## D. Fiber bundles

consider $\mathbb{R}^4$ with $\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4$

let $g_i$ be translation by 1 in $x_i$-direction for $i=1,2,3$

$$g_4(x_1,x_2,x_3,x_4) = (x_1+x_2, x_2, x_3, x_4+1)$$

clearly $g_i^* \omega = \omega$ for $i=1,2,3$

$$g_4^* \omega = (dx_1+dx_2)\wedge dx_2 + dx_3 \wedge dx_4 = \omega$$

so let $E = \mathbb{R}^4/G$ this is a symplectic manifold with $\pi_1(E) = G$

let's figure out what $H_1(E) = G/[G,G]$ is

consider $\pi : \mathbb{R}^4 \to \mathbb{R}^3 : (x_1\ x_2\ x_3\ x_4) \longmapsto (x_2\ x_3\ x_4)$

let $\bar{g}_1 = id_{\mathbb{R}^3}$ $\quad \bar{g}_i =$ translation in $x_i$ direction

$$\begin{tikzcd} \mathbb{R}^4 \arrow[r, "g_i"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circ"] & \mathbb{R}^4 \arrow[d, "\pi"] \\ \mathbb{R}^3 \arrow[r, "\bar{g}_i"] & \mathbb{R}^3 \end{tikzcd}$$

eg. $\pi \circ g_1(x_1,x_2,x_3,x_4) = (x_2,x_3,x_4) = \bar{g}_1$
$\pi \circ g_2(x_1,x_2,x_3,x_4) = (x_2+1, x_3, x_4) = \bar{g}_2$
$\pi \circ g_4(x_1,x_2,x_3,x_4) = (x_2, x_3, x_4+1) = \bar{g}_4$

so we have an induced map

$$\pi : E \to T^3$$

exercise: this is an $S^1$-bundle

note $\pi_* : H_1(E) \to H_1(T^3)$ maps $g_i$ to $\bar{g}_i$ for $i=2,3,4$

so $\pi_*$ onto and $g_i$ infinite order for $i=2,3,4$ and no relations among them

also $(x_1,x_2,x_3,x_4) \xmapsto{g_2} (x_1, x_2+1, x_3, x_4) \xmapsto{g_4} (x_1+x_2+1, x_2+1, x_3, x_4)$
$\xmapsto{g_2^{-1}} (x_1+x_2+1, x_2, x_3, x_4) \xmapsto{g_4^{-1}} (x_1+1, x_2, x_3, x_4)$

so $g_1 = [g_2, g_4]$ $\therefore$ $g_1 = 0$ in $H_1(E)$

$\therefore H_1(E) \cong \mathbb{Z}^3$

we have established most of the following

**Th$^{m}$ 4 (Thurston ~70's, Kodaira unpublished ~'50):**

> there are symplectic manifolds that are not Kähler
> (they can even have a complex structure, of course not compatible with sympl. str.)

**Remark:** This was first such example

**Proof:** for a Kähler manifold $b_{2n+1}$ even but $b_1(E)=3$ $\blacksquare$

**note:** $p: \mathbb{R}^4 \to \mathbb{R}^2 : (x_1\, x_2\, x_3\, x_4) \mapsto (x_3\, x_4)$

gives $E$ the structure of a $T^2$ bundle over $T^2$ with sympl. fibers

generalizing this we have

**Th$^{m}$ 5 (Thurston):**

> Given a symplectic manifold $(M, \omega_M)$ and a compact bundle $E \xrightarrow{p} M$ with fiber $\Sigma$
>
> assume a) if $\dim \Sigma = 2$, $\exists\, \alpha \in H^2_{DR}(E)$ s.t.
>
> $$\int_{\Sigma_x} \alpha > 0 \quad \forall x \in M \text{ where } \Sigma_x = p^{-1}(x)$$
>
> (equivalently $[F_x] \neq 0$ in $H_2(E;\mathbb{R})$)
>
> b) if $\dim \Sigma > 2$, $\exists$ symplectic form $\omega_\Sigma$ on $\Sigma$ such that the structure group of $E$ acts by symplectomorphisms (so $\exists\, \omega_x$ on each $\Sigma_x$) and $\exists\, \alpha \in H^2_{DR}(E)$ s.t. $\alpha|_{\Sigma_x} = [\omega_x]$
>
> Then $\exists$ a symplectic form $\omega_E$ on $E$ s.t.
>
> $$\omega_E|_{\Sigma_x} \text{ symplectic } \forall x$$
>
> Moreover, the image of any prechosen section can be assumed to be symplectic

