Section IV : Almost symplectic structures

an almost symplectic structure on a manifold M is a non-degenerate 2-form η

clearly if we hope for M to be symplectic it is necessary that it be almost symplectic

we will see in this section that one can systematically study when a manifold has an almost symplectic structure

Here is the main conjecture of Eliashberg about the existence of symplectic structures

Conjecture:

If M a manifold of dimension $2k > 4$ with

- a class $h \in H^2(M)$ s.t. $\underbrace{h \cup \dots \cup h}_{k \text{ times}} \neq 0$ and
- a non-degenerate 2-form ω_0

Then $\exists$ a symplectic structure ω on M such that

- ω is homotopic to ω_0 through non-degenerate forms
- $[\omega]$ can be deformed to h keeping its k^{th} power non-zero

Remark: Not true in dimension 4 (we discuss this more later)

But there is an alternate conjecture (discuss later)

We discuss an approach to this conjecture later as well as some interesting test cases in 4D.

A. Linear Symplectic Group

Consider the standard symplectic vector space

$V = \mathbb{C}^n = \mathbb{R}^{2n}$

$e_1 \dots e_n$ a complex basis

$f_j = i e_j$

$e_1, f_1, \dots e_n, f_n$ a real basis

set $J_0 = \begin{pmatrix} \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} & & \\ & \ddots & \\ & & \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \end{pmatrix}$

clearly
$$\omega(u,v) = u^T J_0 v \qquad \text{where } u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{2n} \end{pmatrix} \text{ in basis above}$$

so a linear map $L: V \to V$ is symplectic
$$\Longleftrightarrow$$
$$L^T J_0 L = J_0$$

and we see
$$Sp(2n) = \{\text{linear symplectic maps}\}$$
$$= \{ L \in SL(2n, \mathbb{R}) : L^T J_0 L = J_0 \}$$

lemma 1:

If $L \in Sp(2n)$, then

1) $\det L = 1$
2) λ an eigenvalue of $L \Leftrightarrow \lambda^{-1}$ is (and multiplicities agree)
3) if $Lv = \lambda v$ and $Lu = \lambda' u$ and $\lambda\lambda' \neq 1$ then $\omega(v,u) = 1$

Proof: 1) L preserves ω_{std} so preserves ω_{std}^n ← multiple of volume form
$\therefore$ L preserves volume, ie $\det L = 1$

2) Similar matrices have same eigenvalues, and
$$L^T = J_0 L^{-1} J_0^{-1}$$
(eigenvalues of L^{-1} are inverses of those of L)

3) $\omega(v,u) = \omega(Lv, Lu) = \lambda\lambda' \omega(v,u)$

so $(1 - \lambda\lambda')\,\omega(v,u) = 0$

recall some other linear groups

orthogonal group $\quad O(n) = \{ L \in GL(n, \mathbb{R}) : L^T L = \mathbb{1} \}$ ← identity matrix

$GL(n, \mathbb{C}) = \{\text{Linear maps of } \mathbb{C}^n \text{ that preserve multiplication by } i\}$

thinking of $\mathbb{C}^n$ as $\mathbb{R}^{2n}$

can write

$$GL(n, \mathbb{C}) = \{ L \in GL(2n, \mathbb{R}) : L J_0 = J_0 L \}$$
complex mult.

unitary group $\quad U(n) = \{ L \in GL(2n, \mathbb{R}) : L^T L = \mathbb{1} \text{ and } L J_0 = J_0 L \}$

lemma 2:

$$Sp(2n) \cap O(2n) = Sp(2n) \cap GL(n,\mathbb{C}) = O(2n) \cap GL(n,\mathbb{C}) = U(n)$$

Proof: Clear from above. ▦

Th^m 3:

$U(n) \subset Sp(2n)$ is the maximal compact subgroup of $Sp(2n)$ and the inclusion map is a homotopy equivalence

lemma 4:

let $L \in Sp(2n)$
if L is symmetric and positive definite
then $L^\alpha \in Sp(2n)$ $\forall \alpha \geq 0$

Proof:

Symmetric & pos. def $\Rightarrow$ eigenvalues are positive real numbers $\lambda_1, \dots, \lambda_k$ and eigenvectors span $\mathbb{R}^{2n}$

