D. Bundles and Structure Groups

recall given a fiber bundle
$$\begin{tikzcd} F \arrow[r] & E \arrow[d, "p"] \\ & M \end{tikzcd}$$

we have local trivializations

$U \subset M$ open and diffeomorphisms
$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\phi"] \arrow[dr, "p"'] & & U \times F \arrow[dl, "p_1"] \\ & U & \end{tikzcd}$$

and for any 2 local trivializations $(U_1, \phi_1), (U_2, \phi_2)$

transition maps
$$\begin{tikzcd} (U_1 \cap U_2) \times F \arrow[dr] & p^{-1}(U_1 \cap U_2) \arrow[l, "\phi_1"'] \arrow[r, "\phi_2"] \arrow[d] & (U_1 \cap U_2) \times F \arrow[dl] \\ & U_1 \cap U_2 & \end{tikzcd}$$

$$\phi_2 \circ \phi_1^{-1} : (U_1 \cap U_2) \times F \to (U_1 \cap U_2) \times F$$
$$(x, y) \longmapsto (x, \tau_{21}(x)(y))$$

with $\tau_{21} : U_1 \cap U_2 \to \text{Diffeo}(F)$

transition function or clutching function

note: if $\{(U_\alpha, \phi_\alpha)\}$ a collection of local trivializations such that $M = \bigcup U_\alpha$

then transition maps satisfy
$$\boxed{\begin{aligned} t_{\alpha\alpha}(x) &= id_F \\ t_{\beta\alpha}(x) &= (t_{\alpha\beta}(x))^{-1} \\ t_{\gamma\alpha}(x) &= t_{\gamma\beta}(x) \circ t_{\beta\alpha}(x) \end{aligned}} \quad (*)$$

exercise: Show that if $\{U_\alpha\}$ any cover of M by open sets and $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to \text{Diffeo}(F)$ any maps satisfying $(*)$ then $\exists$ a bundle E that realizes this data

Hint: let $E = \coprod_\alpha (U_\alpha \times F)/\sim$ where $(x,y) \in U_\alpha \times F \sim (x', y') \in U_\beta \times F$ iff $\tau_{\beta\alpha}(x)(y) = y'$, $x = x'$

there is an obvious projection $E \to M$

exercise: Find a cover and the transition maps for

$$\begin{tikzcd} S^1 \arrow[r] & S^{2n+1} \arrow[d] \\ & \mathbb{C}P^n \end{tikzcd} \qquad \begin{tikzcd} \mathbb{R}^n \arrow[r] & TS^n \arrow[d] \\ & S^n \end{tikzcd} \qquad \begin{tikzcd} \mathbb{R}^{2n} \arrow[r] & T\mathbb{C}P^n \arrow[d] \\ & \mathbb{C}P^n \end{tikzcd}$$

suppose $G \subset \mathrm{Diffeo}(F)$ is a sub-Lie group (we will only consider closed subgroups)

If a bundle $\begin{matrix} F \to E \\ \downarrow p \\ M \end{matrix}$ has a collection of transition functions

$$\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to G$$

then we say E has <u>structure group</u> <u>G</u>

if the transition functions can be homotoped to lie in G via a homotopy that always satisfies (*) then we say the structure group <u>reduces</u> to G

<u>note</u>: If G preserves some structure on F then the fibers of E all have this structure!

<u>examples</u>:

1) if $F = \mathbb{R}^n$ and $G = GL(n,\mathbb{R}) \subset \mathrm{Diffeo}(\mathbb{R}^n)$ then each fiber of E has a linear structure i.e. E is a <u>vector bundle</u>

2) if $F = \mathbb{R}^n$ and $G = GL^+(n,\mathbb{R})$, then E is an <u>oriented vector bundle</u>

3) if $F = \mathbb{R}^n$ and $G = O(n,\mathbb{R})$, then E is a <u>vector bundle with a metric</u>

   <u>note</u>: $O(n) \hookrightarrow GL(n,\mathbb{R})$ a homotopy equivalence
   $\Rightarrow$ all vector bundles have metrics!

4) $F = \mathbb{R}^{2n}$, $G = GL(n,\mathbb{C}) \iff E$ <u>complex structure</u>
   $G = Sp(2n) \iff E$ <u>symplectic structure</u>
   $G = U(n) \iff E$ <u>Hermitian structure</u>

   $U(n) \subset GL(n,\mathbb{C})$, $U(n) \subset Sp(2n)$ htpy equivalences
   $\Rightarrow$ all these structures same

5) if $F = \mathbb{R}^n$ we have $GL(k) \times GL(n-k) \subset GL(n)$
$$(A,B) \longmapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

E has structure group $GL(k) \times GL(n-k) \iff E = E_1 \oplus E_2$ where
E_1 a vector $\mathbb{R}^k$ bundle
E_2 a vector $\mathbb{R}^{n-k}$ bundle

specifically $GL(n-k) \hookrightarrow GL(n)$
$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & I_n \end{pmatrix}$$
so E has structure group $GL(n-k)$
$\iff$
E has a <u>k-frame</u> ← k linearly independent sections
$\iff$
$E = E' \oplus \mathbb{R}^k$ with E' a vector $\mathbb{R}^{n-k}$-bundle

So when can you reduce the structure group?

to systematically study this we need a new idea if G is a Lie group then a bundle $\begin{array}{c} G \to P \\ \downarrow p \\ M \end{array}$ is a principal G-bundle

if $\exists$ a smooth right action

$$P \times G \to P$$

s.t. 1) $y \in p^{-1}(x) \Rightarrow y \cdot g \in p^{-1}(x)$ $\forall g, x, y$ (i.e. action preserves fibers)

