general manifold. To this end let $p : E \to M$ be a vector bundle of an n-manifold with fiber dimension m. Let $\Gamma(E)$ denote the smooth sections of E. There are several ways to define Sobolev spaces, the most elegant way involves using connections on a bundle, but the "simplest" way is as follows. Let $\{V_j\}$ be a finite cover of M by coordinate charts homeomorphic to subsets U_j of T^n such that $\overline{U_j}$ is an n-ball embedded in T· Let

$$\phi_j : U_j \to V_j$$

be the inverse of the coordinate chart map and

$$\pi_j : \phi_j^* E|_{V_j} \to \mathbb{R}^m$$

be the projection to $\mathbb{R}^m$ of $U_i \times R^m$ after a fixed identification

$$\phi_j^* E|_{V_j} \cong U_i \times \mathbb{R}^m.$$

Now choose a partition of unity $\{\rho_j\}$ subordinate to $\{V_j\}$. Suppose σ is a section of E. Notice that $\rho_j \sigma$ is a section with support in V_j and $\pi_j \circ (\phi_j^*(\rho_i \sigma))$ is a function on U_j with support on the interior of U_j. Thus we can extend it by 0 to a function $T^n \to \mathbb{R}^m$ and measure

$$\|\pi_j \circ (\phi_j^*(\rho_i \sigma))\|_s$$

using any of the definitions of the Sobolev s-norm for functions on T^n. We now define

$$\|\sigma\|_s = \left(\sum_j \|\pi_j \circ (\phi_j^*(\rho_i \sigma))\|_s^2 \right)^{\frac{1}{2}}. \tag{25}$$

First note that this defines a norm on $\Gamma(E)$. The only non-tivial thing to check is the triangle inequality.

Example 3.14. Prove the triangle inequality.

We now define the **Sobolev space of sections of** E to be the completion of $\Gamma(E)$ using the norm $\|\cdot\|_s$, and denote it $H_s(E)$. For most purposes we can use the above definition and not worry about the fact that the norm depends on lots of data, specifically $\{U_j, V_j, \phi_j, \rho_j, \pi_j\}$. But it is nice to know that this dependence is not important in the following sense.

Lemma 3.15. *For a compact manifold M the Sobolev s-norm on E is independent of the choices made up to equivalence. That is if $\|\cdot\|_s$ is defined using $\{U_j, V_j, \phi_j, \rho_j, \pi_j\}$ and $\|\cdot\|'$ is defined using $\{U'_j, V'_j, \phi'_j, \rho'_j, \pi'_j\}$ then there are constants c, c' such that*

$$c\|\sigma\|_s \le \|\sigma\|'_s \le c'\|\sigma\|_s$$

and hence the topologies induced on $\Gamma(E)$ are the same as are the completions with respect to these norms.

Proof. We first note that if $\mu : M \to \mathbb{C}$ is a smooth function then

$$\begin{aligned}
\|\pi_j \circ (\phi_j^*(\rho_j \mu \sigma)\|_s &= \|\pi_j \circ ((\mu \circ \phi_j)(\phi_j^*(\rho_j \sigma)))\|_s \\
&\qquad \text{(since } \pi_j \text{ is linear on each fiber)} \\
&= \|(\mu \circ \phi_j)(\pi_j \circ (\phi_j^*(\rho_j \sigma)))\|_s \\
&\qquad \text{(by Theorem 3.8)} \\
&\le c\|\pi_j \circ (\phi_j^*(\rho_i \sigma))\|_s.
\end{aligned}$$

We also notice that if $\phi : U \to U'$ is a diffeomorphism between two open subsets of T^n and $f : U' \to \mathbb{C}^m$ then we have

$$\begin{aligned}
\|\phi^* f\|_s^2 &= \sum_{|\alpha|\leq s} \|D^\alpha \phi^* f\|^2 \\
&\quad \text{(by the chain rule and product rule)} \\
&= \sum_{|\alpha|\leq s} (\text{derivatives of the map } \phi)\|\phi^* D^\alpha f\| \\
&\quad \text{(by Theorem 3.8, where } K \text{ is the largest constants associ-} \\
&\quad \text{ated to the sup norms of the derivatives of } \phi) \\
&\leq K \sum_{|\alpha|\leq s} \|\phi^* D^\alpha f\|_s^2 \\
&\quad \text{(by the independence of integrals on coordinates)} \\
&= K \sum_{|\alpha|\leq s} \|D^\alpha f\|_s^2 \\
&= K\|f\|_s^2.
\end{aligned}$$

We can similarly prove that there is a K' such that $\|f\|_s \leq K'\|\phi^* f\|_s$ (just apply the above inequality to $(\phi^{-1})^* f$).

Example 3.16. If E is a bundle over a contractible set $U \subset T^n$ and π and π' are projections of $E|_U$ to $\mathbb{R}^m$ using two different trivializations of $E|_U$ then there are constants c and c' such that

$$c\|\pi \circ \sigma\|_s \leq \|\pi' \circ \sigma\|_s \leq c'\|\pi \circ \sigma\|_s.$$

Hint: This is more or less just like the last estimate proven above.