$\mathbb{R}^{2n} = \bigoplus_{i=1}^{k} E_i$ $\quad E_i$ is λ_i eigenspace

clearly L^α is defined to be mult by λ_i^α on E_i

we just need to check $L^\alpha \in Sp(2n)$

for any $v \in \mathbb{R}^n$ write $v = v_1 + \dots + v_k$ with $v_i \in E_i$

now $\omega(L^\alpha v, L^\alpha u) = \sum_{i,j=1}^{k} (\lambda_i \lambda_j)^\alpha \, \omega(v_i, u_j)$

either $\lambda_i\lambda_j = 1$ or $\omega(v_i, v_j) = 0$ by lemma 1

$$= \sum_{i,j=1}^{k} \omega(v_i, v_j)$$

$$= \sum_{i,j=1}^{k} \lambda_i \lambda_j \, \omega(v_i, u_j) = \omega(v, u)$$

so $L^\alpha \in Sp(2n)$ ▦

Proof of Th^m 3: to show $U(n) \hookrightarrow Sp(2n)$ a homotopy equivalence we build a strong deformation retraction $H: [0,1] \times Sp(2n) \to Sp(2n)$, $(t, L) \longmapsto h_t(L)$

that is $h_0 = \mathbb{1}_{Sp(2n)}$

$h_t = \mathbb{1}_{U(n)} \quad \forall t$

$h_1(Sp(2n)) = U(n)$

to this end set $h_t(L) = L\,\underbrace{(L^TL)}^{-t/2}$ ← note L^TL is symmetric and positive def.

$\therefore$ so is $(L^TL)^{-1}$ and use lemma 4

clearly h_t is continuous

$L \in U(n)$ then $L^TL = \mathbb{1}$ so $h_t(L)$ fixed $\forall t$

and $L \in Sp(2n)$ then $h_1(L) \in Sp(2n)$ s.t.

$$[(L^TL)^{-1/2}]^T \underbrace{L^TL}\,\underbrace{(L^TL)^{-1/2}} = (L^TL)^{-1/2}(L^TL)^{-1/2} L^TL = \mathbb{1}$$

commute (same eigenspaces)

so $L_1(L) \in U(n)$

see McDuff-Salamon for $U(n)$ max cpt subgroup

(not hard, but we don't really need it) ▦

B. Linear Complex Structures

a complex structure on a real vector space V is a linear map

$$J : V \to V$$

such that

$$J^2 = -\mathbb{1}_V$$

given J on V then V gets the structure of a $\mathbb{C}$ vector space by defining $(x+iy)v = xv + yJ(v)$

so clearly V must be even dimensional

the matrix J_0 above is the standard complex str. on $\mathbb{R}^{2n}$

exercise: if V a $2n$-dimensional vector space and J a complex structure on V, then $\exists$ an isomorphism $\phi : \mathbb{R}^{2n} \to V$ s.t.

$$J \circ \phi = \phi \circ J_0$$

i.e. ϕ conjugates J to J_0

(in appropriate basis J given by J_0)

lemma 5:

The space of complex structures on $\mathbb{R}^{2n}$, denoted $\mathcal{J}(\mathbb{R}^n)$, is diffeomorphic to the homogeneous space $GL(2n,\mathbb{R})/GL(n,\mathbb{C})$.

so
- $\mathcal{J}(\mathbb{R}^{2n})$ has two components
- component containing J_0, denoted $\mathcal{J}^+(\mathbb{R}^{2n})$, is $GL^+(2n,\mathbb{R})/GL(n,\mathbb{C})$ ← pos det.

