D. Handlebody Theory

an n-dimensional **k-handle** is

$$h^k = D^k \times D^{n-k}$$

Set $\partial_- h^k = (\partial D^k) \times D^{n-k}$ **attaching region**

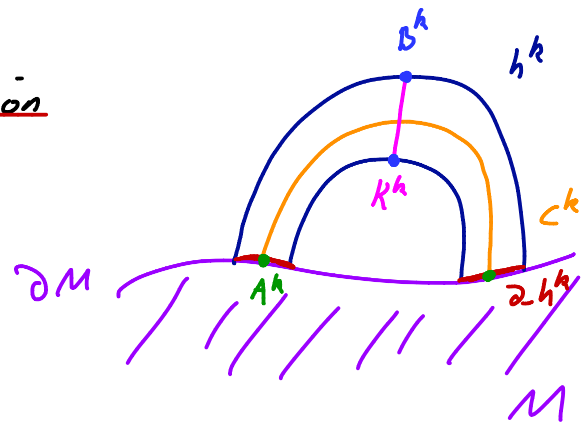
$\partial_+ h^k = D^k \times (\partial D^{n-k})$

$A^k = (\partial D^k) \times \{0\}$ **attaching sphere**

$C^k = D^k \times \{0\}$ **core**

$K^k = \{0\} \times D^{n-k}$ **co-core**

$B^k = \{0\} \times (\partial D^{n-k})$ **belt sphere**



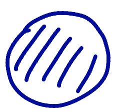
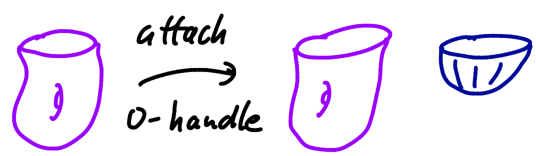
given an n-manifold M and an embedding

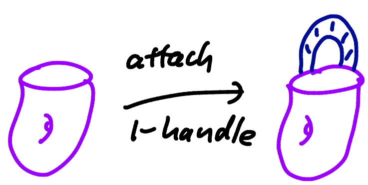
$$\phi : \partial_- h^k \to \partial M$$

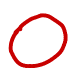
we attach h^k to M by forming the identification space

$$M \sqcup h^k \big/ (x \in \partial_- h^k) \sim (\phi(x) \in \partial M)$$

eg dimension 2:

$\underline{k=0}$:  $\partial_- h^0 = \emptyset$ so 

$\underline{k=1}$:  $\partial_- h^1 =$  so 

$\underline{k=2}$:  $\partial_- h^2 =$  so 

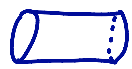
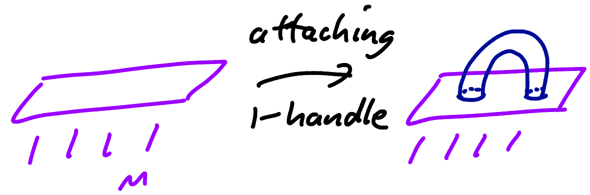
Remark:

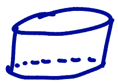
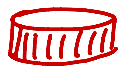
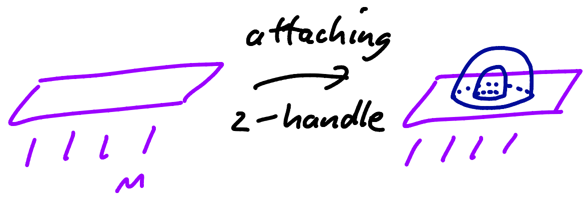
1) In all dimensions attaching a 0-handle is just taking disjoint union with D^n

2) In all dimensions n attaching an n-handle is just "capping off" an S^{n-1} boundary component.

dimension 3:

$\underline{h=0}$: ✓

$\underline{k=1}$:  $\partial_- h^1 =$  so 

$\underline{k=2}$:  $\partial_- h^2 =$  

$\underline{h=3}$: ✓

<u>Remark</u>: Note when a handle is attached one has a manifold with "corners" there is a standard way to smooth them out (see Wall "Differential Topology")

<u>exercises</u>:

1) if $\phi_0, \phi_1 : \partial_- h^k \to \partial M$ are isotopic, then the result of attaching a handle to M via ϕ_0 is diffeomorphic to attaching a handle to M via ϕ_1