Now we have

$$\begin{aligned}
\|\sigma\|_s^2 &= \sum_j \|\pi_j \circ (\phi_j^*(\rho_j \sigma))\|_s \\
&= \sum_j \|\pi_j \circ (\phi_j^*((\sum_{j'} \rho'_{j'})\rho_j \sigma))\|_s \\
&\leq \sum_{j,j'} \|\pi_j \circ (\phi_j^*(\rho'_{j'}\rho_j \sigma))\|_s \\
&\quad \text{(by the exercise above)} \\
&\leq c \sum_{j,j'} \|\pi'_j \circ (\phi_j^*(\rho'_{j'}\rho_j \sigma))\|_s \\
&\quad \text{(by the second estimate proved above)} \\
&\leq c' \sum_{j,j'} \|((\phi'_{j'} \circ \phi_j^{-1})^*(\pi'_j \circ (\phi_j^*(\rho'_{j'}\rho_j \sigma)))\|_s \\
&\quad \text{(Exercise: check that pull back commutes with a fixed } \pi'_j) \\
&= c' \sum_{j,j'} \|\pi'_{j'} \circ ((\phi'_{j'})^*(\rho'_{j'}\rho_j \sigma))\|_s \\
&\quad \text{(by the first estimate proved above)}
\end{aligned}$$

$$\leq c'' \sum_{j'} \|\pi'_{j'} \circ ((\phi'_{j'})^*(\rho'_{j'}\sigma))\|_s = c''(\|\sigma\|'_s)^2$$

We note that in all the constants above we were constantly taking the largest constant involved in the all the different norms in the sum. It is because we have a finite number of terms in the sums that this is a finite fixed constant. Of course the other inequality is similar. □

We now generalize all the results for Sobolev spaces to an arbitrary closed manifold.

Theorem 3.17. *For a closed manifold M and bundles E and F over M we have the following.*

(1) *For any smooth section $\sigma \in \Gamma(E)$ and number $s \leq t$ we have*
$$\|\sigma\|_s \leq \|\sigma\|_t.$$
Thus the identity map on $\Gamma(E)$ induces a bounded inclusion
$$H_t(E) \subset H_s(E).$$

(2) *For all s and linear differential operator of order k, $L : \Gamma(E) \to \Gamma(F)$ there is a constant c such that*
$$\|L\sigma\|_s \leq c\|\sigma\|_{s+k} \tag{26}$$
and thus extends to a bounded linear operator
$$L : H_{s+k}(E) \to H_s(F).$$

(3) *(Sobolev inequality) If $s > \frac{n}{2} + k$ then there is a constant c such that for any smooth section $\sigma \in \Gamma(E)$ we have*
$$\|\sigma\|_{C^k} \leq c\|\sigma\|_s.$$
Thus the identity map on $\Gamma(E)$ induces a bounded inclusion
$$H_s(E) \subset C^k(E),$$
that is any element in $H_s(E)$ has k continuous derivatives.

(4) *(Rellich's lemma) if $s < t$ then any sequence σ_j in $H_t(E)$ for which $\|\sigma_j\|_t \leq K$ for some fixed K, there is a subsequence that is Cauchy (and hence convergent) in $H_s(E)$. That is the bounded linear inclusion map*
$$H_t(E) \subset H_s(E)$$
is a compact operator.

(5) *(Peter-Paul inequality) Let $t' < t < t''$ then for any $\epsilon > 0$ there is a constant c (depending on ϵ, t, t' and t'') such that*
$$\|\sigma\|_t \leq \epsilon\|\sigma\|_{t''} + c\|\sigma\|_{t'}, \tag{27}$$
for all $\sigma \in H_{t''}(E)$.

Proof. For the first statement just fix the data $\{U_j, V_j, \phi_j, \rho_j, \pi_j\}$ used to define the Sobolev norm and note
$$\|\sigma\|_s^2 = \sum_j \|\pi_j \circ (\phi_j^*(\rho_j\sigma)\|_s^2 \leq c \sum_j \|\pi_j \circ (\phi_j^*(\rho_j\sigma)\|_t^2 = c\|\sigma\|_t^2$$
where the inequality comes form the inequality proved on T^n.

For the second statement we have

$$\begin{aligned}
\|L\sigma\|_s^2 &= \|L\sum_j \rho_j\sigma\|_s^2 \\
&\leq \sum_j \|L(\rho_j\sigma)\|_s^2 \\
&\quad \text{(Exercise: applying } \pi_j\circ\phi_j^* \text{ to the } s\text{-norm can be estimated since the terms integrated will only involved fixed derivatives of } \phi_j^*\phi_j \text{ and } \rho_j\text{)} \\
&\leq c\sum_j \|\pi_j\circ\phi_j^*(L(\rho_j\sigma))\|_s^2 \\
&\quad \text{(there is an operator } L'_j \text{ of the same order as } L \text{ such that } (\pi_j\circ\phi_j^*)L = L'_j(\pi_j\circ\phi_j)\text{)} \\
&= c\sum_j \|L'_j(\pi_j\circ\phi_j^*(\rho_j\sigma))\|_s^2 \\
&\quad \text{(by Theorem 3.8)} \\
&\leq c'\sum_j \|\pi_j\circ\phi_j^*(\rho_j\sigma)\|_{s+k}^2 = c'\|\sigma\|_{s+k}^2
\end{aligned}$$

For the Sobolev inequality notice that we just need a point wise bound on σ (and its derivatives, by the way, what is the C^k norm on sections of E!).

Exercise 3.18. Given an inner product, and hence norm $|\cdot|_E$, on E we can estimate $|\rho_j(x)\sigma(x)|_E$ by the Euclidean norm on $\mathbb{R}^n$, $|\pi_j\circ\phi_j^*(\rho_j(x)\sigma(x))|$.