  $\therefore$ htpy equivalent to $SO(2n)/U(n)$
- $\mathcal{J}^+(\mathbb{R}^2) \simeq \{*\}$
- $\mathcal{J}^+(\mathbb{R}^4) \simeq S^2$

Proof: exercise above $\Rightarrow$ $GL(2n,\mathbb{R})$ acts trasitively on $\mathcal{J}(\mathbb{R}^{2n})$

by $A \longmapsto A^{-1}J_0A$

and the stabilizer of J_0 is $GL(n,\mathbb{C})$

proof of Th^{m} 3 gives $GL^+(2n,\mathbb{R})/GL(n,\mathbb{C}) \simeq SO(2n)/U(n)$

for $\mathcal{J}^+(\mathbb{R}^2) \simeq SO(2)/U(1) \cong S^1/S^1 \simeq \{*\}$

for $\mathcal{J}^+(\mathbb{R}^4)$, upto homotopy any J is $A^{-1}J_0A$ with $A\in SO(4)$ (so $A^{-1}=A^T$)

thus $v\cdot Jv = Av\cdot J_0Av = \omega_{std}(Av, Av) = 0$

so Jv orthogonal to v

<u>note:</u> J determined by $J\left(\frac{\partial}{\partial x_1}\right) \in \left\{(x_1,y_1,x_2,y_2) \,\middle|\, \begin{array}{l} x_1^2+y_1^2+x_2^2+y_2^2=1 \\ x_1=0 \end{array}\right\} = S^2$

since $J\left(\frac{\partial}{\partial x_1}\right)\in S^2$ and if $E=\text{span}\left\{\frac{\partial}{\partial x_1}, J\frac{\partial}{\partial x_1}\right\}$

then J on $E^{\perp}$ is rot^{n} by $\pi/2$ (exercise)

↑ inner product perp

If (V,ω) a symplectic vector space we say a complex structure J on V is <u>compatible</u> with ω if

$$\omega(Jv, Ju) = \omega(v,u) \qquad (J^*\omega = \omega)$$

$$\omega(v, Jv) > 0 \quad \forall v \neq 0 \qquad (\text{called } \underline{\text{tame}})$$

<u>note:</u> 1) $g_J(v,u) = \omega(v, Ju)$ is an innerproduct:

- $g_J(v,u) = \omega(v, Ju) = \omega(Jv, J^2u) = -\omega(Jv, u)$
  $= \omega(u, Jv) = g(u,v)$

and
- $g_J(v,v) = \omega(v, Jv) > 0$ if $v \neq 0$

2) $g_J(Jv, Ju) = g_J(v,u)$ (ie. $J^* g_J = g_J$)

lemma 6:

> let V be a vector space
> ω a symplectic structure on V
> J a complex structure on V
>
> Then the following are equivalent
>
> 1) J is compatible with ω
> 2) $g_J(v,u) = \omega(v, Ju)$ is a J-invariant innerproduct
> 3) (V, ω) has a basis $v_1, Jv_1, \dots v_n, Jv_n$ s.t. in this basis ω given by J_0
> 4) $\exists$ isomorphism $\phi: \mathbb{R}^{2n} \to V$ such that $\phi^*\omega = \omega_{std}$ $\phi^* J = J_0$
> 5) ω tamed by J and $\forall$ Lagrangian subspaces $L \subset V$ the space JL also Lagrangian

Proof:

1) $\Rightarrow$ 2) above

2) $\Rightarrow$ 1) $\omega(v, Jv) = g_J(v,v) > 0$ if $v \neq 0$ by non-degen of g_J
symmetry of $g_J \Rightarrow$ symmetry of ω /

3) $\Leftrightarrow$ 4) clear

4) $\Rightarrow$ 1) clear

1) $\Rightarrow$ 5) let $L \subset V$ be a Lagrangian subspace
let $v_1 \dots v_n$ be a g_J orthogonal basis for V

you can check that $v_1, Jv_1, \ldots, v_n, Jv_n$ is the desired basis ✓

1) $\Rightarrow$ 5) L Lagrangian means $L^\perp = L$

$J^*\omega = \omega \Rightarrow \left(v \in L^\perp \Leftrightarrow Jv \in (JL)^\perp\right)$

so $(JL)^\perp = J(L^\perp) = JL$ ✓

5) $\Rightarrow$ 2) let $g_J(u,v) = \omega(u, Jv)$

we check g_J a J-inv't innerproduct

if g_J is not J-invariant then $\exists\, u,v$ st.

