Symplectic Geometry

Given a manifold M

a symplectic structure on M is a 2-form $\omega \in \Omega^2(M)$

s.t. 1) ω is non-degenerate

(i.e. $\forall v \in T_xM$, $v \neq 0$, $\exists u \in T_xM$, s.t. $\omega_x(v,u) \neq 0$)

2) $d\omega = 0$

the study of symplectic manifolds (M,ω) grew out of classical mechanics, but now is a thriving area of study with interesting connections to (low-dimensional) topology, Riemannian geometry and many other subjects

this class will cover the basics of symplectic geometry and then focus on understanding when a manifold admits a symplectic structure (area of current research)

if time permits we will also discuss contact manifolds (an odd dimentional version of symplectic manifolds).

I Symplectic Linear algebra

a symplectic vector space is a (finite dimensional) real vector space V with a non-degenerate, skew-symmetric bilinear form

$$\omega : V \times V \to \mathbb{R}$$

1) $\omega(v,u) = -\omega(u,v)$

2) $\omega(v+cu, w) = \omega(v,w) + c\,\omega(u,w)$ $\quad \forall c \in \mathbb{R},\ v,u,w \in V$

3) $\omega(v,u) = 0 \ \forall u \in V \Rightarrow v = 0$

lemma 1:

a bilinear pairing $\omega : V \times V \to \mathbb{R}$ is non-degenerate

$\Longleftrightarrow$

the linear map $\phi_\omega : V \to V^* : v \mapsto (f_v : V \to \mathbb{R} : u \mapsto \omega(v,u))$ is an isomorphism

Proof:

($\Rightarrow$) $\phi_\omega(v) = 0$ then $f_v : V \to \mathbb{R} : u \mapsto \omega(v,u)$

is the zero map and $v = 0$ by non-degeneracy

$\therefore$ ϕ_ω injective, $\therefore$ isomorphism since $\dim V = \dim V^*$

($\Leftarrow$) if $\omega(v,u) = 0 \ \forall u \in V$ then $\phi_\omega(v) = 0$ and $v = 0$ since

ϕ_ω an isomorphism ▦

example: $V = \mathbb{C}^n = \mathbb{R}^{2n}$

$h(v,u) = \sum \bar{v}_j u_j$ for $v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$, $u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$

Hermitian form

note: $h(v,u) = \overline{h(u,v)}$

set $\langle v,u \rangle = \operatorname{Re} h(v,u)$ — symmetric

$\omega(v,u) = \operatorname{Im} h(v,u)$ — skew-symmetric

$\Big\}$ both non-degenerate

so $\langle , \rangle$ is an inner product on V

ω is a symplectic structure on V

note: $\omega(v,u) = \langle iv, u \rangle$

if $\{e_1 \dots e_n\}$ is the standard basis for V over $\mathbb{C}$

and $f_j = i e_j$

then $\{e_1, f_1, \dots e_n, f_n\}$ is a basis for V over $\mathbb{R}$

(positively oriented)

clearly $\omega(e_j, f_j) = -\omega(f_j, e_j) = \langle e_j, e_j \rangle = \langle f_j, f_j \rangle = 1$

all other other pairs evaluate to 0

if $\{e_1^*, f_1^*, \dots, e_n^*, f_n^*\}$ is the dual basis for $(\mathbb{R}^{2n})^*$

$$\boxed{\omega_{std} = \sum_{j=1}^{n} e_j^* \wedge f_j^*}$$

$\leftarrow$ standard symplectic structure on $\mathbb{R}^{2n}$

If $(V, \omega_V), (W, \omega_W)$ are symplectic vector spaces then $V \oplus W$ has symplectic structure

$$\omega = \pi_V^* \omega_V + \pi_W^* \omega_W$$

where

$$\begin{tikzcd} & V \oplus W \arrow[dl, "\pi_V"'] \arrow[dr, "\pi_W"] & \\ V & & W \end{tikzcd}$$

are projections

Thm 2:

> If (V, ω) a symplectic vector space then $\exists$ an isomorphism $\phi : V \to \mathbb{C}^n$ s.t
>
> $$\phi^* \omega_{std} = \omega$$

we can immediately conclude

Cor 3:

> 1) any symplectic vector space is <u>even</u> dimensional
> 2) a skew-symmetric bilinear form $\omega : V \times V \to V$
>
> is non-degenerate $\iff \underbrace{\omega \wedge \dots \wedge \omega}_{n \text{ copies if } \dim V = 2n} \neq 0$
>
> 3) any symplectic vector space is cannonically oriented (by $\omega \wedge \dots \wedge \omega$)
>   a) V, W symplectic $\Rightarrow$ sympl. orientation on $V \oplus W$ is directsum orientation
>   b) symplectic orientation on $\mathbb{C}^n$ is standard one

for the proof we need:

- (V, ω) symplectic vector space
- $W \subset V$ subspace

then $W^\perp = \{ v \in V \mid \omega(v, u) = 0 \ \forall u \in W \}$

note: $\omega(v, v) = 0$ so $\dim W = 1 \Rightarrow W \subset W^\perp$

so quite different from inner product $\perp$
but we still have

lemma 4: $\boxed{\dim W + \dim W^{\perp} = \dim V}$

Proof: the map $\phi_\omega : V \to V^*$ is an isomorphism

so given $W \subset V$ the set $\phi_\omega(W^\perp) \subset V^*$ vanishes on W

so we have induced map

$$\tilde{\phi}_\omega : W^\perp \to (V/W)^*$$

we claim $\tilde{\phi}_\omega$ is an isomorphism

if so $\dim W^\perp = \dim(V/W)^* = \operatorname{codim} W$ and done!

<u>injectivity</u>: if $\tilde{\phi}_\omega(w) = 0$, then $\omega(w, v) = 0 \ \forall v \in V$

$\therefore w = 0$

<u>surjectivity</u>: for any elt $\eta \in (V/W)^*$ gives a linear map

$$\eta : V \to \mathbb{R}$$

that vanishes on W

so $\exists v \in V$ s.t. $\phi_\omega(v) = \eta$ and $v \in W^\perp$

$\therefore \tilde{\phi}_\omega(v) = \eta$ ▦

Cor 5: $\boxed{(W^\perp)^\perp = W}$

Proof: $W \subset (W^\perp)^\perp$ and same dimension ▦

some terminology: $W \subset (V, \omega)$

if $W \subset W^\perp$ then W is <u>isotropic</u> $(\dim W \leq \frac{1}{2} \dim V)$

if $W \supset W^\perp$ then W is <u>coisotropic</u> $(\dim W \geq \frac{1}{2} \dim V)$

if $W = W^\perp$ then W is <u>Lagrangian</u> $(\dim W = \frac{1}{2} \dim V)$

if $\omega|_W$ is non-degenerate, then W is a <u>symplectic subspace</u>

exercise: $W \subset (V,\omega)$ is a symplectic subspace

$$\Longleftrightarrow$$

$$W \cap W^\perp = \{0\}$$

$$\Longleftrightarrow$$

$$W \oplus W^\perp = V \ (\text{and } \omega = \pi_W^*(\omega|_W) + \pi_{W^\perp}^*(\omega|_{W^\perp})$$

also note W symplectic $\Rightarrow W^\perp$ is symplectic

Proof of Th^{m}2:

Induct on dimension V

$\dim V = 0$: done

$\dim V > 0$: then $\exists\, v \in V$ s.t. $v \neq 0$

ω non-degenerate $\Rightarrow \exists\, \tilde{u} \in V$ s.t. $\omega(v, \tilde{u}) \neq 0$

let $u = \dfrac{\tilde{u}}{\omega(v,u)}$

note: $\omega(v,u) = 1$, $\omega(v,v) = 0$, $\omega(u,u) = 0$

so $W = \operatorname{span}\{v, u\}$ is a symplectic subspace of V

v, u give isomorphism to $(\mathbb{C}, \omega_{std})$

$\therefore V \cong (\mathbb{C}, \omega_{std}) \oplus (\mathbb{C}^{n-1}, \omega_{std}) \cong (\mathbb{C}^n, \omega_{std})$

by induction
since $\dim W^\perp < \dim V$

if $(V, \omega_V), (W, \omega_W)$ are symplectic vector spaces then a linear map

$$f: V \to W$$

is symplectic if $f^* \omega_W = \omega_V$

note: f symplectic $\Rightarrow f$ is injective

$(v \in \ker f \Rightarrow f(v) = 0 \perp W \Rightarrow v \perp V \Rightarrow v = 0)$