## Remarks:

1) Surface bundles over surfaces (all orientable) have symplectic structures with symplectic fibers

$$\Leftrightarrow [\Sigma_x] \neq 0$$

<u>note</u>: if $\exists$ section $\sigma : M \to E$ then $[\Sigma_x] \neq 0$ since $\sigma(M) \cdot \Sigma_x = pt$

2) $S^3 \times S^1 \xrightarrow{\pi} S^2$ where $\pi = \text{Hopf map} \circ \pi_{S^3}$

is a $T^2$-bundle over $S^2$

but no symplectic structure

3) More generally, $[\Sigma_x] \neq 0$ is automatic unless $\Sigma = T^2$

to see this suppose $\begin{array}{c} \Sigma^2 \to E \\ \downarrow \\ M^{2n} \end{array}$ is a bundle

let $\xi_y = T_y \Sigma_x$ for $y \in \Sigma_x$, $x \in M$

so $\xi \subset TE$ and $\xi$ an oriented $\mathbb{R}^2$-bundle on $E$

so $\exists$ Euler class $e(\xi)$ and $e(\xi)([\Sigma_x]) = \chi(\Sigma)$

so if $\chi(\Sigma) \neq 0$, then $\Sigma_x$ pairs non-trivially with something in cohomology.

## Proof:

<u>Claim</u>: $\exists$ a closed 2-form $\eta \in \Omega^2_{DR}(E)$ st. $\omega|_{\Sigma_x} = \begin{cases} \text{area form} & \text{case a)} \\ \omega_x & \text{case b)} \end{cases}$

given this consider $\omega_t = \pi^*\omega_M + t\eta$ $t > 0$

<u>note</u>: $\omega_t|_{\Sigma_x} = t\eta$ so $\Sigma_x$ symplectic $\forall x$

now let's check $\omega_t$ is symplectic

$$d\omega_t = d\pi^*\omega_M + t\,d\eta = \pi^* d\omega_M + 0 = 0$$

choose any metric on $E$ and note $T_y E = T_y \Sigma_x \oplus T_y \Sigma_x^\perp$ where $\pi(y) = x$

now $d\pi_y$ is an isomorphism from $T_y \Sigma_x^\perp$ to $T_x M$

so $\pi^*\omega_M$ non-degenerate on $T_y \Sigma_x^\perp$

$\therefore\ \omega_t$ non-degenerate on $T_y\Sigma_x^{\perp}$ for small $t$

(non-degen is open condition and $E$ compact)

but we already noticed $\omega_t$ non-degen. on $T_y\Sigma_x$

$\therefore\ \omega_t$ non-degen on $E$ ie. $\omega_t$ symplectic!

now if $\sigma: M \to E$ a section then

$$\sigma^*(\pi^*\omega_M) = \omega_M$$

so $\sigma^*(\omega_t) = \omega_M + t\,\sigma^*\eta$ which is symplectic for small $t$.

so we are left to check claim

**Proof of Claim:**

Cover the base $M$ by local trivializations $\{(U_i, \phi_i)\}$ with $U_i$ contractible and transition maps symplectos in case b)

$$\begin{tikzcd} \pi^{-1}(U_i) \arrow[r, "\phi_i"] \arrow[dr, "p"'] & U_i \times \Sigma \arrow[r, "\pi_\Sigma"] \arrow[d, "\pi_{U_i}"] & \Sigma \\ & U_i & \end{tikzcd}$$

let $\{\rho_i\}$ be a partion of unity subordinate to $\{U_i\}$

In case a) let $\omega_\Sigma$ be any area form s.t. $\int_\Sigma \omega_\Sigma = \int_{\Sigma_x}\alpha$

now set $\eta_i = \phi_i^*\pi_\Sigma^*\omega_\Sigma$ on $\pi^{-1}(U_i)$

so $\eta_i|_{\Sigma_x} = \omega_x$ in case b)

let $\varsigma \in \Omega^2(E)$ represent $\alpha \in H^2_{DR}(E)$

$\eta_i - \varsigma|_{\pi^{-1}(U_i)}$ is closed and

note $\pi^{-1}(U_i) \simeq \Sigma_x$ since $U_i$ contractible

$[\eta_i - \varsigma|_{\pi^{-1}(U_i)}] = [\omega_x] - [\omega_x] = 0$ (same in case a) by $\int$ above)

so $[\eta_i - \varsigma|_{\pi^{-1}(U_i)}] = 0$ in $H^2_{DR}(\pi^{-1}(U_i))$

$\therefore\ \exists$ 1-forms $\theta_i \in \Omega^1(\pi^{-1}(U_i))$ s.t. $d\theta_i = \eta_i - \varsigma|_{\pi^{-1}(U_i)}$