$$g_J(Ju, Jv) \neq g_J(u,v)$$

so $\omega(v, Ju) \neq \omega(u, Jv)$

so $v \neq 0$ and $\omega(v, Jv) > 0$ by tamed condition

set $w = u - \frac{\omega(v, Ju)}{\omega(v, Jv)} v$

so $\omega(v, Jw) = \omega(v, Ju) - \frac{\omega(v, Ju)}{\omega(v, Jv)} \omega(v, Jv) = 0$

but $\omega(w, Jv) = \omega(u, Jv) - \omega(v, Ju) \neq 0$ (by assumption)

exercise: $\exists$ Lagrangian $L \subset V$ containing v, Jw

so $w, Jv \in JL$ so JL not Lagrangian

this $\otimes \Rightarrow g_J$ is J-invariant

ω tamed by $J \Rightarrow g_J$ an innerproduct

an inner product g on V is compatible with a complex structure J if it is J-invariant: $J^* g = g$

note: given g and J compatible, then $\omega_J(u,v) = g(Ju, v)$ is a symplectic structure compatible with J

given an inner product g and symplectic structure ω on V

we get

$$\begin{tikzcd} V \arrow[rr, dashed, "A"] \arrow[dr, "\phi_\omega"', "\cong"] & & V \arrow[dl, "\phi_g", "\cong"'] \\ & V^* & \end{tikzcd}$$

$A = \phi_g^{-1} \circ \phi_\omega$

and
- $\omega(v, u) = g(Av, u)$
- $g(Av, u) = \omega(v, u) = -\omega(u, v) = -g(Au, v)$
  $= -g(v, Au)$

  A skew-adjoint for g

we call g and ω compatible if $A^2 = -\mathbb{1}$

exercise: 1) given any 2 of 3 possible compatible structures show you get a unique 3rd structure that is compatible with the other 2.

2) If g, ω, J are all compatible on V

then $h = g + i\omega$ is a Hermitian structure on V thought of as $\mathbb{C}$-v.s. w/ J

i.e. $h : V \times V \to \mathbb{C}$ is $\mathbb{R}$-bilinear

$h(u, v) = \overline{h(v, u)}$

$h(v, v) > 0$ ← note this is real number

$h(v, Ju) = i\, h(v, u)$ (slightly different than std def^{n} of Hermitian but allows for $\operatorname{Im} h = \omega$)

Th^{m} 7:

1) (V, ω) symplectic vector space

$\mathcal{J}(V, \omega) = \{$compatible complex structures with $\omega\}$

is contractible (and non-empty)

2) (V, J) a vector space with complex structure

$\Omega(V, J) = \{$symplectic structures compatible with $J\}$

is contractible (and non-empty)

Remark: This theorem essentially says "choosing a complex structure on V is more or less the same as choosing a symplectic structure"

Proof:

1) let $I(V) = \{\text{inner products on } V\}$

ω induces a map

$$\begin{aligned} \mathcal{J}(V,\omega) &\xrightarrow{\phi} I(V) \\ J &\longmapsto g_J \end{aligned}$$

we now build a map in other direction: given $g \in I(V)$ we have

$$\begin{tikzcd} V \arrow[rr, "A"] \arrow[dr, "\phi_\omega"', "\cong"] & & V \arrow[dl, "\phi_g", "\cong"'] \\ & V^* & \end{tikzcd}$$

$A = \phi_g^{-1} \circ \phi_\omega$ isomorphism

as above $\omega(v,u) = g(Av,u)$
$g(Av,u) = -g(v,Au)$

so $-A^2$ is self-adjoint and $g(-A^2v,v) = g(Av,Av) > 0$ if $v \neq 0$

so $-A^2$ is a positive definite self-adj map

just as in proof of Lemma 4 $-A^2$ has a square root

set $J_g = (-A^2)^{-1/2}A$

commute

clearly $J_g^2 = (-A^2)^{-1/2}A(-A^2)^{-1/2}A = (-A^2)^{-1}A^2 = -\mathbb{1}$

so J_g a complex structure

and $\omega(v, J_g v) = g(Av, (-A^2)^{-1/2}Av) > 0$ if $v \neq 0$

since $Av \neq 0$ and $(-A^2)^{-1/2}$ pos-def operator

finally $\omega(J_g v, J_g u) = g(AJ_g v, J_g u) = -g(Av, J_g^2 u) = g(Av,u)$
$= \omega(v,u)$

J_g skew-adj w.r.t. g

so $J_g \in \mathcal{J}(V,\omega)$

thus we have $\psi : I(V) \to \mathcal{J}(V,\omega)$

note: $\psi \circ \phi = id$ on $\mathcal{J}(V, \omega)$

$\phi \circ \psi : I(V) \to I(V)$ is not identity but it is homotopic to identity since $I(V)$ is contractible

indeed, choose a basis and $I(V) \simeq$ {positive-def. symmetric matrices}

exercise: This is a convex open subset of all matrices
$\therefore$ contractible.