2) G acts freely and transitively on $p^{-1}(x)$ $\forall x$

Remark: Can more consisely define a principal G-bundle as a smooth manifold P with a smooth right G action $P \times G \to P$ that is free and proper

proper ← $P \times G \to P \times P$ is proper ← inverse image of compact is compact
$(p, g) \mapsto (p \cdot g, p)$

examples:

1) if $\begin{array}{c} F \to E \\ \downarrow \\ M \end{array}$ is a bundle with structure group G

then $\exists$ a cover of M by trivializations $\{(U_\alpha, \phi_\alpha)\}$
with transition functions $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to G$

we can construct a principal G-bundle as follows

$$P_E = \coprod_\alpha U_\alpha \times G / \sim \qquad \text{where } (x, g) \in U_\alpha \times G \sim (x', g') \in U_\beta \times G$$
$$\Leftrightarrow$$
$$g' = \underbrace{\tau_{\beta\alpha}(x)}_{\in G} \cdot (g), \quad x = x'$$

exercise: Check P_E is a principal G-bundle

If E is a vector bundle then P_E is a principal $GL(n, \mathbb{R})$-bundle
it is called the frame bundle because you can think of the fibers of P_E as frames for the fibers of E

exercise: think through this

we denote this bundle $\mathcal{F}(E)$

note: $O(n) \simeq GL(n, \mathbb{R})$ so we could look at orthonormal frame bundle with fiber $O(n)$ (still denote $\mathcal{F}(E)$)

2) $\begin{array}{c} S^1 \to S^{2n+1} \\ \downarrow \\ \mathbb{C}P^n \end{array}$ is a principal S^1-bundle

3) Regular covering spaces of manifolds M are principal bundles

exercise: Check this. What are the fibers?
Can irregular covers be principal bundles over M?

exercises:

1) Show a principal G-bundle is trivial $\Leftrightarrow$ it admits a section

2) If E is a vector bundle then show a section of E is the same as a $GL(n,\mathbb{R})$-equivariant map:

$$v: \mathcal{F}(E) \to \mathbb{R}^n$$
$$v(y\cdot g) = g^{-1} v(y)$$

Hint: given $s: M \to E$
then for each $y \in \mathcal{F}(E)$ let $v(y) = s(p(y))$ expressed in frame y

projection $\mathcal{F}(E) \to M$

Construction

Given $\begin{array}{c} P \\ \downarrow \\ M \end{array}$ a principal G-bundle

and $\rho: G \to G'$ a homomorphism (of Lie groups) where $G' \subset \mathrm{Diffeo}(F)$

then we can construct an F bundle with structure group G'

$$P \times_\rho F = P \times F \big/ (p\cdot g, f) \sim (p, \rho(g)(f))$$

exercises:

1) Describe $P \times_\rho F$ using local trivializations
2) if $F = G'$ then $P \times_\rho G'$ is a principal G'-bundle
3) if E a vector bundle, then $E \cong \mathcal{F}(E) \times_\rho \mathbb{R}^n$ where $\rho = id_{GL(n,\mathbb{R})}$
4) recall $GL(n,\mathbb{R})$ acts on $(\mathbb{R}^n)^*$ in a natural way

i.e. $GL(n,\mathbb{R}) \xrightarrow{\rho^*} GL((\mathbb{R}^n)^*) = GL(n,\mathbb{R})$

check $T^*M = \mathcal{F}(TM) \times_{\rho^*} (\mathbb{R}^n)^*$

5) Similarly $GL(n,\mathbb{R})$ acts on $\Lambda^k(\mathbb{R}^n)^*$ in a natural way

i.e. $GL(n,\mathbb{R}) \xrightarrow{\rho_k} GL(\Lambda^k(\mathbb{R}^n)^*)$

check $\Lambda^k(T^*M) \cong \mathcal{F}(TM) \times_{\rho_k} \Lambda^k(\mathbb{R}^n)^*$

now given a principal G-bundle $\begin{array}{c} P \\ \downarrow \\ M \end{array}$ and a subgroup $H < G$

if $\exists$ a principal H-bundle $P_H \subset P$ then one can check

$P_H \times_H G \longrightarrow P : [f,g] \mapsto f\cdot g$ is a bundle isomorphism

H acts on G by multiplication

this isomorphism shows the transition functions for P could be chosen to have image in H

so we say the structure group of P reduces to H in this case

<u>note</u>: If the structure group of $\mathcal{F}(E)$ reduces from $GL(n,\mathbb{R})$ to H then so does the structure group of E:

$$\mathcal{F}(E)_H \times_H \mathbb{R}^n$$

$\to$ $H < GL(n,\mathbb{R})$ acts on $\mathbb{R}^n$

so we have turned questions about the structure group of E into questions about the structure group of principal bundles

now given a principal G-bundle $\begin{array}{c} P \\ \downarrow \\ M \end{array}$ and a subgroup $H < G$

we get the bundle $\begin{array}{c} P/H \\ \downarrow \\ M \end{array}$ with fibers G/H

<u>lemma 10</u>:

> let P be a principal G-bundle and $H < G$
> reductions of the structure group of P to H are in one-to-one correspondance with sections of P/H

<u>Proof</u>: $(\Rightarrow)$ given a reduction we have

$$\begin{tikzcd} P_H \arrow[rr, hook] \arrow[dr] & & P \arrow[dl] \\ & M & \end{tikzcd}$$

and so

$$\begin{tikzcd} P_H/H \arrow[rr, hook] \arrow[dr, "\approx"'] & & P/H \arrow[dl] \\ & X \arrow[ur, bend left, "\text{section}"] & \end{tikzcd}$$