finally set $\eta = \varsigma + d\left(\Sigma(\rho_i\circ\pi)\,\theta_i\right)$

clearly well-defined and $d\eta = 0$

$$\eta|_{\Sigma_x} = \rho|_{\Sigma_x} + \Sigma \underbrace{(\rho_1 \circ \pi)}_{\text{constant on } \Sigma_x} d\theta_1|_{\Sigma_x} = \rho|_{\Sigma_x} + \Sigma(\rho_1\circ\pi)(\eta_1|_{\Sigma_x} - \rho|_{\Sigma_x})$$

$$= \cancel{\rho|_{\Sigma_x} - \rho|_{\Sigma_x} \underbrace{\Sigma(\rho_1\circ\pi)}_{1}} + \Sigma(\rho_1\circ\pi)\,\eta_1|_{\Sigma_x}$$

$$= \begin{cases} \Sigma(\rho_1\circ\pi)(\text{area form}) & \text{case a)} \\ \Sigma(\rho_1\circ\pi)\,\omega_x & \text{case b)} \end{cases}$$

$$= \begin{cases} \text{area form} & \text{case a)} \\ \omega_x & \text{case b)} \end{cases}$$

## E. Lefschetz Pencils and Fibrations

Briefly a Lefschetz pencil is simply a fibration over $S^2$ except two types of "singularities" are allowed

for a Lefschetz fibration only one type is allowed.

More rigorously, a (topological) Lefschetz pencil on a compact, oriented manifold $M^{2n}$ is

n=2 automatic

called base locus

1) a codimension 4 compact submanifold $B \subset M$ and

2) a smooth map $\pi: (M - B) \to \mathbb{CP}^1 = S^2$

such that

a) for each $p \in B$ there are orientation preserving coordinates about $p$ where $B$ is $z_1 = z_2 = 0$ in $\mathbb{C}^n$ and $\pi$ in the compliment of $B$ is

$$(z_1, \ldots z_n) \longmapsto [z_1 : z_2] \in \mathbb{CP}^1$$

(i.e. $\pi$ on each fiber of normal bundle is projectivization)

b) there are a finite number of critical points $\{c_1, \ldots c_k\}$ such that fore each $c_i$ there are orientation preserving coordinates about $c_i$ and $\pi(c_i)$ in which $\pi$ is given by

$$(z_1, \ldots, z_n) \longmapsto z_1^2 + \ldots + z_n^2$$

(can assume image of $c_i$ disjoint)

a <u>Lefschetz fibration</u> is a Lefschetz pencil with $B = \emptyset$

**Remarks:**

1) for a Lefschetz fibration we can have $\pi : M \to S$ for any oriented surface $S$

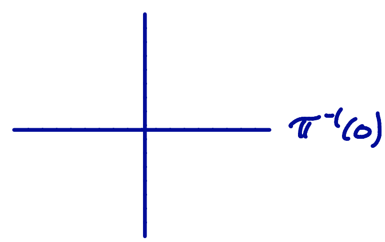
2) The requirement for the critical points is really just that they are non-degenerate, then a complex version of the Morse lemma gives the desired form

   <u>exercise:</u> find a complex change of coordinates taking
   $$z_1^2 - z_2^2 \text{ to } z_1^2 + z_2^2 \text{ or to } z_1 z_2$$

3) Nbhd of critical points

   <u>$n=2$ case:</u> from above assume $\pi(z_1, z_2) = z_1 z_2$

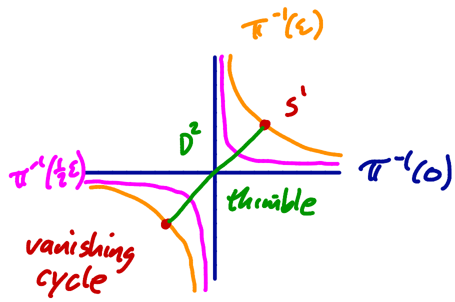
   $\pi^{-1}(0) = \mathbb{C} \times \{0\} \cup \{0\} \times \mathbb{C}$ with singular point $0$




$\pi^{-1}(\varepsilon) \cong S^1 \times \mathbb{R}$ indeed $S^1 \times \mathbb{R} \to \pi^{-1}(\varepsilon)$