In turn, if $s > \frac{n}{2}$, this can be estimated, using the Sobolev inequality in T^n, by $\|\pi_j\circ\phi_j^*(\rho_j\sigma)\|_s$ which is clearly less than $\|\sigma\|_s$. Thus we have

$$|\sigma(x)|_E = |\sum_j \rho_j(x)\sigma(x)| \leq \sum_j |\rho(x)\sigma(x)| \leq c\|\sigma\|_s$$

and hence $\|\sigma\|_{C^0} \leq c\|\sigma\|_s$. Similarly we can get $\|\sigma\|_{C^k} \leq c\|\sigma\|_s$, if $s > \frac{n}{s} + k$.

Rellich's lemma easily follows since there are a finite number of charts V_j in the cover of M. Indeed, suppose we have a sequence σ_n and a K such that $\|\sigma_n\|_t \leq K$ for all n. Then on T^n we certainly have

$$\|\pi_j\circ\phi_j^*(\rho_j\sigma_n)\|_t \leq K$$

for all n (and j). Thus Rellich's lemma on T^n implies there is a subsequence of the σ_n's such that $\pi_j\circ\phi_j^*(\rho_j\sigma_n)$ is Cauchy in the Sovolev s-norm for each j. Thus for each $\epsilon > 0$ there is an N such that for $n, m > N$ we have

$$\begin{aligned}
\|\sigma_n - \sigma_m\|_s^2 &= \sum_j \|\pi_j\circ\phi_j^*(\rho_j(\sigma_n - \sigma_m))\|_s^2 \\
&= \sum_j \|\pi_j\circ\phi_j^*(\rho_j\sigma_n) - \pi_j\circ\phi_j^*(\rho_j\sigma_m))\|_s^2 \leq \epsilon.
\end{aligned}$$

Thus the sequence σ_n is Cauchy in the Sobolev s-norm.

Exercise 3.19. Generalize the Peter-Paul inequality from T^n to M.

□

4. Elliptic Operators

4.1. **Elliptic Operators on T^n.** Recall a differential operator L of order k on $C^\infty(T^n, \mathbb{C}^m)$ is an operator of the form
$$L = P_k(D) + \ldots + P_0(D)$$
where $P_j(D)$ is an $m \times m$ matrix with entries $\sum_{|\alpha|=j} a_\alpha(x) D^\alpha$ where $a_\alpha(x)$ is a smooth function on T^n.

Example 4.1. For $n = 2 = m$ and $k = 1$ we have $L = P_1(D) + P_0(D)$ where
$$P_1(D) = \begin{bmatrix} x^2 \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} & \frac{\partial}{\partial y} \\ \cos y \frac{\partial}{\partial y} & e^x \frac{\partial}{\partial y} \end{bmatrix}$$
and
$$P_0(D) = \begin{bmatrix} x^2 \sin y & 5 \\ 6x^3 + y^2 & e^x \end{bmatrix}.$$

Example 4.2. For $m = 1$ and $k = 2$ we have $L' = P_2(D)$ where
$$P_2(D) = \sum_{j=1}^{n} \left(\frac{\partial}{\partial x^j} \right)^2.$$

The **symbol** of a differential operator$L$ of order k, denoted $\sigma(L)$, is the function
$$\mathbb{R}^n \to M_m(C^\infty(T^n, \mathbb{C}))$$
that takes an element $\xi \in \mathbb{R}^n$ and returns the matrix $P_k(\xi)$ with entries from $C^\infty(T^n, \mathbb{R})$.

Example 4.3. In the first example above for $\xi = (\xi^1, \xi^2)$ we have
$$\sigma(L)(\xi) = P_1(\xi) = \begin{bmatrix} x^2\xi^1 + x\xi^2 & \xi^2 \\ (\cos y)\,\xi^2 & e^x \xi^2 \end{bmatrix}.$$

Example 4.4. In the second example above we have
$$\sigma(L)(\xi) = P_2(\xi) = \sum_{j=1}^{n} (\xi^j)^2.$$

We call L **elliptic** at $x \in T^n$ if for all $\xi \in \mathbb{R}^n$ with $\xi \neq 0$ the matrix $\sigma(L)(\xi)(x)$ is invertible. We say L is **elliptic** if it is elliptic at each point $x \in T^n$. There are two fundamental theorems about elliptic operators.

Theorem 4.5 (Gärding inequality on T^n)**.** *If L is an elliptic operator on $C^\infty(T^n, \mathbb{C}^m)$ of order k then for any integer s we can extend L to $H_{s+k}(T^n)$ such that there is a contact c for which*
$$\|f\|_{s+k} \leq c(\|Lf\|_s + \|f\|_s)$$
for all $f \in H_{s+k}(T^n)$.

Notice that we already know that
$$\|Lf\|_s \leq c\|f\|_{s+k}$$
and of course
$$\|f\|_s \leq \|f\|_{s+k}$$
so we trivially have
$$\|Lf\|_s + \|f\|_s \leq c'\|f\|_{s+k}.$$

In other words the Gärding inequality basically says that the norm $\|\cdot\|_{s+k}$ is equivalent to the norm $\|L\cdot\|_s + \|\cdot\|_s$. That is, controlling Lf is as good as controlling k derivatives of f.