Group of all symplectic linear maps of $(\mathbb{R}^{2n}, \omega_{std})$ is

$$Sp(2n, \mathbb{R})$$

note: $U(n) \subset Sp(2n, \mathbb{R})$ in fact maximal compact subgroup

$U(n) \hookrightarrow Sp(2n, \mathbb{R})$ a homotopy equivalence

Section II: Symplectic manifolds

recall a symplectic structure on a manifold M is a 2-form $\omega \in \Omega^2(M)$

s.t. 1) ω is non-degenerate (on each T_xM)

2) $d\omega = 0$

note: any symplectic manifold M is

1) even dimensional
2) oriented
3) has canonical volume form $\Omega = \omega\wedge\dots\wedge\omega$
4) $\exists\, a \in H^2(M;\mathbb{R})$ s.t. $\underbrace{a\cup\dots\cup a}_{n \text{ times}} \neq 0$ cup product

M closed

since $d\omega = 0 \Rightarrow [\omega] \in H^2_{DR}(M) \cong H^2(M;\mathbb{R})$
and $\omega\wedge\dots\wedge\omega$ is a volume form
so $\neq 0$ in $H^{2n}_{DR}(M)$

examples:

0) S^{2n} not symplectic if $n \neq 1$

$S^2 \times S^{2m}$ not symplectic if $m \neq 0, 1$

1) $\mathbb{C}^n = \mathbb{R}^{2n}$ $(x_1, y_1, \dots x_n, y_n)$ $z_j = x_j + iy_j$

$\omega_{std} = \sum_{j=1}^{n} dx^j \wedge dy^j$ is a symplectic form, called the "standard" structure

$\omega = d\lambda$ where $\lambda = \frac{1}{2}\sum x_j dy_j - y_j dx_j = \frac{i}{2}\sum z_j d\bar{z}_j - \bar{z}_j dz_j$

2) Any oriented surface with an area form ω
($d\omega = 0$ for dim. reasons)

a submanifold N of a symplectic manifold (M,ω) is

Lagrangian (isotropic, symplectic, coisotropic) if each $T_xN \subset T_xM$

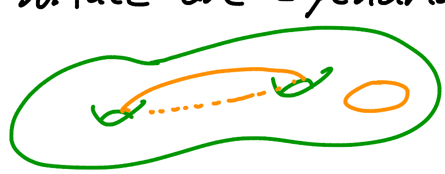
is Lagrangian (isotropic, symplectic, coisotropic)

note: if $N \subset (M,\omega)$ is a symplectic submanifold then $(N, \omega|_{TN})$
is a symplectic manifold.

examples:

1) any 1-dimensional manifold is isotropic

so curves in a surface are Legendrian



2) any codimension 1 submanifold is coisotropic

projections $M_1 \times M_2 \to M_i$

3) $(M_1, \omega_1), (M_2, \omega_2)$ symplectic

then $M_1 \times M_2$ has symplectic structure $\pi_{M_1}^* \omega_1 + \pi_{M_2}^* \omega_2$

moreover $M_1 \times \{p\}$ and $\{q\} \times M_2$ are symplectic submanifolds

if $L_i \subset M_i$ a Lagrangian submanifold of (M_i, ω_i)

then $L_1 \times L_2$ a Lagrangian submanifold of $M_1 \times M_2$

e.g. if Σ_1, Σ_2 surfaces w/area forms then $\Sigma_1 \times \Sigma_2$ has <u>lots</u> of Lagrangian tori