$(\theta, t) \longmapsto (e^{t+i\theta}, \varepsilon e^{-t-i\theta})$

is a diffeomorphism




as $\varepsilon \to 0$, $\pi^{-1}(\varepsilon) \to \pi^{-1}(0)$

note there is an $S^1 \subset \pi^{-1}(\varepsilon)$ that collapses to $0$ as $\varepsilon \to 0$

this $S^1$ is called a <u>vanishing cycle</u>

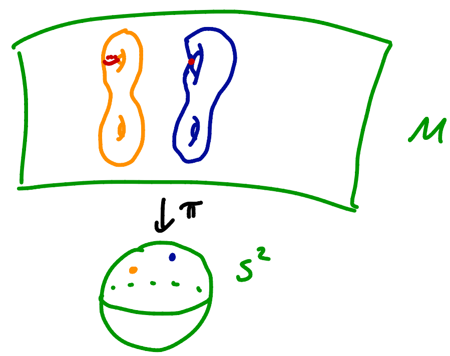
the union of thes $S^1$ as $\varepsilon \to 0$ and critical point is a $D^2$ called the thimble so generic fiber in this nbhd is an annulus and its generator in homology vanishes when included in nbhd

exercise: in higher dimensions show generic fiber is $T^*S^{n-1}$, vanishing cycle is an $S^{n-1}$ and thimble is $D^n$

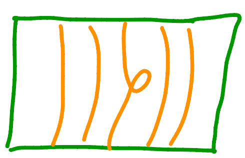
4) If $B = \emptyset$ (so a lefschetz fibration) then generic fiber is a $(2n-2)$-manifold

note: $\pi|_{\pi^{-1}(\pi(\{c_i\}))} : \pi^{-1}(\pi(\{c_i\})) \to (S^2 - \pi(\{c_i\}))$

a fiber bundle




schematically we write $M$



to indicate its a singular fiber bundle

5) Now let's consider $B$

n=2 case: $B = \{b_1 \ldots b_k\}$

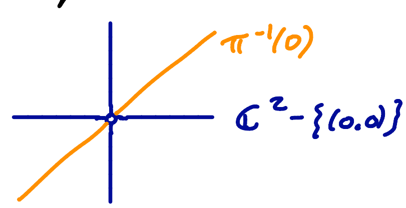
$b_i$ has nbhd $\mathbb{C}^2$ where $\pi$ in complement of $b_i$ is

$$\mathbb{C}^2 - \{(0,0)\} \to \mathbb{C}P^1$$
$$(z_1, z_2) \longmapsto [z_1 : z_2]$$

$b_i$

so $\pi^{-1}(z)$ in this nbhd is a complex line $- \{(0,0)\}$

so the closure of $\pi^{-1}(z)$ is just a copy of $\mathbb{C}$

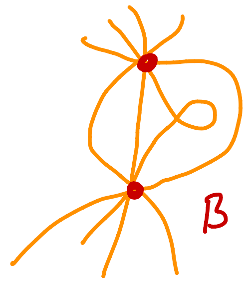



thus $\pi^{-1}(z) \subset M$ has closure a surface going throug all $b_i$

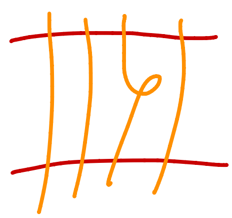
so $\pi : (M - B) \to \mathbb{C}P^2$ is a fibration with fibers $\Sigma_z$ being punctured surfaces whose closures $\overline{\Sigma}_z$ in $M$ is an embedded surface containg $B$

thus a Lefschetz pencil fills $M$ with (singular) surfaces all disjoint except at $B$

schematically




note: if we blow up each point of $B$ we get $M \#_k \overline{\mathbb{C}P^2}$



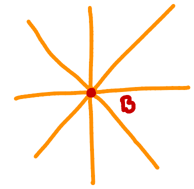

then $\pi$ extends over blow up nbhd to give a Lefschetz fibration $M \#_k \overline{\mathbb{C}P^2} \to \mathbb{C}P^1$ and we have $k$ sections which are the $\mathbb{C}P^1 \subset \overline{\mathbb{C}P^2}$

exercise: in higher dimensional case $B$ has a nbhd that is a $\mathbb{C}^2$ bundle over $B$

we can replace this with a $(\overline{\mathbb{C}P^2} - B^4)$-bundle over $B$ and extend $\pi$ to a Lefschetz fibration of this new manifold (we "blow up $B$", ie a "parameterized blow up")

let's now see some examples where Lefschetz pencils naturally arise examples:

1) Case where we have no critical points
Complex projective lines $\mathbb{C}P^1$ through a point $B$ in $\mathbb{C}P^2$




exercise:

1) for each $[t_0:t_1] \in \mathbb{C}P^1$ show

$$L_{[t_0:t_1]} = \{[x:y:z] \in \mathbb{C}P^2 : t_0 x = t_1 y\}$$

is a well-defined copy of $\mathbb{C}P^1$ in $\mathbb{C}P^2$
all of which contain $B = [0:0:1]$

Hint: for $t_1 \neq 0$ consider $\mathbb{C}P^1 \to \mathbb{C}P^2$
$[z_1:z_2] \mapsto [z_1 : \frac{t_0}{t_1} z_1 : z_2]$