$\therefore \mathcal{J}(V, \omega) \simeq I(V)$ is contractible

proof of 2 similar

exercise:

1) Show $I(\mathbb{R}^n) \cong GL(n)/O(n)$

($\therefore$ since I contractible $O(n) \hookrightarrow GL(n)$ htp. equiv.)

2) If $\mathcal{H}(\mathbb{C}^n) = \{$Hermitian str. on $\mathbb{C}^n\}$

then $\mathcal{H}(\mathbb{C}^n) \cong GL(n, \mathbb{C})/U(n)$

and $\mathcal{H}(\mathbb{C}^n)$ contractible

($\therefore U(n) \hookrightarrow GL(n, \mathbb{C})$ htpy equiv.)

3) $\mathcal{J}(\mathbb{R}^{2n}, \omega_{std}) \cong Sp(2n)/U(n)$

so Thm 3 says $\mathcal{J}(\mathbb{R}^{2n}, \omega_{std})$ contractible

C. "Almost" Structures

let M be an n-manifold

an almost complex structure on M is a bundle map

$$\begin{tikzcd} TM \arrow[rr, "J"] \arrow[dr] & & TM \arrow[dl] \\ & M & \end{tikzcd}$$

s.t. $J^2 = -id_{TM}$ (i.e. fiberwise J_x a complex structure on T_xM)

an almost symplectic structure on M

is a non-degenerate 2-form ω (i.e. fiberwise ω_x is a sympl. str. on T_xM)

a (Riemannian) metric on M is a smoothly varying inner product on each T_xM

we call any 2 of the above compatible if they are compatible on each tangent space

an almost Hermitian structure on M is an almost complex structure and together with a smoothly varying Hermitian form on each T_xM

Th$^{\underline{m}}$ 8:

M any smooth manifold

1) space of metrics $\mathcal{M}(M)$ is non-empty and contractible

2) given an almost symplectic structure ω on M the space

$$\mathcal{J}(M,\omega) = \left\{\begin{array}{l}\text{almost complex str on } M \\ \text{compatible with } \omega\end{array}\right\}$$

is non-empty and contractible

3) given an almost complex structure J on M the space

$$\Omega(M,J) = \left\{\begin{array}{c}\text{almost symplectic str on } M \\ \text{compatible with } J\end{array}\right\}$$

is non-empty and contractible

4) given an almost complex structure J on M the space

$$\mathcal{H}(M,J) = \{\text{almost Hermitian str. on } (M,J)\}$$

is non-empty and contractible.

Remark: So upto homotopy

a) any manifold has a unique metric!

b) almost complex, almost symplectic, almost Hermetian structures are the same! (upto homotopy)

Proof: 1) note if $g_1, \dots g_n$ are inner products on V

and $t_1, \dots t_n \geq 0$ s.t. $\sum t_i = 1$

then $\sum t_i g_i$ an inner product on V

if $\{U_\alpha\}$ a cover of M by coordinant charts

let g_α be any metric on U_α (i.e. $TU_\alpha = U_\alpha \times \mathbb{R}^n$ take any I.P. on))

and $\{\rho_\alpha\}$ a partition of unity subordinant to $\{U_\alpha\}$

then $\sum \rho_\alpha g_\alpha$ is a metric on M

if g_0, g_1 are metrics on M, then so is $t g_1 + (1-t) g_0$

exercise: Show this implies $\mathcal{M}(M)$ is contractible

2), 3) follow directly from Th$^{\underline{m}}$ 7 (proof)

4) just like for 1) ∎

if (M, J) an almost complex manifold then $Y \subset M$ an almost complex submanifold if $J(TY) \subset TY$ (so $J|_{TY}$ an almost complex str on N)

exercise: 1) if J is compatible with ω on M and N an almost complex submanifold of (M,J) then N is also a symplectic submanifold