2) the isotopy class of $\phi : \partial_- h^k \to \partial M$ is determined by

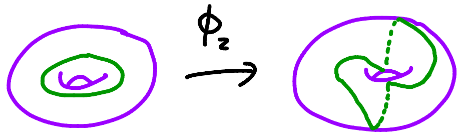
   1) isotopy class of $\phi|_{A^k}$ $\quad (A^k = S^{k-1} \times \{0\})$

   (i.e. a S^{k-1} knot in ∂M)

   2) the "framing" of the normal bundle of $\phi(A^k)$
   i.e. an identification of $\nu(\phi(A^k))$
   with $S^{k-1} \times D^{n-k}$

e.g. notice that $S^1 \times D^2$ has an integers worth of framings

$$S^1 \times D^2 \xrightarrow{\phi_n} S^1 \times D^2$$
$$(\phi, (r,\theta)) \longmapsto (\phi, (r, \theta + n\phi))$$



3) more generally show the framings on a k-dimensional sphere in Y^n is in one-to-one correspondence with $\pi_k(O(n-k))$

dim of normal bundle

so we see to attach an n-dimensional k-handle one must specify

1) an S^{k-1} knot in ∂M and
2) "elt" of $\pi_{k-1}(O(n-k))$

to really get such an element need a canonical "zero" framing

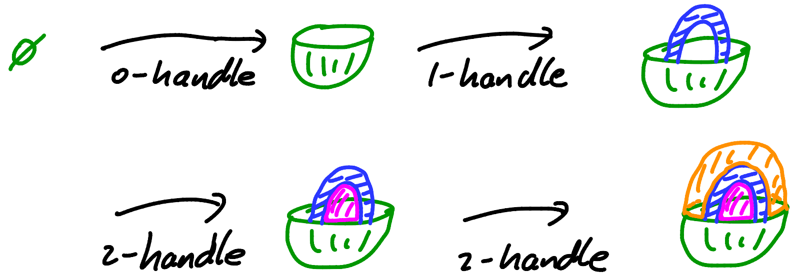
A handle decomposition of an n-manifold M is a sequence of manifolds $M_0, M_1, \ldots M_\ell$ such that

1) $M_0 = \emptyset$ and $M_\ell \cong M$
2) M_{i+1} is obtained from M_i by a k-handle attachment for some k

example:

handle decompositions of S^2

1) 

2) 

Th$^{\underline{m}}$:

Any smooth compact manifold has a handle decomposition

a brief sketch of the proof goes as follows.

recall a Morse function

$$f : M \to \mathbb{R}$$

is a function all of whose critical points are non-degenerate

i.e. if $p \in M$ a critical point, then in locall coordinates about p the matrix $\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(p)\right)$ is invertable

exercise: 1) Show p is a non-degenerate critical point of f $\iff$ df is transverse to the zero section of T^*M at $df(p)$

2) Every function $f: M \to \mathbb{R}$ can be perturbed to be a Morse function

3) If p is a non-degenerate critical point of $f: M \to \mathbb{R}$ then $\exists$ coordinates about p such that f takes the form

Fundamental lemma of Morse theory

$$f(x_1, \dots, x_n) = f(p) - x_1^2 - \dots - x_k^2 + x_{k+1}^2 + \dots + x_n^2$$

k is called the index of p

Main Th$^{\underline{m}}$ of Morse Theory:

let $f: M \to \mathbb{R}$ be a Morse function

I) if $[a,b]$ contains no critical values then

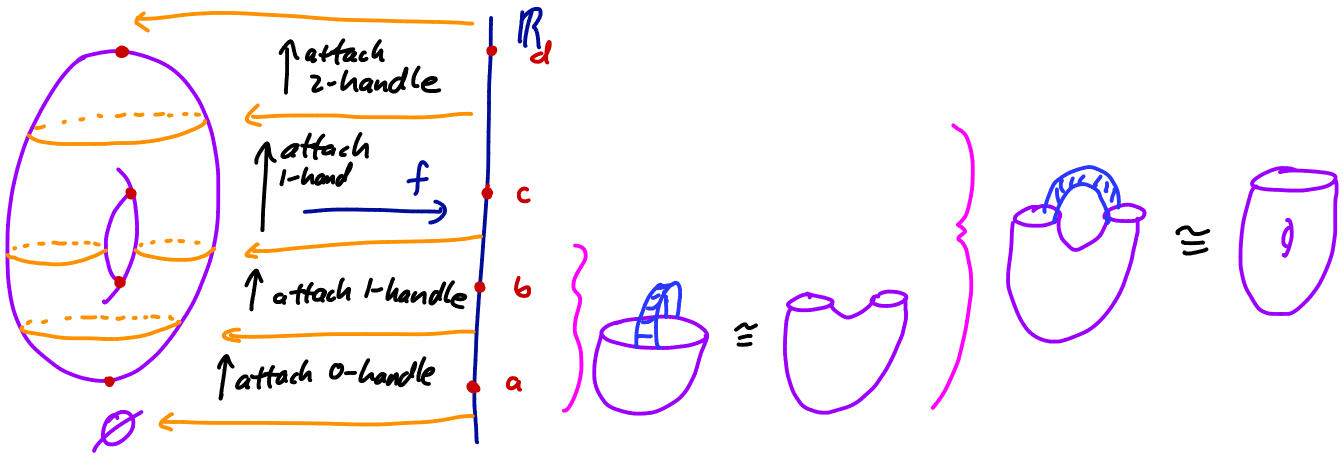
$$f^{-1}([a,b]) = f^{-1}(a) \times [a,b]$$