Theorem 4.6 (Elliptic regularity on T^n)**.** *If L is an elliptic operator on $C^\infty(T^n, \mathbb{C}^m)$ of order k then given $u \in H_0(T^n)$ and $v \in H_t(T^n)$ such that*

$$Lu = v$$

we have

$$u \in H_{t+k}(T^n).$$

Recall, that the bigger t for which u sits in $H_t(T^n)$ the smoother u is. So this theorem says that elliptic operators can improve the regularity, or smoothness, of solutoins. Thus if v is smooth then any solution to $Lu = v$ is smooth (since $u \in \cap_s H_s(T^n) = C^\infty(T^n, \mathbb{C}^m)$).

Proof of Gärding's inequality on T^n. There are three steps in the proof. We first will easily show that if L is a contact coefficient operator of order k (and the $P_j(D) = 0$ for $j < k$) then the inequality is true. Then we show that for functions supported near a point (thus the coefficients of $P_k(D)$ only vary a little on the support of the function) then the inequality is true and finally we prove the general version of the inequality.

Step 1: Inequality when $L = P_k(D)$ and the entries of of $P_k(D)$ are of the form $\sum_{|\alpha|=j} a_\alpha D^\alpha$ where the a_α are constants. We are in the situation wher $P_k(\xi) \in M_m(\mathbb{C})$ and by ellipticity this $m \times m$ matrix is invertible. Since the unit sphere in R^m is compact we see that there is some constant $c > 0$ such that

$$|P_k(\xi)u|^2 \geq c$$

for all $|\xi| = 1 = |u|$. Thus for any $\xi \neq 0 \neq u$ we have

$$\left| P_k\left(\frac{\xi}{|\xi|}\right)\frac{u}{|u|}\right|^2 \geq c$$

and since the entries in P_k are homogeneous of order k in the ξ variables we have

$$|\frac{1}{|\xi|}P_k(\xi)\frac{u}{|u|}|^2 \geq c.$$

Thus

$$|P_k(\xi)u|^2 \geq c|\xi|^{2k}|u|^2$$

(note this also holds for $\xi = 0$ or $u = 0$). Thus for all $f \in C^\infty(T^n, \mathbb{C}^m)$ we have

$$\|Lf\|_s^2 = \sum_\xi (1+|\xi|^2)^s |P_k(\xi) f_\xi|^2 \geq c \sum_\xi (1+|\xi|^2)^s |\xi|^{2k} |f_\xi|^2.$$

So

$$\begin{aligned}(\|Lf\|_s + \|f\|_s)^2 &\geq \|Lf\|_s^2 + \|f\|_s^2 \\ &\geq \sum_\xi (1+|\xi|^2)^s (c|\xi|^{2k} + 1)|f_\xi|^2 \\ &\qquad \text{(taking } c' \text{ smaller than } c \text{ or } 1) \\ &\geq c' \sum_\xi (1+|\xi|^2)^s (|\xi|^{2k} + 1)|f_\xi|^2\end{aligned}$$

Notice that since if $\xi \neq 0$ then $|\xi|^2 \geq 1$ we have

$$\begin{aligned}(1+|\xi|^2)^k &= \sum \binom{k}{l} |\xi|^{2l} \\ &\leq \sum \binom{k}{l} |\xi|^{2k} \leq 1 + k|\xi|^{2k} \\ &\qquad \text{(taking } k' \text{ greater than 1 or } k) \\ &\leq k'(1+|\xi|^{2k})\end{aligned}$$

Combining the last two estimates we have some constant c'' such that

$$\begin{aligned}(\|Lf\|_s + \|f\|_s)^2 &\geq c' \sum_\xi (1+|\xi|^2)^s (|\xi|^{2k}+1)|f_\xi|^2 \\ &\geq c'' \sum_\xi (1+|\xi|^2)^s (|\xi|^2+1)^k |f_\xi|^2 \\ &= c'' \sum_\xi (1+|\xi|^2)^{k+s} |f_\xi|^2 = c'' \|f\|_{s+k}^2.\end{aligned}$$

Thus using Inequality (24) we have

$$c'''(\|Lf\|_s + \|f\|_s) \geq \|f\|_{s+k}.$$

Step 2: Inequality for a general L but with "locally" supported functions. Specifically we show that for each point $x \in T^n$ there is an open set U such that for functions f with support in U the inequality is true. Set $P_k^0(\xi) = P_k(\xi)(p)$ and L_0 the order k differential operator with constant coefficients $P_k^0(D)$. From Step 1 we know there is a constant c such that

$$\|f\|_{s+k} \leq c(\|L_0 f\|_s + \|f\|_s) \leq c(\|Lf\|_s + \|(L_0 - L)f\|_s + \|f\|_s).$$

Of course we need to control the second term on the right. To that end there is a neighborhood U' of x such that the sup norm of the first s derivatives of the coefficients of $L_0 - L$ in U' are less than or equal to ϵ. Now define the operator L' to be $L_0 - L$ on a subset U of U', such that $\overline{U} \subset U'$, to be 0 on the complement of U' and any interpolating operator in between. Thus if f has support in U then immediately from the inequality above we have