2) Show converse not true

3) If $L \subset (M, \omega)$ Lagrangian and J is compatible with ω, then $J(TL)$ is the normal bundle of L in M

(note this completes the proof of Cor III.5 about nbhds of Lagrangian submanifolds)

We will use this theorem to show there are lots of manifolds that can't be symplectic, but first we discuss how to "remove the almost"

of course, an almost symplectic structure ω is symplectic if $d\omega = 0$

recall a complex manifold is simply a (Hausdorff, 2^{nd} countable) space M with a maximal atlas of coordinant charts $\{\phi_\alpha : U_\alpha \to V_\alpha\}$ where $V_\alpha \subset \mathbb{C}^n$ and transition maps are holomorphic

complex differentiable

exercise: Show a complex manifold has an almost complex structure

an almost complex structure J on M is called integrable if M has a complex structure inducing J.

recall an almost Hermitian structure on an almost complex manifold (M,J) gives (and is determined by) a compatible almost symplectic structure ω. If J is integrable and $d\omega = 0$ then (M,J,ω) is called a Kähler manifold

example: $\mathbb{C}P^n = \mathbb{C}^{n+1} - \{(0,\dots 0)\} / \mathbb{C} - \{0\}$

has coordinate charts $\phi_i : U_i \to V_i$ where $U_i = \{[z_0 : \dots z_n] \mid z_i \neq 0\}$, $V_i = \mathbb{C}^n$

you can easily see transition maps are holomorphic

$\phi([z_0 : \dots : z_n]) = \left(z_0/z_i, \dots, \widehat{z_i/z_i}, \dots z_n/z_i\right)$

$\phi^{-1}(z_1 \dots z_n) = [z_1 : \dots : 1 : \dots : z_n]$

i^{th} entry

so $\mathbb{C}P^n$ a complex manifold

exercise: Check ω_{FS} is compatible with complex structure so $(\mathbb{C}P^n, \omega_{FS})$ a Kähler manifold and all complex submanifolds are too.

Remark: later we will see not all complex and not all symplectic manifolds are Kähler.

recall a function $f : \mathbb{C}^n \to \mathbb{C}^m : (z_1, \dots, z_n) \longmapsto (f_1(z_1 \dots z_n), \dots f_m(z_1, \dots z_n))$ with $f_j = u_j + i v_j$ is holomorphic if coordinant functions satisfy

$$\boxed{\begin{aligned} \frac{\partial u_j}{\partial x_k} &= \frac{\partial v_j}{\partial y_k} \\ \frac{\partial u_j}{\partial y_k} &= -\frac{\partial v_j}{\partial x_k} \end{aligned}} \qquad \forall k, j$$

Cauchy-Riemann Eq$^{\underline{n}}$s

exercise: If J_1 is almost complex structure on $\mathbb{C}^n$ (coming from i)
and J_2 " " " " $\mathbb{C}^m$ (" ")
then $f: \mathbb{C}^n \to \mathbb{C}^m$ is holomorphic

$$\Longleftrightarrow$$

$$df \circ J_1 = J_2 \circ df$$

now if (M, J) and (M', J') are almost complex manifolds then a function $f: M \to M'$ is called (pseudo) holomorphic (or J-holomorphic or (J,J')-holomorphic ...) if

$$\boxed{df \circ J = J' \circ df} \quad *$$

from exercise above we see for complex manifolds this is the same as being holomorphic

Remarks:

1) Given $f: M \to M'$ as above with $\dim M = 2n$, then when you write out $*$ in local coordinates you see that if
   - $n > 1$ system of P.D.E.s is overdetermined
   - $n = 1$ " " is elliptic
2) So for $n > 1$ expect No solutions. It's a miracle of complex geometry that there are any such functions but for generic almost complex structures expect no sol$^{\underline{n}}$s
3) for $n = 1$ expect sol$^{\underline{n}}$s to be "nice"
   In 1985 Gromov observed this and revolutionize symplectic geometry by studying pseudo-holomorphic curves (more on this later)
4) In 1996 Donaldson noticed you could study functions that almost solve $*$ and learn a lot about symplectic manifold (more on this later)