2) for distinct points $[t_0:t_1] \neq [s_0:s_1]$

$$L_{[t_0:t_1]} \cap L_{[s_0:s_1]} = \{B\}$$

3) for any $P \neq B$ in $\mathbb{C}P^2$ $\exists!$ $[t_0:t_1]$ s.t. $P \in L_{[t_0:t_1]}$

from above we have a map

$$\pi : (\mathbb{C}P^2 - \{B\}) \to \mathbb{C}P^1$$
$$P \longmapsto [t_0:t_1] \text{ s.t. } P \in L_{[t_0:t_1]}$$

in coords about $B$ we see $\pi$ is

$$(\mathbb{C}^2 - \{(0,0)\}) \to \mathbb{C}P^1$$
$$(z_1, z_2) \longmapsto [z_1 : z_2]$$

so this is a Lefschetz pencil!

if we blow up base locus $B$ we get a Lefschetz fibration

$$\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2 \to \mathbb{C}P^1$$

with no singular fibers i.e. $S^2$-bundle over $S^2$

exercise: $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$ is not diffeomorphic to $S^2 \times S^2$

## 2) Cubic pencil of $\mathbb{CP}^2$ and the elliptic surface $E(1)$

exercise:

1) If $P(z_1, z_2, z_3)$ is a non-constant homogeneous polynomial, then

$$V_p = \{[z_0 : z_1 : z_2] \in \mathbb{CP}^2 : P(z_0, z_1, z_2) = 0\}$$

is well-defined

2) for a generic $P$, $V_p$ is a surface of genus

$$g = \frac{(d-1)(d-2)}{2}$$

now consider 2 generic degree 3 polynomials $p_0, p_1$

For $[t_0 : t_1] \in \mathbb{CP}^1$, let

$$V_{[t_0:t_1]} = \{[z_0 : z_1 : z_2] \in \mathbb{CP}^2 : t_0 p_0(z_0, z_1, z_2) + t_1 p_1(z_0, z_1, z_2) = 0\}$$

exercise:

1) $V_{[0:1]} \cap V_{[1:0]} = \{9 \text{ points}\} = B$

and any $V_{[t_0:t_1]}$ contains $B$

2) if $[t_0 : t_1] \neq [s_0 : s_1]$ then $V_{[t_0:t_1]} \cap V_{[s_0:s_1]} = B$

3) for any $P \notin B$, $\exists! [t_0 : t_1]$ s.t. $P \in V_{[t_0:t_1]}$

Fact: for most $p_0, p_1$, $T_{[t_0:t_1]}$ will be smooth tori except for 12 points

from above we have a map

$$\pi : (\mathbb{CP}^2 - B) \to \mathbb{CP}^1$$

that near pts in $B$ looks like

$$(\mathbb{C}^2 - \{(0,0)\}) \longrightarrow \mathbb{CP}^1$$
$$(z_1, z_2) \longmapsto [z_1 : z_2]$$

moreover $\pi$ has 12 non-degenerate critical points so $\pi$ is a Lefschetz pencil of $\mathbb{CP}^2$

if we blow up $B$ we get a Lefschetz fibration

$$\pi : \mathbb{CP}^2 \#_9 \overline{\mathbb{CP}}^2 \to \mathbb{CP}^1$$

with elliptic ($T^2$) fibers

we call this manifold $E(1)$

from construction $E(1)$ is symplectic and to torus fibers are also symplectic

exercise: Show $E(1)$ - (regular fiber) is simply connected

Hint: consider section coming from blow up

Remark: Existence of $E(1)$ completes proof of Cor 3 about realizing all finitely presented groups as $\pi_1$ of a symplectic manifold.

Th$^{m}$ (Donaldson):

$(M, \omega)$ a symplectic manifold

suppose $[\omega] \in H^2_{DR}(M)$ is an integral class

For sufficiently large integers $k$ there is a topological Lefschets pencil on $M$ whose fibers are symplectic and homologous to the Poincaré dual of $k[\omega]$

We will prove this later, but for now we turn to

Th$^{m}$ 6:

I) Any 4-manifold with a Lefschetz pencil (such that each irreducible component of each fiber intersects the base locus non-trivially) has a symplectic structure with symplectic fibers

II) A 4-manifold $M$ with a Lefschetz fibration has a symplectic structure with symplectic fibers

$\Longleftrightarrow$

a generic fiber is non-trivial in $H_2(M;\mathbb{R})$

Remark: 1) There is a higher dimensional version of this but a bit complicated to state

2) in case II) can let base be any surface $S$ not just $S^2$ as in case I)
(note: def$^n$ of Lefschetz fibration works here)