manifold since a regular value

II) if $\exists!$ critical point $p \in f^{-1}([a,b])$ s.t. $f(p) \in (a,b)$

then $f^{-1}([a,b])$ is obtained from $f^{-1}(a) \times [a, a+\varepsilon]$ by attaching a k-handle to $f^{-1}(a) \times \{a+\varepsilon\}$

example:

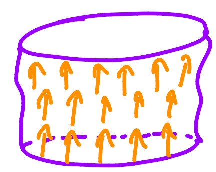


Remark: handle decomposition theorem clearly follows

Idea of proof of "Main Thm":

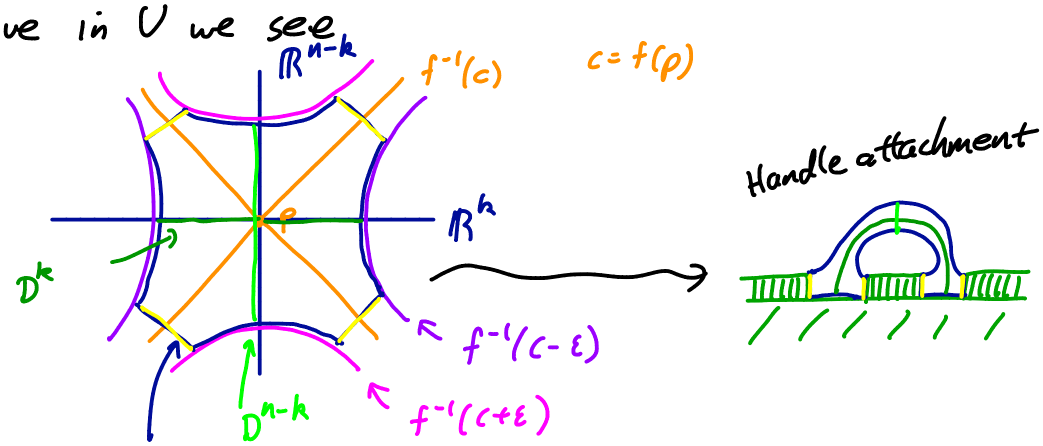
I) let $\Phi_t : M \to M$ be the (normalized) gradient flow of f

then
$$f^{-1}(a) \times [0, b-a] \to M$$
$$(p,t) \longmapsto \Phi_t(p)$$



is an embedding onto $f^{-1}([a,b])$

II) let U be nbhd about p where f has the form as in exercise 3 above in U we see



this is essentially a k-handle attachment

exercise: finish proof of II)

D. Weinstein Manifolds

given a function $\phi : M \to \mathbb{R}$ we say a vector field v is <u>gradient like</u> if

$$v \cdot \phi \geq \delta \left(\|v\|_g^2 + \|d\phi\|_g^2 \right)$$

for some $\delta > 0$ and some metric g

this means ϕ is increasing on flowlines of v and $v = 0 \Leftrightarrow \|d\phi\|_g = 0$

let $p \in M$ be a nondegenerate zero of v

Set $T_p^{\pm} M = \text{span}\{ \text{eigenvalues of } \left(\frac{\partial v_i}{\partial x_j}(p) \right) \text{ have } \pm \text{ real part} \}$ in local coords

let $\psi_t : M \to M$ be the flow of v (where defined)