$$\begin{aligned}\|f\|_{s+k} &\leq c(\|Lf\|_s + \|L'f\|_s + \|f\|_s) \\ &\qquad \text{(by the standard estimate for } k^{\text{th}} \text{ order operators, Inequality 26)} \\ &\leq c(\|Lf\|_s + c'\|f\|_{s+k} + \|f\|_s) \\ &\qquad \text{(by the Peter-Paul Inequality with } \epsilon = \frac{1}{cc'2}) \\ &\leq c(\|Lf\|_s + \frac{1}{2c}\|f\|_{s+k} + c''\|f\|_{s+k-1} + \|f\|_s) \\ &\qquad \text{(by the Peter-Paul Inequality with } \epsilon = \frac{1}{cc''4}) \\ &\leq c(\|Lf\|_s + \frac{1}{2c}\|f\|_{s+k} + \frac{1}{4c}\|f\|_{s+k} + c'''\|f\|_s + \|f\|_s) \\ &= \frac{3}{4}\|f\|_{s+k} + c(\|Lf\|_s + (c'''+1)\|f\|_s)\end{aligned}$$

$$\leq \frac{3}{4}\|f\|_{s+k} + k(\|Lf\|_s + \|f\|_s).$$

Moving the first term to the left hand side we and multiplying by 4 we have the desired inequality

$$\|f\|_{s+k} \leq k'(\|Lf\|_s + \|f\|_s).$$

Step 3: Inequality in the general case. We will need a technical lemma.

Lemma 4.7. *For smooth functions $f : T^n \to \mathbb{C}^n$ and $u : T^n \to C$ and a differential operator of order k there is a constant c such that*

$$|\langle Lu^2 f, Lf\rangle_s - \langle Luf, Luf\rangle_s| \leq c\|f\|_{s+k}\|f\|_{s+k-1}. \tag{28}$$

Proof. First we note that

$$\begin{aligned}|\langle Lu^2 f, Lf\rangle_s - \langle Luf, Luf\rangle_s| &\leq |\langle uLuf, Lf\rangle_s - \langle Luf, uLf\rangle_s| \\ &\quad + |\langle Luf, (uL - Lu)f\rangle_s| + |\langle (Lu - uL)uf, Lf\rangle_s|.\end{aligned}$$

Now the second term on the left is

$$\begin{aligned}|\langle Luf, (uL - Lu)f\rangle_s| &\leq c\|Luf\|_s\|(uL - Lu)f\|_s \\ &\quad \text{(since } uL - Lu \text{ is a differential operator of order} \\ &\quad \leq k-1 \text{ applying Inequality (26) twice)} \\ &\leq c\|uf\|_{s+k}\|f\|_{s+k-1} \leq c'\|f\|_{s+k}\|f\|_{s+k-1}.\end{aligned}$$

We can similarly bound the last term and the first term is appropriately bounded by Inequality (21) □

Let ρ_j be a partition of unity subordinate to the finite cover U_j of T^n, such that Step 2 applies to any function supported in U_j. Now wet $u_j = \sqrt{\rho_j}$.

$$\begin{aligned}\|f\|^2_{s+k} &= \langle f, f\rangle_{s+k} = \langle \sum u_j^2 f, f\rangle_{s+k} \\ &\quad \text{(by Inequality (21))} \\ &\leq \sum \langle u_j f, u_j f\rangle_{s+k} + c\|f\|_{s+k}\|f\|_{s+k-1} \\ &= \sum \|u_j f\|^2_{s+k} + c\|f\|_{s+k}\|f\|_{s+k-1} \\ &\quad \text{(by Step 2)} \\ &\leq \sum (\|Lu_j f\|^2_s + \|u_j f\|^2_s) + c\|f\|_{s+k}\|f\|_{s+k-1} \\ &\quad \text{(by Inequality (28) on the first term and Inequality (26)} \\ &\quad \text{on the second)} \\ &\leq c' \sum \langle Lu_j^2 f, Lf\rangle_s + c''\|f\|^2_s + c'''\|f\|_{s+k}\|f\|_{s+k-1} \\ &\quad \text{(by Inequality (23) on the last term with } \epsilon = \frac{1}{2c'''}) \\ &\leq c'\|Lf\|^2_s + c''\|f\|^2_s + \frac{1}{2}\|f\|^2_{s+k} + C\|f\|^2_{s+k-1} \\ &\quad \text{(by the Peter-Paul Inequality)} \\ &\leq c'\|Lf\|^2_s + c''\|f\|^2_s + \frac{3}{4}\|f\|^2_{s+k} + C'\|f\|^2_s\end{aligned}$$

Thus we have

$$\|f\|^2_{s+k} \leq C''(\|Lf\|^2_s + \|f\|^2_s)$$

and hence

$$\|f\|_{s+k} \leq C''(\|Lf\|_s + \|f\|_s).$$

□

We need some preliminary notions and results to prove elliptic regularity on T^n. In particular we need to understand when the derivative of a function in is a Sobolev space. To do this we work with difference quotients. Given $h \in \mathbb{R}^n$ we define

$$T_h : C^\infty(T^n, C^m) \to C^\infty(T^n, \mathbb{C}^m)$$

by

$$T_h(f)(x) = f(x+h).$$

Notice that

$$\begin{aligned}
(T_h(f))_\xi &= \frac{1}{(2\pi)^n} \int_{T^n} f(x+h) e^{-ix\cdot\xi}\, dx \\
&= \frac{1}{(2\pi)^n} \int_{T^n} f(x) e^{-i(x-h)\cdot\xi}\, dx \\
&= e^{ih\cdot x} \frac{1}{(2\pi)^n} \int_{T^n} f(x) e^{-ix\cdot\xi}\, dx \\
&= e^{ih\cdot x} f_\xi.
\end{aligned}$$

Thus if we define

$$f^h = \frac{f(x+h) - f(x)}{|h|} = \frac{T_h(f)(x) - f(x)}{|h|},$$

then

$$(f^h)_\xi = \left(\frac{e^{ih\cdot\xi} - 1}{|h|} \right) f_\xi.$$

We can use these formulas to define T_h and $(\cdot)^h$ on $H_s(T^n)$. (That is thinking of the Sobolev space as a subset of the sequences $l^2(\mathbb{Z}^n, \mathbb{C}^n)$ the operators are defined by the appropriate multiplication of the terms in the sequence.)