So when is an almost complex manifold complex?

given an almost complex manifold (M, J) define

$$N_J(v,u) = [Jv, Ju] - [v,u] - J[v, Ju] - J[Jv, u]$$

for vector fields $v, u \in \mathfrak{X}(M)$

exercise: 1) N is a tensor (i.e. $N(fv, gu) = fg\, N(v,u)$)

2) if J is integrable show $N = 0$

3) for any diffeomorphism $\phi: M \to M$: $N_{\phi^* J}(\phi^* v, \phi^* u) = \phi^* N_J(v,u)$

N_J is called the Nijehuis tensor of J

Thm (Newlander-Nirenberg '57):

an almost complex structure J on M is integrable if and only if $N_J = 0$

Proof is beyond this course (mainly P.D.E.)

let's understand this more geometrically

if J an almost complex structure on M

then eigenvalues of $J_x : T_xM \to T_xM$ are $\pm i$ (root of e.v. of $-id_{T_xM}$)

so can't diagonalize over $\mathbb{R}$

let $T_{\mathbb{C}}M = TM \otimes \mathbb{C}$ complexified tangent bundle

now we can write $T_{\mathbb{C}}M = (TM)^{(1,0)} \oplus (TM)^{(0,1)}$

eigenspace of i eigenspace of $-i$

exercise:

Show $TM \to (TM)^{(1,0)}$, $v \longmapsto \frac{1}{2}(v - iJv)$ and $TM \to (TM)^{(0,1)}$, $v \mapsto \frac{1}{2}(v + iJv)$

are linear isomorphisms

the dual of $J_x : T_x M \to T_x M$ is

$$J_x^* : T_x^* M \to T_x^* M$$

so we have a "complex structure" on $T^* M$ as well

as above we have

$$T_{\mathbb{C}}^* M = T^* M \otimes \mathbb{C} = (T^* M)^{(1,0)} \oplus (T^* M)^{(0,1)}$$

and

$$\Lambda^k T_{\mathbb{C}}^* M = \Lambda^k (T^* M \otimes \mathbb{C}) = \bigoplus_{p+q=k} \Lambda^{(p,q)} T^* M$$

$$\Omega^{p,q}(M) = \Gamma(\Lambda^{(p,q)} T^* M)$$

now if $v_1 \dots v_n$ is a basis for $T_x M$ then

$v_1, Jv_1, \dots v_n, Jv_n$ is a real basis for $(T_{\mathbb{C}} M)_x$

and

$$w_j = \tfrac{1}{2}(v_j - iJv_j) \quad j = 1 \dots n \text{ span } (TM)^{(1,0)}$$
$$\bar{w}_j = \tfrac{1}{2}(v_j + iJv_j) \quad j = 1 \dots n \text{ span } (TM)^{(0,1)}$$
$$w_j^* = v_j^* + iJ^* v_j^* \quad j = 1 \dots n \text{ span } (T^* M)^{(1,0)}$$
$$\bar{w}_j^* = v_j^* - iJ^* v_j^* \quad j = 1 \dots n \text{ span } (T^* M)^{(0,1)}$$

moreover $\eta \in \Omega^{(p,q)}(M)$ can be written

$$\eta = \sum_{|A|=p,\ |B|=q} \eta_{A,B}\, w_A^* \wedge \bar{w}_B^*$$

where $\eta_{A,B}$ functions A, B multi-index (eg $A = (i_1 \dots i_p)$, $i_j \in \{1, \dots, n\}$)

and $w_A^* = w_{i_1}^* \wedge \dots \wedge w_{i_p}^*$ etc.

exercise:

1) if $\eta \in \Omega^{p,q}(M)$ then $d\eta \in \Omega^{p+2,q-1}(M) \oplus \Omega^{p+1,q}(M) \oplus \Omega^{p,q+1}(M) \oplus \Omega^{p-1,q+2}(M)$

2) if (M, J) integrable then $d\eta \in \Omega^{p+1,q}(M) \oplus \Omega^{p,q+1}(M)$

denote $\partial : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M)$

$\bar{\partial} : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M)$ the composition of d with appropriate projection

so you showed if M complex $d = \partial + \bar{\partial}$

3) if M complex show $\partial^2 = \bar\partial^2 = \partial\circ\bar\partial + \bar\partial\circ\partial = 0$