$(M, \omega_M), (N, \omega_N)$ symplectic

a map $f: M \to N$ is called <u>symplectic</u> if $f^* \omega_N = \omega_M$

<u>note</u>: this implies df_x injective $\forall x \in M$

$\therefore f$ an immersion

a symplectic diffeomorphism f is called a <u>symplectomorphism</u> and is the natural equivalence relation on sympl. manifolds (note, f^{-1} also symplectic)

<u>example</u>:

$(M_1, \omega_1), (M_2, \omega_2)$ symplectic

then $M_1 \times M_2$ has symplectic structure

$$\omega_{\lambda_1, \lambda_2} = \lambda_1 \pi_{M_1}^* \omega_1 + \lambda_2 \pi_{M_2}^* \omega_2$$

for any $\lambda_1, \lambda_2 \in \mathbb{R} - \{0\}$

given a map $f: M_1 \to M_2$, the graph of f is

$$\Gamma_f = \{ (x, f(x)) : x \in M_1 \} \subset M_1 \times M_2$$

<u>exercise</u>: a diffeomorphism $f: M_1 \to M_2$ is a symplecto.

$\Longleftrightarrow$

Γ_f is Lagrangian in $(M_1 \times M_2, \omega_{1,-1})$

Important example:

let M be any smooth manifold

$\pi : T^*M \to M$ the projection map

the Liouville 1-form λ on T^*M

$\lambda \in \Omega^1(T^*M)$

is defined as follows:

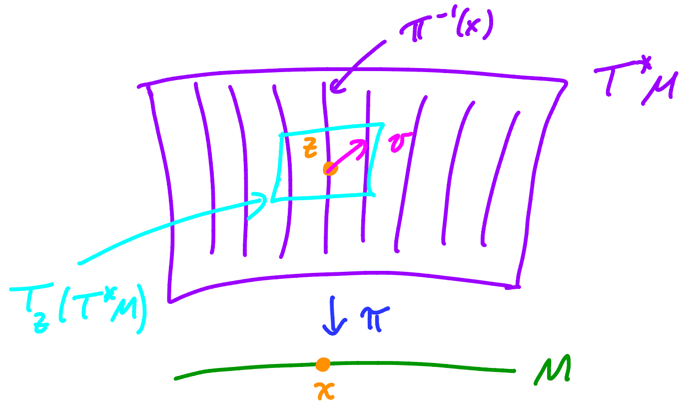
if $z \in T_x^*M$, then

$z : T_{\pi(x)} M \to \mathbb{R}$

if $v \in T_z(T^*M)$, then

$d\pi_z(v) \in T_{\pi(x)} M$

so $\boxed{\begin{array}{l} \lambda_z : T_z(T^*M) \to \mathbb{R} \\ \qquad v \longmapsto z(d\pi_z(v)) \end{array}}$



exercise:

1) if $q_1, \dots q_n$ are local coordinates on $U \subset M$

then any $z \in T^*U \subset T^*M$ can be written

$z = \sum_{i=1}^{n} p_i \, dq_i$ some $p_1 \dots p_n \in \mathbb{R}$

so $\{q_1 \dots q_n, p_1 \dots p_n\}$ are coordinates on T^*U

and $\pi(q_1 \dots q_n, p_1 \dots p_n) = (q_1 \dots q_n)$

note: $dq_i = \pi^* dq_i$

q_i coordinate on T^*U so $dq_i \in \Omega^1(T^*U)$

q_i coordinate on U so $dq_i \in \Omega^1(U)$

in our notation this makes sence but if confusing write $d\tilde{q}_i = \pi^* dq_i$

Show $\boxed{\lambda = \sum_{i=1}^{n} p_i \, dq_i}$ $\left(= \sum_{i=1}^{n} p_i \, d\tilde{q}_i\right)$

so $\omega = -d\lambda$ is a symplectic form on T^*M

2) If $\alpha \in \Omega^1(M)$ then $\alpha : M \to T^*M$

Show $\boxed{\alpha^* \lambda = \alpha}$ (λ also called canonical 1-form)