Lemma 4.8. *We have the following*

1. *If $f \in H_l(T^n)$ then $T_h(f) \in H_l(T^n)$ and*
$$\|T_h(f)\|_l = \|f\|_l.$$
2. *If $f \in H_{l+1}(T^n)$ then $f^h \in H_l(T^n)$ and there is a constant c such that*
$$\|f^h\|_l \leq c\|f\|_{l+1}.$$
3. *if $f \in H_l(T^n)$ and there is some c such that for all h sufficiently small*
$$\|f^h\|_l \leq c,$$
then $f \in H_{l+1}(T^n)$.

Proof. The first statement is obvious. For the second statement

$$\begin{aligned}
\|f^h\|_l &= \sum_\xi (1+|\xi|^2)^l \left| \left(\frac{e^{ih\cdot\xi}-1}{|h|}\right) f_\xi\right|^2 \\
&\qquad (\text{since } e^{ih\cdot\xi} = \cos h\cdot\xi + i\sin h\cdot\xi) \\
&= \sum_\xi (1+|\xi|^2)^l \left[\left(\frac{\cos ih\cdot\xi - 1}{|h|}\right)^2 + \left(\frac{\sin ih\cdot\xi}{|h|}\right)^2 \right] |f_\xi|^2 \\
&= \sum_\xi (1+|\xi|^2)^l \left(\frac{\cos^2 h\cdot\xi + \sin^2 h\cdot\xi - 2\cos h\cdot\xi + 1}{|h|^2}\right) |f_\xi|^2 \\
&= \sum_\xi (1+|\xi|^2)^l \left(\frac{2-2\cos h\cdot\xi}{|h|^2}\right) |f_\xi|^2 \\
&= \sum_\xi (1+|\xi|^2)^l \left(\frac{4\sin^2(\frac{1}{2}h\cdot\xi)}{|h|^2}\right) |f_\xi|^2 \\
&\leq \sum_\xi (1+|\xi|^2)^l \left(\frac{(h\cdot\xi)^2}{|h|^2}\right) |f_\xi|^2 = \sum_\xi (1+|\xi|^2)^l \left(\frac{h}{|h|}\cdot\xi\right)^2 |f_\xi|^2 \\
&\leq \sum_\xi (1+|\xi|^2)^l (1+|\xi|^2)|f_\xi|^2 = \|f\|_{l+1}.
\end{aligned}$$

For the last statement consider the truncated Fourier series

$$f_N = \sum_{|\xi|\leq N} f_\xi e^{ih\cdot\xi}.$$

If we can show that there is a c such that

$$\|f_N\|_{l+1} \leq c$$

for all N then clearly

$$\|f\|_{l+1} \leq c$$

and we are done. Now let $e_1, \ldots, e_n$ be the standard orthonormal basis for $\mathbb{R}^n$ and set $h = te_j$ for any j. Then for a fixed ξ we have

$$\left|\frac{e^{ih\cdot\xi}-1}{|h|}\right|^2 = \left|\frac{e^{ih\cdot\xi}-1}{t}\right|^2 \to |\xi_i|^2 \quad \text{as } t\to 0$$

since the Taylor expansion of the second term is $i\xi_j + \frac{1}{2}(i\xi_j)^2 t + \ldots$. So for any finite number of ξ's and a given $\epsilon > 0$ there is a $\delta > 0$ such that $t \leq \delta$,

$$\left| \left|\frac{e^{ih\cdot\xi}-1}{|h|}\right|^2 - |\xi_j|^2 \right| \leq \epsilon.$$

By hypothesis there is a $\delta' > 0$ such that for all $t \leq \delta'$,

$$\sum_{|\xi|\leq N} (1+|\xi|^2)^l \left|\left(\frac{e^{ih\cdot\xi}-1}{|h|}\right)\right|^2 |f_\xi|^2 \leq \sum_\xi (1+|\xi|^2)^l \left|\left(\frac{e^{ih\cdot\xi}-1}{|h|}\right)\right|^2 |f_\xi|^2 < k^2.$$

For appropriately chosen ϵ and $t \leq \min\{\delta, \delta'\}$ we have

$$\sum_{|\xi| \leq N} (1 + |\xi|^2)^l |\xi_j|^2 |f_\xi|^2 \leq k^2$$

and hence

$$\sum_{|\xi| \leq N} (1 + |\xi|^2)^l |\xi_|^2 |f_\xi|^2 \leq k^2 n.$$

Notice this bound is independent of N (of course for different N we might need to choose different δ). We now have

$$\begin{aligned}
\|f_N\|_{l+1} &= \sum_{|\xi| \leq N} (1 + |\xi|^2)^{l+1} |f_\xi|^2 \\
&= \sum_{|\xi| \leq N} (1 + |\xi|^2)^l |f_\xi|^2 + \sum_{|\xi} | \leq N (1 + |\xi|^2)^l |\xi|^2 |f_\xi|^2 \\
&\leq \sum_{\xi} (1 + |\xi|^2)^l |f_\xi|^2 + nk^2 \\
&\leq \|f\|_l + nk^2.
\end{aligned}$$