$(\Leftarrow)$ $P \xrightarrow{\pi} P/H$ is a principal H-bundle

if $s: M \to P/H$ a section, then $\bigcup_{x \in M} \pi^{-1}(s(x)) \subset P$

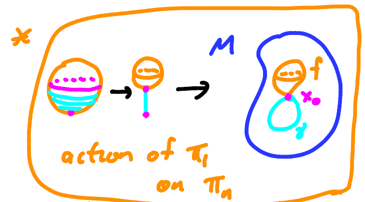
is a principal H-bundle

example:

since $GL(n,\mathbb{R})/O(n)$ is contractible and bundles with contractible fibers always have sections (check this) we see $\mathcal{F}(E)/O(n)$ has sections and so all vector bundles have metrics!

So how can we tell if P/H has sections?

E. Obstruction Theory



We want to study sections of a fiber bundle
$$\begin{array}{ccc} F & \to & E \\ & & \downarrow p \\ & & M \end{array}$$

we assume: A) M is a CW complex (always true for manifolds)

B) F is n-simple for all n (examples H-spaces, so Lie groups, loop spaces...)

i.e. action of $\pi_1(F,x_0)$ on $\pi_n(F,x_0)$ trivial *

$\Rightarrow$ $\pi_1(F,x_0)$ abelian and $\pi_n(F,x_0) \cong [S^n, F]$ homotopy classes of maps $S^n \to F$

canonical so π_n indep. of x_0

$$\begin{array}{ccc} & & E \\ & \nearrow & \downarrow \\ [0,1] & \xrightarrow{\gamma} & M \end{array}$$

Homotopy lifting $\Rightarrow$ $\tilde{\gamma}: F\times[0,1]\to E$

$\tilde{\gamma}(x,1): F\to F$ induces map $\tilde{\gamma}(\cdot,1)_*: \pi_n(F)\circlearrowleft$ **

not so important assumption

C) action of $\pi_1(M)$ on $\pi_n(F)$ trivial **

so $\pi_n(p^{-1}(x))$ canonically independent of $x\in M$

(can get around this using "cohomology with local coeff.")

Denote the n-skeleton of M by $M^{(n)}$

assuming we have a section $s_k: M^{(k)}\to E$ we define a cohomology cochain
$$\tilde{o}(s_k)\in C^{k+1}(M;\pi_k(F))$$

as follows

recall $\tau\in C^{k+1}(M;\pi_k(F))$ simply a homomorphism $\tau: C_{k+1}(M)\to\pi_k(F)$

where $C_{k+1}(M)$ is the CW-chain group

which is generated by $k+1$ cells $e_1^{k+1},\dots e_\ell^{k+1}$ of M

(recall $M^{(k+1)} = M^{(k)}\cup e_1^{k+1}\cup\dots\cup e_\ell^{k+1}/\sim$

where $e_i^{k+1} = D^{k+1}$ and $\exists\ a_i: \partial e_i^{k+1}\to M^{(k)}$

s.t. $\sim$ above is gluing e_i^{k+1} to $M^{(k)}$ by a_i)

let $I_i: e_i^{k+1}\to M$ be "inclusion"

$$\begin{array}{ccc} I_i^*E & \cong & D^{k+1}\times F \\ \downarrow & \swarrow & \\ e_i^{k+1} = D^{k+1} & & \end{array}$$
since e_i^{k+1} contractible

s_k pulls back to a section of $I_2^* E$ along ∂e_2^{k+1}

so $p_2 \circ s_k : \underset{\overset{\shortparallel}{S^k}}{\partial e_2^{k+1}} \to F$ gives an element of $\pi_k(F)$

(here $p_2 : e_2^{k+1} \times F \to F$ is projection)

now define $\tilde{\sigma}(s_k)(e_2^{k+1}) = [p_2 \circ s_k]$

assumptions above say this is well-def

so $\tilde{\sigma}(s_k) \in C^{k+1}(M; \pi_k(F))$

exercises:

1) $\tilde{\sigma}(s_k)$ invariant under homotopies of s_k
2) $\tilde{\sigma}(s_k) = 0 \iff s_k$ extends over $M^{(k+1)}$
3) $\delta \tilde{\sigma}(s_k) = 0$ (i.e. $\tilde{\sigma}(s_k)$ a cocycle)
4) if s_k and s_k' are sections of E over $M^{(k)}$ that are homotopic on $M^{(k-1)}$ then

$$\tilde{\sigma}(s_k) - \tilde{\sigma}(s_k') = \delta \tau(s_k, s_k')$$

for some $\tau(s_k, s_k') \in C^k(M; \pi_n(Y))$

5) by varying homotopy class of s_k on $M^{(k)}$ (relative to $M^{(k-1)}$) we can change $\tilde{\sigma}(s_k)$ by any coboundary

the above proves

Thm:

given a bundle $\begin{matrix} F \to E \\ \downarrow \\ M \end{matrix}$ satisfying A)–C) above

and a section $s_k : M^{(k)} \to E$ then $s_k|_{M^{(k-1)}}$ extends to $M^{(k+1)}$

$$\iff$$

$$\sigma(s_k) = [\tilde{\sigma}(s_k)] = 0 \in H^{k+1}(M; \pi_k(F))$$

So we have an obstruction to a section existing!