3) image of zero section of T^*M Lagrangian

more generally, if $\alpha \in \Omega^1(M)$, then

$$\text{image}(\alpha) \text{ Lagrangian} \iff d\alpha = 0$$

4) fibers of $\pi : T^*M \to M$ are Lagrangian

5) If $f : M \to N$ a diffeomorphism, then

$$f^* : T^*N \to T^*M$$

is a symplectomorphism

Remark: This means you can try to distinguish smooth manifolds using symplectic geometry of their cotangent bundles!

interesting research problem: can you use this distinguish exotic 4-manifolds? other homeomorphic, but not-diffeomorphic pairs?

example: Abouzaid showed if M a $(4n+1)$-manifold s.t. T^*M and T^*S^{4n+1} are symplectomorphic then M a homotopy sphere that bounds a manifold with trivial tangent bundle

This $\Rightarrow$ symplectic geometry of cotangent bundles can distinguish 6 of 7 exotic smooth structures on S^9 from standard S^9

Major Open Question:

are M, N diffeomorphic

$\iff$

T^*M and T^*N are symplectomorphic

more Lagrangians in $(T^*M, d\lambda)$

if $S^k \subset M^n$ is a submanifold, then its conormal bundle

is $N^*S = \{ \eta \in T^*M : \pi(\eta) \in S,\ \eta(v) = 0\ \forall v \in T_{\pi(\eta)} S \}$

covectors that vanish on S

this is a bundle over S

and a properly embedded submanifold of T^*M

exercise:

Hint: choose coordinates adapted to S at $x \in S$

1) $\dim N^*S = n$ (so fibers have dim $n-k$)

2) N^*S is Lagrangian (even more $\iota^*\lambda = 0$ where $\iota : N^*S \to T^*M$)

example: if $x \in M$, then $N^*\{x\} = \pi^{-1}(x)$

note: if we have an isotopy S_t of S, then N^*S_t undergoes a proper isotopy through Legendrian submanifolds

so symplectic invariants of the Legendrian isotopy class of $N^*S \subset T^*M$ are invariants of the smooth isotopy class of $S \subset M$!

So we see T^*M contains lots of Lagrangian submanifolds, but conjecturally not lots of compact exact Lagrangian submanifolds

a Lagrangian submanifold $L \subset (T^*M, d\lambda)$ is exact if $\exists$ a function $f : L \to \mathbb{R}$ such that $df = \iota^*\lambda$ (where $\iota : L \to T^*M$ inclusion)

Major Open Question:

if yes to this then yes to question on previous page.

> let M be a compact manifold
> if $L \subset (T^*M, d\lambda)$ a compact, orientable, exact Lagrangian
> then L can be deformed through exact Lagrangians to the zero section?

this is Arnold's "nearby Lagrangian conjecture"

for our next example we need a few more ideas

recall an isotopy is a smooth map $\Phi : M \times (-a,a) \to M$ such that $\phi_t = \Phi(\cdot, t) : M \to M$ is a diffeomorphism and $\phi_0 = id_M$ (a usually taken to be ∞)

given ϕ_t we get a time dependent vector field

$$v_t(p) = \frac{d}{ds} \phi_s(q)\Big|_{s=t} \quad \text{where } q = \phi_t^{-1}(p)$$

i.e. $\boxed{v_t \circ \phi_t = \frac{d\phi_t}{dt}}$ *

conversely given a time dependent vector field v_t (with compact support) then $\exists!$ isotopy $\Phi: M \times \mathbb{R} \to M$ satisfying * called the flow of v_t

if v is time independent then flow satisfies

$$\phi_s \circ \phi_t = \phi_{s+t}$$

exercise:

the Lie derivative, defined by $\mathcal{L}_{v_t} \underbrace{\eta}_{k\text{-form}} = \frac{d}{dt} (\phi_t)^* \eta \big|_{t=0}$,

satisfies $\boxed{\mathcal{L}_v \eta = \iota_v d\eta + d\iota_v \eta}$ Cartan magic formula