And as mentioned above this bounds $\|f\|_{l+1}$ so we have shown $f \in H_{l+1}(T^n)$. $\square$

Elliptic Regularity for T^n. The theorem follows if we can show that $u \in H_s(T^n)$ and $Lu = v \in H_{s-l+1}(T^n)$ then $u \in H_{s+1}(T^n)$. To this end supose $h \in \mathbb{R}^n$ and $h \neq 0$. Let L^h be the operator obtained from L by replacing all the coefficients $a(x)$ with $\frac{a(x+h)-a(x)}{h}$ and let $T_h(L)$ be the operator obtained by replacing the coefficients of L with $a(x+h)$. Notice

$$\begin{aligned}
\left[L(u^h) - (L(u))^h\right](x) &= L\left(\frac{u(x+h) - u(x)}{|h|}\right) \\
&\qquad - \frac{(T_h(L)(u))(x+h) - (Lu)(x))}{|h|} \\
&= \frac{L(u(x+h)) - (T_h L)(u(x+h))}{|h|} \\
&= -L^h(T_h(u)).
\end{aligned}$$

Thus using Gärding's inequality we have

$$\begin{aligned}
\|u^h\|_s &\leq c(\|Lu^h\|_{s-l} + \|u^h\|_{s-l}) \\
&\leq c'(\|(Lu)^h\|_{s-l} + \|L^h(T_h u)\|_{s-l} + \|u\|_{s-l+1}) \\
&\qquad \text{(since } L^h \text{ is a differential operator of order } l) \\
&\leq c'(\|(Lu)^h\|_{s-l} + \|T_h u\|_s + \|u\|_{s+l-1}) \\
&\leq c''(\|Lu\|_{s-l+1} + \|u\|_s + \|u\|_s) \\
&\leq c'''(\|v\|_{s-1+1} + \|u\|_s).
\end{aligned}$$

Thus $\|u^h\|_s$ is bounded independent of h so $u \in H_{s+1}(T^n)$. $\square$

4.2. **Elliptic Operators a general closed manifold.** Suppose E and F are bundles over a manifold M. An operator $L : \Gamma(E) \to \Gamma(F)$ is a **differential operator of order** k if in local coordinates (homeomorphic to a subset of T^n) it can be expressed as a differential operator on an open subset of T^n. We similarly call the operator **elliptic** if in local coordinate about each point $x \in M$ it is elliptic at x. This definition is somewhat cumbersome to work with in practice so we will reformulate it but first notice.

Exercise 4.9. Using the local expression for the Laplacian on functions given in Section 1.2 show that it is elliptic.

Suppose $L : C^\infty(T^n, \mathbb{C}^m) \to C^\infty(T^n, \mathbb{C}^m$ is a differential operator of order k on T^n. We claim that L is elliptic at x if and only if for all

$$L(\phi^k u) \neq 0$$

for all $u \in C^\infty(T^n, \mathbb{C}^m)$ with $u(x) \neq 0$ and $\phi : T^n \to \mathbb{R}$ with $\phi(x) = 0$ and $d\phi_x \neq 0$. To see this notice that

$$\begin{aligned} L(\phi^k u)(x) &= P_k(D)(\phi^k u)(x) \\ &\qquad + (\text{terms with} \leq k \text{ derivatives of } \phi^k \text{ at } x \text{ so they} = 0) \\ &= P_k(c\, d\phi_x) u(x) \end{aligned}$$

for some nonzero c. Thus by our choice of ϕ and u the operator L is elliptic if and only if the last term is nonzero for all suitable choices of ϕ and u. This criterion easily transverse to manifolds and bundles. In particular we see that $L : \Gamma(E) \to \Gamma(F)$ is a differential operator of order k then it is elliptic at x if and only if for all $\alpha \in \Gamma(E)$ with $\alpha(x) \neq 0$ and $\phi : M \to \mathbb{R}$ with $\phi(x) = 0$ and $d\phi_x \neq 0$ we have

$$L(\phi^k \alpha)(x) \neq 0.$$

Exercise 4.10. Show that the above construction yields, for each x, a well defined linear map

$$T_x^* M \to Hom(E_x, F_x)$$

thus we get a bundle map

$$T^* M \to Hom(E, F).$$

The bundle map constructed in the exercise is called the **symbol** of L and is denoted $\sigma(L)$. In terms of this symbol we can say L is elliptic if and only if $\sigma(L)_\xi$ is a isomorphism $E_x \to F_x$ (where ξ is a covector above x) for all $\xi \neq 0$. This is the definition of ellipticity that is easiest to deal with.

Theorem 4.11. *The Laplacian*

$$\Delta : \Omega^p(M) \to \Omega^p(M)$$

is an elliptic operator or order 2.