Remark: if $\pi_k(F) = 0$ for $k < \dim B$, then $\exists$ a section of $\begin{matrix} F \to E \\ \downarrow \\ M \end{matrix}$!

note: $\sigma(s_k)$ depends on $s_k|_{M^{(k-1)}}$ i.e. it is no just about whether there is a section of E over $M^{(k+1)}$ but wheather our choice of section on $M^{(k)}$ (when restricted to $M^{(k-1)}$) extends to $M^{(k+1)}$

luckily the "first obstruction" is independent of any choices

Th$^{\underline{m}}$:

given $\begin{array}{c} F \to E \\ \downarrow \\ M \end{array}$ satisfying A)–C)

if $\pi_k(F) = 0$ for $k < n$, then $\exists$ a section $s_n : M^{(n)} \to E$

and the obstruction $\sigma(s_n)$ does not depend on s_n

(well-defined indep. of choices)

so we denote it $\gamma^{n+1}(E)$ (called primary obstruction)

and if $f : N \to M$ a map then $\gamma^{n+1}(f^*E) = f^*\gamma^{n+1}(E)$

example

recall a vector bundle $\begin{array}{c} \mathbb{R}^n \to E \\ \downarrow \\ M \end{array}$

has a k-frame $\Leftrightarrow$ structure group reduces to $GL(n-k)$

or first put a metric on E so structure group is $O(n)$ then

E has an orthonormal k-frame $\Leftrightarrow$ structure group reduces to $O(n-k)$

in terms of principal bundles, let $\mathcal{F}(E)$ be the o.n. frame bundle

(i.e. principal $O(n)$ bundle associated to E)

now E has an orthonormal k-frame $\Leftrightarrow$ $\mathcal{F}(E)/O(n-k)$ has a section

fibers of $\mathcal{F}(E)/O(n-k)$ are $O(n)/O(n-k)$ $\Leftarrow$ called Stiefel manifold $V_k(\mathbb{R}^n)$ = space of o.n. k-frames in $\mathbb{R}^n$

exercise: $\pi_i(V_k(\mathbb{R}^n)) = 0 \quad i < n-k$

$$\pi_{n-k}(V_k(\mathbb{R}^n)) \cong \begin{cases} \mathbb{Z} & n-k \text{ even or } k=1 \\ \mathbb{Z}/2\mathbb{Z} & n-k \text{ odd} \end{cases}$$

$O(n)/O(n-1) \cong S^{n-1}$

unfortunately $\pi_1(M)$ does not necissarily act trivially on $\pi_{n-k}(V_k(\mathbb{R}^n))$ when it is $\mathbb{Z}$ (see Steenrod book)

but if we take this mod 2 it will (only autom. of $\mathbb{Z}/2\mathbb{Z}$ is id.)

so we have a primary obstruction to a k-frame over the $n-k+1$ skeleton of M

$$\gamma_{n-k+1}(E) \in H^{n-k+1}(M; \pi_{n-k}(V_k(\mathbb{R}^n) \bmod 2)$$

set $w_\ell(E) = \gamma_\ell(E) \in H^\ell(M; \mathbb{Z}/2\mathbb{Z})$

this is called the ℓ^{th} Stiefel-Whitney class of E

when ℓ even (so $n-k$ odd where $\ell = n-k+1$) $w_\ell(E)$ is the primary obstruction to $\exists$ of a $n-\ell+1$ frame on $M^{(\ell-1)}$ that extends to $M^{(\ell)}$.

in general it is a "reduction" of this

Fact: (Steenrod) the w_i determine all the primary invariants

exercises:

1) given $\begin{array}{c} \mathbb{R}^n \to E \\ \downarrow \\ M \end{array}$, $w_1(E) = 0 \Leftrightarrow \exists$ an n-frame over $M^{(0)}$ that extends over $M^{(1)}$

$\Leftrightarrow E$ is orientable

2) if E orientable, then $w_2(E) = 0 \Leftrightarrow \exists$ an $(n-1)$-frame over $M^{(1)}$ that extends over $M^{(2)}$

$\Leftrightarrow \exists$ an n-frame over $M^{(1)}$ that extends over $M^{(2)}$

this is called a spin structure

another way to think of Stiefel-Whitney classes

$\exists$ unique functions $w_i : \mathrm{Vect}(M) \to H^i(M; \mathbb{Z}/2)$ $\forall M$

satisfying

1) $w_i(f^*E) = f^* w_i(E)$ $\forall f: M \to N$

2) $w_0(E) = 1$, $w_i(E) = 0$ $\forall i >$ fiber dim E

3) $w(E_1 \oplus E_2) = w(E_1) \cup w(E_2)$

where $w(E) = 1 + w_1(E) + w_2(E) + \ldots$

4) $w_1(\gamma) \neq 0$ where γ is the universal line bundle over $\mathbb{R}P^\infty$

for 4) recall $\gamma_n = \{(\ell, v) \in \mathbb{R}P^n \times \mathbb{R}^{n+1} : v \in \ell\}$

this is a line bundle over $\mathbb{R}P^n$

4) $\Leftrightarrow$ $w_1(\gamma_n)$ generates $H^1(\mathbb{R}P^n; \mathbb{Z}/2)$ $\forall n$

sample computation recall $H^*(\mathbb{R}P^n; \mathbb{Z}/2) \cong \mathbb{Z}_2[a]/_{a^{n+1}=1}$

degree $a = 1$

$$w(T\mathbb{R}P^n) = (1+a)^{n+1} = 1 + \binom{n+1}{1} a + \binom{n+1}{2} a^2 + \ldots$$

eg $w_1 = (n+1)a$ so $\mathbb{R}P^n$ orientable $\Leftrightarrow$ n odd

example: if $\begin{array}{c} \mathbb{R}^n \to E \\ \downarrow \\ M \end{array}$ an oriented bundle

then $\pi_1(M)$ acts trivially on $\pi_{n-1}(V_1(\mathbb{R}^n)) \cong \mathbb{Z}$

exercise: check this

so we get an obstruction $e(E) \in H^n(M; \mathbb{Z})$

to the existence of a non-zero section of E

$e(E)$ is called the Euler class of E

exercise: 1) if $s: M \to E$ is any section of E that is transverse to the zero section Z, then

$$e(E) = P.D.[s^{-1}(Z)]$$

2) $e(TM)([M]) = \chi(M)$

fundamental class

Euler characteristic

example: let $\mathbb{C}^n \to E$ be a vector bundle with structure group $GL(n;\mathbb{C})$
$\qquad\qquad\downarrow$
$\qquad\qquad M$

that is a "complex bundle"