3) in case II), fiber non-trivial in $H_2(M;\mathbb{R})$ unless it is $T^2$ and no critical point
(for $T^2$ same argt as for fibrations, for critical points consider what we know about elliptic fibrations)

Proof:

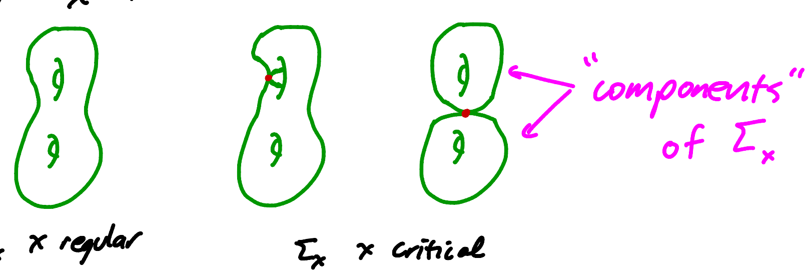
let $(\pi, B)$ be a Lefschetz pencil of $M$ over $S = S^2$ (I)

or a Lefschetz fibration of $M$ over a surface $S$ (II)

in both cases a "regular fiber" is a surface $\Sigma$

denote: $\overline{\pi^{-1}(x)}$ by $\Sigma_x$ for $x \in S$

so $\Sigma_x$ is




Step 1: $\exists\, \alpha \in H^2_{DR}(M)$ s.t. $\int_\Sigma \alpha > 0$ $\forall\, \Sigma$ "components" of $\Sigma_x$

Step 2: Define form near base locus $B$ and critical points $\{c_i\}$

**Step 3:** Define form near fibers

**Step 4:** Patch forms above together to get singular symplectic structure

**Step 5:** Alter form near $B$ to get desired symplectic structure

**Proof of 1:**

case I) let $\alpha$ = Poincaré dual of $[\Sigma_x]$, $x$ regular value

now if $\Sigma \subset \Sigma_{x'}$ then $\Sigma \cap \Sigma_x \neq \emptyset$ subset $B$

and all $\cap$ pts positive so

$$\int_\Sigma \alpha = \Sigma_x \cdot \Sigma > 0$$

case II) let $\tilde{\alpha} \in H^2_{DR}(M)$ st. $\langle \alpha, [\Sigma_x] \rangle = 1$, for $x$ regular value

(ok since $[\Sigma_x] \neq 0$)

if $x'$ critical, $\Sigma_{x'} = \Sigma_1 \cup \Sigma_2$ and for, say $\Sigma_1$,

$$\langle \tilde{\alpha}, \Sigma_1 \rangle = 0$$

then note

$$\Sigma_1 \cdot \Sigma_2 = 1 \quad \text{(recall } \Sigma_1 \cap \Sigma_2 \text{ transversally)}$$

so let $\alpha = \tilde{\alpha} + c \cdot$ Poincaré Dual$[\Sigma_2]$ some small $c$

now
$$\langle \alpha, \Sigma_x \rangle = \langle \tilde{\alpha}, \Sigma_x \rangle + c(\Sigma_2 \cdot \Sigma_x) = 1$$
$$\langle \alpha, \Sigma_1 \rangle = \langle \tilde{\alpha}, \Sigma_1 \rangle + c(\Sigma_1 \cdot \Sigma_2) = c > 0$$
$$\langle \alpha, \Sigma_2 \rangle = \langle \tilde{\alpha}, \Sigma_{x'} \rangle - \langle \tilde{\alpha}, \Sigma_1 \rangle = 1 - c > 0$$

$[\Sigma_{x'}] = [\Sigma_x]$

(do for other singular fibers too)

**Proof of 2:**

let $U_i$ be nbhds of points $b_i \in B$ from definition of Lefschetz Pencil

$V_i$ " " critical points $c_i$ " "

and set $V = (\cup U_i) \cup (\cup V_i)$

define $\omega_V$ to be $\omega_{std}$ on $\mathbb{C}^2$ using coord charts for $U_1$ and $V_i$

note: $\omega_V$ is symplectic on $\Sigma_y \cap V \quad \forall y$

Proof of 3:

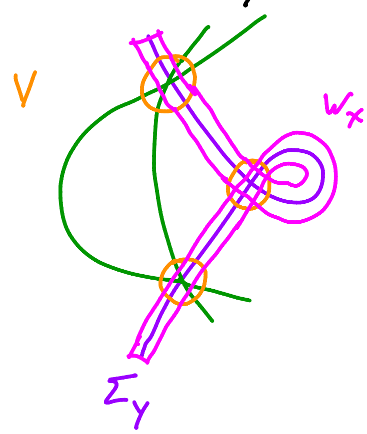
for each $y \in S$ let $\omega_y$ be a symplectic form on $\Sigma_y$