Proof. Clearly Δ is of order two. For ellipticity consider any $x \in M$ and any non-zero elements $v \in (\wedge_p^* M)_x$ and $\xi \in T_x^* M$. Let $\alpha \in \Omega^p(M)$ and $\phi : M \to \mathbb{R}$ be such that

$$\alpha(x) = v$$

and

$$\phi(x) = 0, \quad d\phi_x = \xi.$$

Now

$$\begin{aligned}
\delta d(\phi^2\alpha) &= (-1)^{np+1} * d * d(\phi^2\alpha) \\
&= (-1)^{np+1} * d * (2\phi d\phi \wedge \alpha + \phi^2 d\alpha) \\
&= (-1)^{np+1} * d(2\phi * (d\phi \wedge \alpha) + \phi^2 * d\alpha) \\
&= (-1)^{np+1}(2d\phi \wedge *(d\phi \wedge \alpha) + 2\phi d * (d\phi \wedge \alpha) \\
&\qquad + 2\phi d\phi \wedge *d\alpha + \phi^2 d * d\alpha).
\end{aligned}$$

Evaluating this at x we get

$$\delta d(\phi^2\alpha) = (-1)^{np+1}(2 * \xi \wedge (*(\xi \wedge v))).$$

Similarly

$$d\delta(\phi^2\alpha) = (-1)^{n(p+1)+1}(2\xi \wedge *(\xi \wedge *v))$$

and thus

$$\Delta(\phi^2\alpha) = (-1)^{np+1}(2 * \xi \wedge (*(\xi \wedge v))) + (-1)^{n(p+1)+1}(2\xi \wedge *(\xi \wedge *v)).$$

To see Δ is elliptic we need to see that this is nonzero. To this end consider the map

$$W_\xi : \wedge_k^*(M) \to \wedge_{k+1}^*(M)$$

given by

$$W_\xi(v) = \xi \wedge v.$$

We claim that the adjoint of W_ξ is

$$W_\xi^*(v) = (-1)^{nk} * \xi \wedge *v.$$

Thus we can write

$$\Delta(\phi^2\alpha) = (-1)^{n+1}2(W_\xi^* W_\xi + W_\xi W_\xi^*)v.$$

Below we will see that this is nonzero, but first let us check our claim about the adjoint of W_ξ. Indeed

$$\begin{aligned}
\langle W_\xi(v), w\rangle &= \langle \xi \wedge v, w\rangle \\
&\qquad \text{(using the formula for inner products on } \wedge_k(M)\text{, see Equation (1))} \\
&= *((\xi \wedge v) \wedge *w) = *(v \wedge (-1)^k(\xi \wedge *w)) \\
&= *(v \wedge * * (-1)^{k(n-k)}(-1)^k(\xi \wedge *w)) \\
&= *(v \wedge *[(-1)^{kn} * (\xi \wedge *w)]) \\
&= \langle v, (-1)^{kn} * (\xi \wedge *w)\rangle
\end{aligned}$$

and thus the adjoint is as claimed.

We now observe a simple linear algebra fact. Suppose U, V and W are vector spaces with inner products and

$$U \xrightarrow{A} V \xrightarrow{B} W$$

exact at V. Let A^* and B^* be the adjoints of A and B, respectively. We claim that

$$B^*B + AA^* : V \to V$$

is an isomorphism. Indeed suppose $v \neq 0$. If $(B^*B + AA^*)v = 0$ then

$$0 = \langle (B^*B + AA^*)v, v\rangle = \langle Bv, Bv\rangle + \langle A^*v, A^*v\rangle.$$

Thus $A^*v = 0$ and $Bv = 0$. Since $Bv = 0$ we have some u such that $Au = v$, but

$$\langle v, v\rangle = \langle Au, Au\rangle = \langle u, A^*Au\rangle = \langle u, A^*v\rangle = 0$$

which of course implies that $u = 0$ and hence $v = Au = 0$, a contradiction. Thus $B^*B + AA^*$ is an isomorphism as claimed.

Thus we will have completed our proof once we check that

$$\wedge^*_{k-1}(M) \stackrel{W_\xi}{\to} \wedge^*_k(M) \stackrel{W_\xi}{\to} \wedge^*_{k+1}(M)$$

is exact at $\wedge^*_k(M)$ for any $\xi \neq 0$. Clearly we have

$$W_\xi W_\xi v = \xi \wedge \xi \wedge v = 0.$$

So we are left to show that if $W_\xi v = 0$ then $v = W_\xi u$ for some $u \in \wedge^*_{k-1}(M)$. For this if $\xi \neq 0$ we choose a basis for T^*M, $e^1, \ldots, e^n$ where $e^1 = \xi$. Now

$$v = \sum a_{i_1,\ldots,i_k} e^{i_1} \wedge \ldots \wedge e^{i_k},$$

where we only consider $i_i < i_2 < \ldots < i_k$. Since $e^{i_1} \wedge \ldots \wedge e^{i_{k+1}}$, with $i_i < i_2 < \ldots < i_{k+1}$ is a basis for $\wedge^*_{k+1}(M)$ we see that if any of the terms in v does not have e^1 in it then $\xi \wedge v = e^1 \wedge v \neq 0$. There for if we set

$$u = \sum a_{i_1,\ldots,i_k} e^{i_2} \wedge \ldots \wedge e^{i_k},$$

then clearly $W_\xi(u) = v$. □

4.3. **Properties of Elliptic Operators a general closed manifold.**

Theorem 4.12. *If M is a closed manifold, E and F are a vector bundles over M with the same fiber dimension and L is an elliptic operator from $\Gamma(E)$ to $\Gamma(F)$ of order k then*

(1) *(Gärding inequality.) There is a constant c for which*

$$\|\alpha\|_{s+k} \leq c(\|L\alpha\|_s + \|\alpha\|_s)$$

for all $\alpha \in H_{s+k}(E)$.

(2) *(Elliptic regularity.) Given $u \in H_0(E)$ and $v \in H_t(F)$ such that*

$$Lu = v$$

we have

$$u \in H_{t+k}(E).$$