(from above we can assume $U(n)$)

so the frame bundle can be taken to be a principal $U(n)$-bundle

as in the real case E will have a complex k-frame

$\Leftrightarrow$

$\mathcal{F}(E)/U(n-k)$ has a section (this as a $U(n)/U(n-k)$ bundle)

exercise:

1) $\pi_i\left(U(n)/U(n-k)\right) = \begin{cases} 0 & i < 2(n-k) \\ \mathbb{Z} & i = 2(n-k)+1 \end{cases}$

2) $\pi_1(M)$ acts trivially on $\pi_{2(n-k)+1}\left(U(n)/U(n-k)\right)$ where we think of $U(n)/U(n-k)$ as the fiber of $\mathcal{F}(E)/U(n-k)$

thus the primary obstruction to a complex k-frame is

$$\gamma_{2(n-k)+2} \in H^{2(n-k)+2}(M; \underbrace{\pi_{2(n-k)+1}(U(n)/U(n-k))}_{\mathbb{Z}})$$

we define $c_k(E) = \gamma_{2k}(E) \in H^{2k}(M;\mathbb{Z})$

this is the k^{th} Chern class of E

clearly $c_k(E)$ is the obstruction to a complex $(n-k+1)$ frame on $M^{(2k-1)}$ that extends to $M^{(2k)}$

exercise: if E a complex $\mathbb{C}^n$-bundle over M

1) $c_n(E) = e(E)$
2) $w_{2i+1}(E) = 0$ $\qquad$ ($\Rightarrow$ complex bundles are oriented)
3) $w_{2i}(E) = c_i(E) \mod 2$
4) $c_1(E) = 0 \Leftrightarrow$ structure group of E reduces to $SU(n)$
"complex orientation"
5) if $\bar{E}$ is E with "conjugate complex structure" (ie. $-J$) then $c_i(\bar{E}) = (-1)^i(E)$ Hint: easy for $c_n(E)$, reduce to this. See Milnor-Stasheff

another way to think of Chern classes

$\exists$ unique functions $\quad c_i : \mathrm{Vect}_{\mathbb{C}}(M) \to H^{2i}(M;\mathbb{Z}) \quad \forall M$

satisfying

1) $c_i(f^*E) = f^* c_i(E) \quad \forall f: M \to N$
2) $c_0(E) = 1, \; c_i(E) = 0 \quad \forall i >$ fiber $\mathbb{C}$-dim E
3) $c(E_1 \oplus E_2) = c(E_1) \cup c(E_2)$

   where $c(E) = 1 + c_1(E) + c_2(E) + \dots$
4) $c_1(\gamma)$ generates $H^2(\mathbb{C}P^\infty)$, where γ is the universal $\mathbb{C}$-line bundle over $\mathbb{C}P^\infty$

sample computation $\quad$ recall $\quad H^*(\mathbb{C}P^n;\mathbb{Z}) \cong \mathbb{Z}[a]/a^{n+1}=1$

degree $a = 2$

when $E=TM$ we denote $c(TM)$ by $c(M)$

$$c(\mathbb{C}P^n) = (1+a)^{n+1} = 1 + \binom{n+1}{1} a + \binom{n+1}{2} a^2 + \dots$$

eg $\quad c_1 = (n+1)a \quad$ so $\quad c_1(\mathbb{C}P^1) = 2a \quad (\chi(S^2) = 2)$

$c_1(\mathbb{C}P^2) = 3a \quad$ and $\quad c_2(\mathbb{C}P^2) = 3a^2$

there is one more "standard" characterisic class

given a real vector bundle $\quad \mathbb{R}^n \to E \to M$

then $E \otimes_{\mathbb{R}} \mathbb{C}$ is a complex vector bundle

the i^{th} Pontrjagin class of E is

$$p_i(E) = (-1)^i c_{2i}(E \otimes \mathbb{C})$$

exercises:

1) Show $E \otimes \mathbb{C}$ and $\overline{E \otimes \mathbb{C}}$ are isomorphic

   use this to show $c_{2i+1}(E \otimes \mathbb{C})$ is 2-torsion

2) $\left(1 - p_1(\mathbb{C}P^n) + p_2(\mathbb{C}P^n) - \dots \pm p_n(\mathbb{C}P^n)\right) = (1 - a^2)^{n+1}$

   where a generates $H^2(\mathbb{C}P^n)$

   so $\quad p_1(\mathbb{C}P^2) = 3a^2, \; p_1(\mathbb{C}P^3) = 4a^2, \; \dots$

3) if E an oriented $2n$-bundle then $p_n(E) = e(E) \cup e(E)$

4) if E is complex bundle and $E^{\mathbb{R}}$ denotes underlying real bundle then $E^{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \cong E \oplus \bar{E}$