4) Nijehuis tensor $N_J = 0 \iff (TM)^{(1,0)}$ closed under Lie bracket

$\iff \bar\partial^2 = 0$ on functions

$\iff d = \partial + \bar\partial$

$\iff d(\Omega^{p,q}(M)) \subset \Omega^{p+1,q}(M) \oplus \Omega^{p,q+1}(M)$

so all of these $\iff$ J integrable

If M is a complex manifold then $\bar\partial^2 = 0$, and we can define the Dolbeault cohomology of M to be

$$H^{p,q}_{\bar\partial}(M) = \frac{\ker(\bar\partial : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M))}{\operatorname{im}(\bar\partial : \Omega^{p,q-1}(M) \to \Omega^{p,q}(M))}$$

exercise: if $f : M \to N$ a holomorphic map between complex manifolds then $f^* : \Omega^k(N) \to \Omega^k(M)$ induces a map

$$f^* : \Omega^{p,q}(N) \to \Omega^{p,q}(M)$$

s.t. $f^*\circ\bar\partial = \bar\partial\circ f^*$

so we get a map $\quad f^* : H^{p,q}_{\bar\partial}(N) \to H^{p,q}_{\bar\partial}(M)$

Hodge Th^{m}:

> for a compact complex manifold
> $H^{p,q}_{\bar\partial}(M)$ is finite dimensional

proof is all about elliptic PDE and beyond this course but I put notes on course web page.

More Facts: if M is a compact, connected complex n-manifold with a Hermitian form, then

$$H^{n,n}_{\bar\partial}(M) \cong \mathbb{C}$$

and

$$H^{p,q}_{\bar\partial}(M) \times H^{n-p,n-q}_{\bar\partial}(M) : (\alpha,\beta) \mapsto \int_M \alpha\wedge\beta$$

is nondegenerate, so

$$H^{n-p,n-q}_{\bar\partial}(M) \cong (H^{p,q}_{\bar\partial}(M))^*$$

we write $h^{p,q} = \dim H^{p,q}_{\bar\partial}(M)$ called Hodge numbers

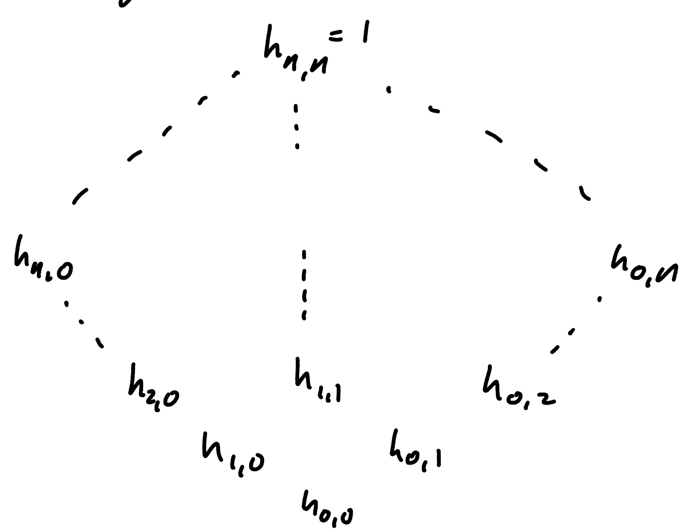
from above we have
$$h^{p,q} < \infty$$
$$h^{n,n} = 1$$
$$h^{n-p,n-q} = h^{p,q}$$

More Facts: if M is a compact Kähler n-manifold, then we have the Hodge decomposition
$$H^r_{DR}(M) \cong \bigoplus_{p+q=r} H^{p,q}_{\bar\partial}(M)$$

and
$$H^{p,q}_{\bar\partial}(M) \cong \overline{H^{q,p}_{\bar\partial}(M)}$$

and
$$h^{k,k} > 0$$

so we have the Hodge diamond:



symmetric about middle line

Cor 9:

$$\boxed{\text{for a Kähler manifold } M \text{ we have } \dim H^{2k+1}_{DR}(M) \text{ is even}}$$

Proof: Clear from symmetry of diamond ∎