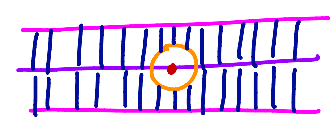
that extends $\omega_V$ on $\Sigma_y \cap V$ and ($\omega_y$ not necessarily smooth in $y$)

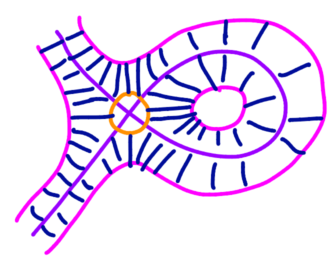
$\int_\Sigma \omega_y = \alpha(\Sigma) \quad \forall$ components $\Sigma \subset \Sigma_y$ (might need to shrink $V$)

we now extend $\omega_y$ to a nbhd $W_y$ of $\Sigma_y$ as a closed 2-form $\eta_y$

for this let $f_y : W_x \to \Sigma_x \cup V$ be a retraction that is identity near $B \cup \{c_i\}$




near $b \in B$ 

near $c_i$: 

set $\eta_y = f_y^*(\omega_y \cup \omega_V)$

by choosing smaller nbhds if necessary can assume

$\exists\, U_y \in S$ a nbhd of $y$ s.t. $W_y = \pi^{-1}(U_y) \cup V$

and no critical points in $W_y - \Sigma_y$

and $U_y$ contractible

and $\forall z \in U_y$, $\eta_y|_{\Sigma_z}$ is symplectic

(since non-degeneracy is open condition)

Proof of 4: let $\rho$ be a 2-form representing $\alpha$

$(\eta_y - \rho|_{W_y})(\Sigma_y) = \alpha(\Sigma_y) - \alpha(\Sigma_y) = 0$

since $H^2(W_Y) \cong H^2(\Sigma_Y)$ we see $\eta_Y - \zeta|_{W_Y} = 0$ in $H^2(W_Y)$

$\therefore \exists\, \theta_Y$ s.t. $d\theta_Y = \eta_Y - \zeta|_{W_Y}$

$\exists$ a finite number $\{U_{Y_0}, \dots U_{Y_k}\}$ of $U_Y$ covering $S$

note $d(\theta_{Y_i} - \theta_{Y_0}) = (\eta_{Y_1} - \zeta|_{W_{Y_i}}) - (\eta_{Y_0} - \zeta|_{W_{Y_0}}) = \omega_V - \omega_V$ near $B$
$= 0$

$\therefore$ in nbhd of $B$ $\exists\, f_i$ s.t. $df_i = \theta_{Y_i} - \theta_{Y_0}$
(cut-off $f_i$ outside small nbhd) replace $\theta_{Y_i}$
with $\theta_{Y_i} - df_i$

so we can assume $\theta_{Y_i} = \theta_{Y_j}$ in nbhd of $B$ $\forall i, j$

let $\{\rho_i\}$ be a partition of unity subordinate to $\{U_{Y_i}\}$

set $\eta = \zeta + d\sum(\rho_i \circ \pi)\theta_{Y_i}$ on $M - B$

$\eta$ closed 2-form and $\eta|_{\Sigma_Y - B}$ symplectic $\forall Y \in S$

near $B$, $\eta = \zeta + d\theta_{Y_0} = \zeta + \eta_{Y_0} - \zeta|_{W_{Y_0}} = \eta_{Y_0} = \omega_V$

$\therefore \eta$ can be extended over $B$ by $\omega_V$

near critical points of $\pi$ only one $\rho_i \neq 0$

$\therefore \eta = \zeta + d\theta_{Y_i} = \omega_V$ near critical point

so $\eta$ global closed 2-form
$\eta|_{\Sigma_Y}$ symplectic
$\eta$ near $B \cup \{c_i\}$ is $\omega_V$

set $\omega_t = \pi^* \omega_S + t\eta$ $\quad t > 0$ small $\quad$ Note: only defined on $M - B$ !

$\uparrow$ area form on $S$

just as in proof of Theorem 5 $\omega_t$ symplectic outside of $V$

near a critical point we have a chart $\mathbb{C}^2$ and $\mathbb{C}$ s.t.

$$\pi(z_1,z_2) = z_1^2 + z_2^2$$

and $\omega_t = \pi^* \omega_{\mathbb{C}} + t\, \omega_{\mathbb{C}^2}$

if $J$ is almost complex str on $\mathbb{C}^2$ and $\mathbb{C}$

then $d\pi \circ J = J \circ d\pi$ since $\pi$ is holomorphic

$$\text{so } \omega_t(v, Jv) = \omega_{\mathbb{C}}(d\pi(v), d\pi(Jv)) + t\, \omega_{\mathbb{C}^2}(v, Jv)$$

$$= \underbrace{\omega_{\mathbb{C}}(d\pi(v), J d\pi(v))}_{\geq 0} + t\, \underbrace{\omega_{\mathbb{C}^2}(v, Jv)}_{>0 \text{ if } v \neq 0}$$