as complex bundles

5) if E is a $\mathbb{C}^n$ bundle then

$$(1 - p_1(E) + p_2(E) \dots \pm p_n(E)) = (1 + c_1(E) + \dots + c_n(E)) \cup (1 - c_1(E) \dots \pm c_n(E))$$

eg. $p_1(E) = c_1(E) \cup c_1(E) - 2c_2(E)$

Recall, if M is a closed, oriented $4n$-manifold then

$$\hat{H}^{2n}(M) \times \hat{H}^{2n}(M) \xrightarrow{I} \mathbb{Z} \qquad \text{where } \hat{H}^{2n}(M) = H^{2n}(M)/\text{torsion}$$

$$(\alpha, \beta) \longmapsto \underbrace{\alpha \cup \beta}_{H^{4n}(M)} (\underbrace{[M]}_{H_{4n}(M)})$$

is a symmetric non-degenerate pairing

this can be diagonalized over $\mathbb{R}$

the signature of M is

$$\sigma(M) = \#\text{positive eigenvalues of } I - \#\text{ negative eigenvalues of } I$$

Hirzebruch Signature Theorem:

> If M is a closed oriented smooth $4n$ manifold, then there is a polynomial L_n in the Pontrjagin classes s.t.
> $$\sigma(M) = L_n([M])$$

for example $L_1 = \frac{1}{3} p_1(M)$

$$L_2 = \frac{1}{45}(7p_2(M) - p_1(M) \cup p_1(M))$$

note: $Th^{m} \Rightarrow$ you get integer classes even though there are fractions in formula!

can use this to show there are manifolds homeomorphic but not diffeomorphic to the 7-sphere (and higer dim'l spheres)

Application: S^4 does not have an almost complex structure!

to see this suppose it does, then

$$p_1(S^4) = c_1^2(S^4) - 2c_2(S^4)$$

evaluate on $[S^4]$ to get

$$3\sigma(S^4) = c_1^2(S^4)([S^4]) - 2\chi(S^4)$$

since $H^2(S^4) = 0$ we see $c_1^2(S^4)([S^4]) = 0$, so

$$3\sigma(S^4) = -2\chi(S^4)$$

but $\sigma(S^4) = 0$ and $\chi(S^4) = 2$ $\times$ so S^4 has no almost $\mathbb{C}$-str.

Fact: Can use same argument to see S^{4n} doesn't have a complex structure

with more work can show

$$\boxed{S^n \text{ has an almost complex structure} \iff n = 2, 6}$$

"Open" Problem: Does S^6 have a complex structure

Characteristic classes, in general, do not determine a bundle, but we do have

I) Complex line bundles are determined by c_1
Moreover, any $\alpha \in H^2(M)$ is c_1 of some complex line bundle

II) $\mathbb{C}^2$-bundles are determined by c_1 and c_2
Moreover, $\forall (\alpha, \beta) \in H^2(M) \times H^4(M)$ $\exists$ a $\mathbb{C}^2$-bundle $\begin{matrix} E \\ \downarrow \\ M \end{matrix}$ s.t. $c_1(E) = \alpha$, $c_2(E) = \beta$

III) $SO(3)$ bundles are isomorphic $\iff$ w_2 and p_1 agree

IV) $SO(4)$ bundles are isomorphic $\iff$ w_2, p_1 and e agree

exercise: prove the above. I)–II) "easy" (we will do I) later)
III)–IV) harder

F. Existence of Almost Complex Structures

We want to see when M admits an almost complex structure (and hence an almost symplectic one)

We start with an oriented manifold M of dimension $2n$

so we can assume its structure group is $SO(2n)$ and $\mathcal{F}(TM)$ is a principal $SO(2n)$ bundle

thus M has an almost complex structure $\Leftrightarrow$ $\mathcal{F}(TM)/U(n)$ has a section

the fibers of this bundle are $SO(2n)/U(n)$ so we need

for $i < 2n-1$

$$\pi_i\left(SO(2n)/U(n)\right) = \begin{cases} \mathbb{Z} & i \equiv 2 \mod 4 \\ \mathbb{Z}/2\mathbb{Z} & i \equiv 0,7 \mod 8 \\ 0 & \text{otherwise} \end{cases}$$

Bott '59

and

$$\pi_{2n-1}\left(SO(2n)/U(n)\right) = \begin{cases} \mathbb{Z} + \mathbb{Z}/2\mathbb{Z} & n \equiv 0 \mod 4 \\ \mathbb{Z}/(n-1)!\,\mathbb{Z} & n \equiv 1 \mod 4 \\ \mathbb{Z} & n \equiv 2 \mod 4 \\ \mathbb{Z}/\frac{(n-1)!}{2}\mathbb{Z} & n \equiv 3 \mod 4 \end{cases}$$

Massey '61

so there are obstruction to M having an almost complex structure in

$\gamma_i^{\mathbb{C}} \in H^i(M;\mathbb{Z})$ for $i < 2n$ and $i \equiv 3 \mod 4$

$\gamma_i^{\mathbb{C}} \in H^i(M;\mathbb{Z}/2\mathbb{Z})$ for $i < 2n$ and $i \equiv 0,1 \mod 8$

recall the exact sequence $0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$ gives rise to a long exact sequence

$$\cdots \to H^n(M;\mathbb{Z}) \to H^n(M;\mathbb{Z}) \to H^n(M;\mathbb{Z}/2\mathbb{Z}) \xrightarrow{\beta} H^{n+1}(M;\mathbb{Z}) \to \cdots$$

β is called the Bockstein homomorphism

the integral Steifel-Whitney classes of a bundle $\begin{matrix} E \\ \downarrow \\ M \end{matrix}$ are

$$W_i(E) = \beta(w_i(E)) \in H^i(M;\mathbb{Z})$$

note: $W_i(E)$ is obstruction to integral lift of $w_{i-1}(E)$

Theorem (Massey):

let M be an oriented $2n$-manifold

let $s: M^{(4k+2)} \to \mathcal{F}(M)/U(n)$ be a section

Then
$$W_{4k+3}(M) = \begin{cases} (2k)!\ \gamma^{\mathbb{C}}_{4k+3}(s) & k \text{ even} \\ \frac{1}{2}(2k)!\ \gamma^{\mathbb{C}}_{4k+3}(s) & k \text{ odd} \end{cases}$$

Remarks:

1) Amazing $\gamma^{\mathbb{C}}_{4k+3}(s)$ doesn't really depend on s!