and $\boxed{\frac{d}{dt} \phi_t^* \eta = \phi_t^* \mathcal{L}_{v_t} \eta}$

now given a symplectic form ω on M, the linear isomorphism $\phi_{\omega_x}: T_x M \to T_x^*(M)$ gives an isomorphism

$$\Phi_\omega : \mathcal{X}(M) \to \Omega^1(M)$$
$$v \longmapsto \iota_v \omega$$

vector fields on M

call a vector field v symplectic if $\iota_v \omega$ is closed

note: if v_t symplectic and ϕ_t its flow then

$$\frac{d}{dt} \phi_t^* \omega = \phi_t^* \mathcal{L}_{v_t} \omega = \phi_t^* (\iota_{v_t} \overbrace{\cancel{d\omega}}^{0} + \overbrace{\cancel{d\iota_{v_t}\omega}}^{0}) = 0$$

so ϕ_t preserves ω

i.e. $\phi_t : M \to M$ is a symplectomorphism

exercise: if $\phi_t: M \to M$ flow of $v_t \in \mathcal{X}(M)$, then

ϕ_t a symplectomorphism $\forall t \iff v_t$ symplectic $\forall t$

a special type of symplectic vector field is a Hamiltonian vector field

given a function $H: M \to \mathbb{R}$

get $dH \in \Omega^1(M)$ a closed 1-form

let X_H be unique vector field such that

$$\iota_{X_H} \omega = dH \quad (\text{i.e. } \Phi^{-1}(dH))$$

X_H is called the Hamiltonian vector field of the energy (or Hamiltonian) function H

note: If $\gamma(t)$ a flow line of X_H, then $\gamma'(t) = X_H(\gamma(t))$

$$\text{so} \quad \frac{d}{dt}\big[H(\gamma(t))\big] = dH_{\gamma(t)}(\gamma'(t)) = \omega(X_H(\gamma(t)), \gamma'(t))$$
$$= \omega(X_H(\gamma(t)), X_H(\gamma(t))) = 0$$

so $\boxed{\text{flow of } X_H \text{ tangent to level sets of } H}$

i.e. energy is conserved along flow

Physics asside:

in local coordinates $(q_1 \dots q_n, p_1 \dots p_n)$ on $T^*\mathbb{R}^n = \mathbb{R}^{2n}$

we have $\omega = -d\lambda = -\sum dp_i \wedge dq_i$

$$dH = \sum\left(\frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q_i} dq_i\right)$$

and

$$\iota_{X_H}\omega = -\sum\left(dp_i(X_H)\, dq_i - dq_i(X_H)\, dp_i\right)$$

so p_i-coord of $X_H = -\frac{\partial H}{\partial q_i}$

q_i-coord of $X_H = \frac{\partial H}{\partial p_i}$

or if $\gamma(t) = (q_1(t), \dots q_n(t), p_1(t), \dots p_n(t))$ is a flow line of X_H then

$$\boxed{\begin{aligned} \dot{p}_i &= -\frac{\partial H}{\partial q_i} \\ \dot{q}_i &= \frac{\partial H}{\partial p_i} \end{aligned}} \quad \text{Hamilton's Equations}$$

now if $V: M \to \mathbb{R}$ is some "potential energy" of some system

exerts a "force" $F = -\nabla V$

then the "total energy" is $H(q,p) = \underbrace{\frac{\|p\|^2}{2m}}_{\text{Kinetic energy}} + V(p)$

in local coordinates get flow lines

satisfy
$$\dot{q}_i = \frac{\partial H}{\partial p_i} = \frac{p_i}{m} \Rightarrow p_i = m\dot{q}_i \quad \text{(momentum = mass x velocity)}$$
$$\dot{p}_i = -\frac{\partial H}{\partial q_i} = -\frac{\partial V}{\partial q_i} \Rightarrow m\ddot{q}_i = \dot{p}_i = -\nabla V = F$$

Newton's equations!