$> 0$ so $\omega_t$ non-degenerate for $t > 0$

So we are done in case II)

near $b \in B$ we have a chart $\mathbb{C}^2$ s.t.

$$\pi : (\mathbb{C}^2 - \{(0,0)\}) \longrightarrow \mathbb{C}P^1$$
$$(z_1, z_2) \longmapsto [z_1 : z_2]$$

$$\omega_t = \underbrace{\pi^* \omega_{\mathbb{C}P^1}}_{\text{singular at } (0,0)} + t\, \omega_{\mathbb{C}^2}$$

so $\omega_t$ symplectic on $M - B$ and diverges at $B$

<u>Proof of 5:</u>

consider $\mathbb{C}^2 - \{(0,0)\} \longrightarrow \mathbb{C}P^1$

$$\mathbb{C}^2 - \{(0,0)\} = S^3 \times (0,\infty)$$

radial coord $r$

Let $L$ denote any $\mathbb{C}$-line in $\mathbb{C}^2$ through $0$

now $T_p(\{r\} \times S^3) = T_pL \oplus T_pL^{\perp_g} = L \oplus L^{\perp_g}$ where $g$ is std metric on $\mathbb{C}^2$

$L \cap (\{r\} \times S^3) = S^1$ there is a vector field $v$ generating $S^1$-action on all $\{r\} \times S^1$

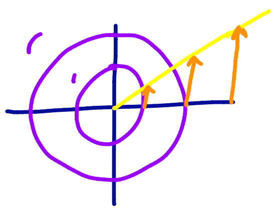
let $\beta$ be 1-form on $\mathbb{C}^2 - \{(0,0)\}$ s.t. $\beta(v) = 1$ and $\beta(v^{\perp_g}) = 0$

on any $L$, $\beta|_L = d\theta$

so $\omega_{\mathbb{C}^2}|_L = r\,dr \wedge \beta|_L = d(\tfrac{1}{2}r^2) \wedge \beta|_L$

now $T_p(\{1\} \times S^3) = L^{\perp g} \oplus \mathrm{span}\{v\}$

so $\pi^* \omega_{\mathbb{CP}^1}|_{\{1\}\times S^3} = \omega_{\mathbb{C}^2}$ on $L^\perp$ (by definition of $\omega_{\mathbb{CP}^1}$)



<u>note:</u> $\pi \circ (\text{mult by } r) = \pi$

so $(\text{mult by } r)^* \circ \pi^* = \pi^*$ and $(\text{mult by } r)^* dx_1 = r\,dx_1$

$(\text{mult by } r)^* dy_1 = r\,dy_1$

$\therefore$ on $L^\perp$, $\omega_{\mathbb{C}^2} = \pi^* \omega_{\mathbb{CP}^1}|_{\{1\}\times S^3} = (\text{mult by } r)^* \circ (\pi^* \omega_{\mathbb{CP}^1})|_{\{r\}\times S^3}$

$= r^2 \pi^* \omega_{\mathbb{CP}^1}$

and on $L$, $\pi^* \omega_{\mathbb{CP}^1} = 0$

finally we have $\boxed{\omega_{\mathbb{C}^2} = r^2 \pi^* \omega_{\mathbb{CP}^1} + d(\tfrac{1}{2}r^2) \wedge \beta}$

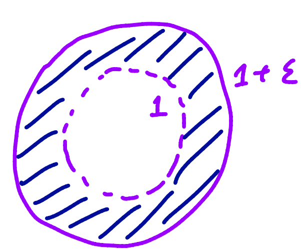
thus $\omega_t = \pi^* \omega_{\mathbb{CP}^1} + t\,\omega_{\mathbb{C}^2}$

$= (1 + t r^2) \pi^* \omega_{\mathbb{CP}^1} + t\, d(\tfrac{1}{2}r^2) \wedge \beta$

if we set $R = 1 + tr^2$ then

$\omega_t = R^2 \pi^* \omega_{\mathbb{CP}^1} + d(\tfrac{1}{2}R^2) \wedge \beta$

so on $B^4 - \{0\}$ $\omega_t$ is symplectomorphic to $\omega_{\mathbb{C}^2}$ on




using $R$ coordinant

$\therefore$ can glue $B_1^4$ to $M - B$ and extend $\omega_t$ over $B_1^4$ by $\omega_{\mathbb{C}^2}$