2) If $H^{4k+3}(M;\mathbb{Z})$ has no p-torsion for any prime $p \le 2k$

   then $W_{4k+3} = 0 \Rightarrow \gamma^{\mathbb{C}}_{4k+3}(s) = 0$

3) $\gamma^{\mathbb{C}}_3(s) = W_3(M)$ and $\gamma^{\mathbb{C}}_7(s) = W_7(M)$

note:

a) no obstruction to oriented surface having almost complex structure

b) when M is an oriented 4-manifold we have

$$\pi_k\left(SO(4)/U(2)\right) = \begin{cases} 0 & k=1 \\ \mathbb{Z} & k=2,3 \end{cases}$$

so a 4-manifold always has an almost complex structure on $M^{(2)}$

obstruction to extending over $M^{(3)}$ is

$$\gamma^{\mathbb{C}}_3(M) = W_3(M)$$

so have almost complex structure on $M^{(3)}$
$$\Longleftrightarrow$$
$\exists$ integral lift of $w_2(M)$

we consider obstruction to extension to M later

c) When M is an oriented 6-manifold we have

$$\pi_k\left(SO(6)/U(3)\right) = \begin{cases} 0 & k=1,2,4,5 \\ \mathbb{Z} & k=3 \end{cases}$$

so the first and only obstruction to an almost complex structure on M^6 is

$$\gamma^{\mathbb{C}}_3(TM) = W_3(M)$$

When M is $4n$-dimensional the top dimensional obstruction is in

$$H^{4n}(M; \underbrace{\pi_{2n-1}(O(4n)/U(2n))}_{=\begin{cases} \mathbb{Z} & n \text{ odd} \\ \mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z} & n \text{ even}\end{cases}})$$

let $\gamma^{\mathbb{Z}}_{4n}(M)$ be $\gamma^{\mathbb{C}}_{4n}(M)$ if n odd and the $H^{4n}(M;\mathbb{Z})$ component of $\gamma^{\mathbb{C}}_{4n}(M)$ if n even

recall if TM is a complex bundle

$$p_k(M) = (-1)^k \sum_{i+j=2k} (-1)^i c_i(M) \cup c_j(M)$$

Th$^{\underline{m}}$ (Massey):

> let M be an oriented $4n$-manifold
>
> let $s: M^{(4n-1)} \to \mathcal{F}(M)/U(n)$ be a section
>
> Then
>
> $$4\cdot\gamma^{\mathbb{Z}}_{4n}(s) = (-1)^{n+1} p_n(M) + \sum_{i+j=2n} (-1)^i c_i(M) \cup c_j(M)$$
>
> where the $c_i(M)$ are the Chern classes of $TM|_{M^{(2n-1)}}$ coming from the complex structure induced by s and $c_{2n}(M) = e(M)$

The discussion above covers all the "integral obstructions" to an almost complex structre the non-integral obstructions are harder

But we do get

Th$^{\underline{m}}$ II:

> A closed oriented 4-manifold M has an almost complex structure
>
> $\Leftrightarrow$
>
> $\exists$ a cohomology class $a \in H^2(M;\mathbb{Z})$ such that
>
> $w_2(M) \equiv a \mod 2$ and
>
> $a^2([M]) = 3\sigma(M) + 2\chi(M)$
>
> Moreover, any such a is c_1 of an almost complex structure

sometimes called Wu's theorem

Proof: ($\Rightarrow$) follows from discussion of characteristic classes from Section E

($\Leftarrow$) follows from 2 th$^{\underline{m}}$'s of Massey above for the last statement, prove using last exercise in Section E (about $SO(4)$ and $U(2)$ bundles)

exercise: Do this and prove $\Leftarrow$ result in Th^{m} in the same way ▦

Th^{m} 12:

A closed oriented 6-manifold M has an almost complex structure

$\Leftrightarrow$

$\exists$ a cohomology class $a \in H^2(M;\mathbb{Z})$ such that

$w_2(M) \equiv a \mod 2$

Moreover, if $H^2(M;\mathbb{Z})$ has no 2-torsion, then there is a one-to-one correspondence

$$\left\{\begin{matrix}\text{almost complex} \\ \text{structures on } M\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}a \in H^2(M;\mathbb{Z}) \text{ with} \\ w_2(M) \equiv a \mod 2\end{matrix}\right\}$$

and complex structure corresponding to a has

$c_1 = a, \quad c_2 = (a^2 - p_1(M))/2, \quad \text{and } c_3 = e(M)$

Proof: $\Leftarrow$ follows as in last proof

the one-to-one correspondance follows from more obstruction theory ▦

Recall: All statements in Th^{m} 11 and 12 also hold for almost symplectic manifolds

—— // ——

So what closed oriented 4-manifolds might have symplectic structures?