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now given a Hamiltonian $H: M \longrightarrow \mathbb{R}$ for (M, ω)

from above X_H is tangent to level sets $H^{-1}(c)$

assume c a regular value so $H^{-1}(c)$ a manifold

Claim: X_H spans $(T(H^{-1}(c)))^{\perp\omega}$

indeed $v \in T_x(H^{-1}(c))$, then $\exists\, \gamma: (-\varepsilon, \varepsilon) \to H^{-1}(c)$ s.t. $\gamma(0) = x$, $\gamma'(0) = v$

and $0 = \frac{d}{dt}(H(\gamma(t))) = dH_{\gamma(t)}(\gamma'(t)) = \omega(X_H, v)$

so $X_H \in (TH^{-1}(c))^\perp$

but $\dim (TH^{-1}(c))^\perp = 1$

also note flow ϕ_t^H of X_H preserves ω

Important example:

recall complex projective space is

$\mathbb{C}P^n = \mathbb{C}^{n+1} - \{(0,\dots,0)\} / \mathbb{C}-\{0\}$ where $\mathbb{C}-\{0\}$ acts on $\mathbb{C}^{n+1}$ by multiplication

$\cong S^{2n+1} / S^1$ ← unit sphere in $\mathbb{C}$

unit sphere in $\mathbb{C}^{n+1}$

now if we set $H: \mathbb{C}^{n+1} \to \mathbb{R} : (z_0, \dots z_n) \mapsto \sum_{i=0}^{n} |z_i|^2$ $\qquad z_j = x_j + iy_j$

then $S^{2n+1} = H^{-1}(1)$ ← regular value

note $dH = \sum 2(x_i\, dx_i + y_i\, dy_i)$

so $X_H = 2\sum(-x_i \frac{\partial}{\partial y_i} + y_i \frac{\partial}{\partial x_i})$

now if $\gamma(t) = e^{it}(x_1 + iy_1, \dots)$ is an orbit of S^1-action

then $\gamma'(0) = (-y_1 + ix_1, \dots)$

so vector field generating S^1-action is $v = \sum -y_i \frac{\partial}{\partial x_i} + x_i \frac{\partial}{\partial y_i}$

since X_H proportional to v, orbits of X_H are orbits of S^1

so $\mathbb{C}P^n = S^{2n+1} / \text{orbits of } X_H$

exercise:

1) if (V, ω) symplectic vector space and $W \subset V$ coisotropic ($W^\perp \subset W$)

then ω induces a symplectic structure on $W/W^\perp$

2) $x \in \mathbb{C}P^n$ and $z \in S^{2n+1}$ s.t. $\pi(z) = x$ $\quad (\pi: S^{2n+1} \to \mathbb{C}P^n$ projection$)$

then $T_x\mathbb{C}P^n \cong T_z S^{2n+1} / T_z(S^1\text{-orbit}) = T_z(H^{-1}(1)) / (T_z(H^{-1}(1)))^\perp$

gets a symplectic structure from ω_z

since flow of X_H preserves ω show symplectic structure on $T_x\mathbb{C}P^n$

is independent of choice of z

also show this gives a smooth closed 2-form ω_{FS} on $\mathbb{C}P^n$

i.e. $\mathbb{C}P^n$ is a symplectic manifold

Fubini-Study

this is an example of "symplectic reduction" so $(\mathbb{C}P^n, \omega_{FS})$ symplectic manifold

define later

$\mathbb{C}P^n$ is also a complex manifold and complex structure is "compatible" with ω_{FS}

so any complex submanifold of $\mathbb{C}P^n$ is a symplectic submanifold

example: given a collection of homogeneous complex polynomials in $\mathbb{C}^{n+1}$

$$p(\lambda z) = \lambda^d p(z)$$

they have a well-defined zero locus in $\mathbb{C}P^n$

this is called a complex algebraic variety

if it is a manifold then it is a symplectic manifold

e.g. $\{\sum z_i^d = 0\}$ degree d hypersurfaces in $\mathbb{C}P^n$

Later we will consider many other constructions of symplectic manifolds but for now move onto the "local theory".