1) S^4 : No for many reasons: no $a \in H^2(S^4)$ with $a \cup a \neq 0$

and if $a \in H^2(S^4)$ with $w_2 \equiv a \mod 2$

then $2\chi + 3\sigma = 4 \neq a^2[S^4] = 0$

so no almost symplectic structure

2) $S^1 \times S^3$: does has almost symplectic structure since

$w_2(S^1 \times S^3) = 0$ so $a = 0 \in H^2(S^1 \times S^3)$ is a lift of w_2

and

$2\chi + 3\sigma = 0 = a^2([S^1 \times S^2])$

so by Th^{m} $\exists$ almost complex/symplectic structure

(in fact easy to put complex structure on $S^1 \times S^3$)

but no $a \in H^2(S^1 \times S^2)$ with $a \cup a \neq 0$ so No symplectic structure

3) $\#_n \mathbb{CP}^2$: when does it have an almost symplectic structure?

$H^2(\#_n \mathbb{CP}^2) = \oplus_n \mathbb{Z}$ Intersection pairing $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$

so $\sigma = n$ and $\chi = n+2$

thus $2\chi + 3\sigma = 2n+4+3n = 5n+4$

also $w_2(\#_n \mathbb{CP}^2) = (1,1,\dots 1) \in H^2(\#_n \mathbb{CP}^2; \mathbb{Z}/2\mathbb{Z}) \cong \oplus_n \mathbb{Z}/2\mathbb{Z}$

so $\#_n \mathbb{CP}^2$ almost symplectic $\Leftrightarrow \exists a \in H^2(\#_n \mathbb{CP}^2; \mathbb{Z})$ with

$(1,1,\dots 1) \equiv a \mod 2$ and

$a^2([\#_n \mathbb{CP}^2]) = 5n+4$

suppose $a = (a_1, \dots a_n) \in \oplus_n \mathbb{Z}$

$$a^2 = \sum_{i=1}^n a_i^2$$

so we need $\underline{a_i \in \mathbb{Z}}$ s.t. $\boxed{\sum_{i=1}^n a_i^2 = 5n+4 \text{ and } a_i \text{ odd}}$

<u>$n=1$:</u> $a_1^2 = 9$ so must have $a_1 = \pm 3$

$\therefore \mathbb{CP}^2$ admits 2 almost complex/symplectic structures (only one upto conjugation)

of course $\mathbb{CP}^2$ is also symplectic! (Kähler)

<u>$n=2$:</u> need $a_1^2 + a_2^2 = 14$ no sol$^{\underline{n}}$!

so $\mathbb{CP}^2 \# \mathbb{CP}^2$ has no almost symplectic structure!

$\therefore$ no symplectic str.

even though $\exists \alpha \in H^2(\mathbb{CP}^2 \# \mathbb{CP}^2)$ s.t. $\alpha^2 > 0$

<u>$n=3$:</u> need $a_1^2 + a_2^2 + a_3^2 = 19$ sol$^{\underline{n}}$s $(\pm 3, \pm 3, \pm 1)$ (and permutations)

so $\mathbb{CP}^2 \# \mathbb{CP}^2 \# \mathbb{CP}^2$ has almost symplectic structures (all are diffeomorphic)

also has $\alpha \in H^2$ s.t. $\alpha^2 > 0$

<u>Is $\mathbb{CP}^2 \# \mathbb{CP}^2 \# \mathbb{CP}^2$ symplectic?</u> <u>No</u> but very hard to show!

<u>Taubes:</u> Symplectic manifolds with $b_2^+ \geq 2$ have non-vanishing Seiberg-Witten invariants

number of pos. eigenvalues in intersection pairing

<u>Kotschick:</u> If $X^4 = X_1 \# X_2$ is simply connected and $b_2^+(X_i) \geq 1$, then Seiberg-Witten invariants vanish.

so no $\#_n \mathbb{C}P^2$ symplectic if $n>1$!

exercise: $\#_n \mathbb{C}P^n$ is almost symplectic $\Leftrightarrow$ n odd

e.g. $n=5$ get sol^{n}s $(5,1,1,1,1)$ and $(3,3,3,1,1)$ (these are really different)

lemma 13:

If M a 4-manifold with an almost symplectic structure then $b_2^+(M)$ and $b_1(M)$ have opposite parity

Proof: Wu showed that for any $c \in H^2(M;\mathbb{Z})$

$$c \cup c \equiv w_2(M) \cup c \mod 2$$

suppose a is an integral lift of $w_2(M^4)$ then

$$c \cup c \equiv a \cup c \mod 2$$

such an a is called a characteristic element

not to hard to show for any characteristic element

$$c \cup c([M]) \equiv \sigma \mod 8$$

so if M has an almost complex structure then $c_1(M) \equiv w_2 \mod 2$

and

$$c_1^2 \equiv \sigma \mod 8$$

$$c_1^2 = 3\sigma + 2\chi$$

so

$$\sigma + 8k = 3\sigma + 2\chi \quad \text{some } k$$

$$8k = 2(\sigma + \chi)$$

$$2k = \frac{\sigma+\chi}{2}$$

i.e. $$\frac{\sigma+\chi}{2} \equiv 0 \mod 2$$

$$\frac{b_2^+ - b_2^- + 2 - 2b_1 + b_+ - b_-}{2} \equiv 0 \mod 2$$

so $\boxed{b_2^+ - b_1 + 1 \text{ is even}}$ if M almost complex

exercise: 1) let M be a simply connected oriented 4-manifold.

Show M has an almost symplectic structure

$\Leftrightarrow$

b_2^+ is odd

2) at most 2 of M_1^4, M_2^4, and $M_1 \# M_2$ have an almost symplectic structure