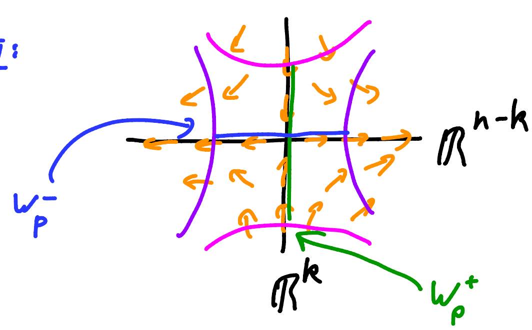
let $W^{\mp}(p) = \{ x \in M \text{ st. } \lim_{t \to \pm\infty} \psi_t(x) = p \}$

$W^-(p)$ is called the <u>stable manifold</u> "points flowing to p"

$W^+(p)$ " " <u>unstable manifold</u> "points flowing away from p"

exercise: $T_p W^{\pm}(p) = T_p^{\pm} M$

eg:



$f(x_1, \ldots x_n) = -x_1^2 \ldots - x_k^2 + x_{k+1}^2 + \ldots + x_k^2$

so core of handle part of W_p^+

co-core part of W_p^-

a <u>Weinstein manifold</u> is a manifold M together with

1) a Morse function $\phi : M \to \mathbb{R}$ with boundary being regular level sets

2) a symplectic structure ω

3) a vector field v that is gradient like for ϕ and expanding ω (ie. $\mathcal{L}_v \omega = \omega$)

exercise:

1) Show $\mathcal{L}_v \omega = \omega \Rightarrow \psi_t^* \omega = e^t \omega$ where ψ^t flow of v

and if $\lambda = \iota_v \omega$ then $\psi_t^* \lambda = e^t \lambda$

lemma 7:

$$\boxed{\begin{array}{l}\text{if } (M, \phi, \omega, v) \text{ is a Weinstein manifold and } p \text{ a critical point of } \phi \text{ then } W_p^- \text{ is isotropic} \\ \text{in particular, index}(p) \le n \text{ if } \dim M = 2n\end{array}}$$

Remark: This implies M can't be closed

(find another proof of this!)

exercise: Under hypothesis of lemma show W_p^+ is coisotropic.

Proof: We first note we can linearize v at p:

$$A = \text{"}dv(p)\text{"} : T_pM \to T_pM$$

(in local coords $\underset{p=0}{}$ $v: \mathbb{R}^n \to \mathbb{R}^n$ (really $\begin{array}{l}\mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n \\ x \mapsto (x, v(x))\end{array}$)

$dv(0) = T_0\mathbb{R}^n \to T_0\mathbb{R}^n$)

note the flow $\psi_t : M \to M$ induces a flow $d\psi_t : T_pM \to T_pM$

exercise: 1) This flow is given by $v \mapsto e^{tA} v$

2) $A = \left(\frac{\partial v_i}{\partial x_j}\right)$ in local coords

so $\pm$ eigenspaces of A are $T_p^{\pm}M$

now we see for $u, w \in T_p^- M$

$$e^t \omega(u, w) = \psi_t^* \omega(u, w) = \omega(d\psi_t(u), d\psi_t(w)) = \omega(e^{At}u, e^{At}w)$$

R.H.S. is bounded since $u, w \in T_p^- M$ so $\omega(u, v) = 0$

i.e. $T_p^- M$ is isotropic!

<u>exercise</u>: if $v \in T_x W^-$, then $(d\psi_t)_x(v) \to 0$ as $t \to \infty$

<u>Hint</u>: near p real part of $(d\psi_t)_x$ on $T_x W^-$ is negative (now consider $\psi_t \circ \psi_s = \psi_{s+t}$)

now if $u \in T_x W^+$, then we have

$$e^t \lambda_x(u) = (\psi_t^* \lambda)_x(u) = \lambda_{\psi_t(x)}(\underbrace{(d\psi_t)_x(u)}_{\text{in } TW^-}) = 0$$

so as $t \to \infty$, $= 0$

$\therefore \lambda_x(u) = 0$ & $\omega = d\lambda = 0$ on W_p^-

Weinstein handles:

consider $\mathbb{R}^{2n} = T^*\mathbb{R}^k \times \mathbb{R}^{2(n-k)}$

$\mathbb{R}^k$ has coords $q_1 \dots q_n$

$T^*\mathbb{R}^k$ " " $q_1 \dots q_n\, p_1 \dots p_n$

$\mathbb{R}^{2(n-k)}$ " " $x_1, y_1, \dots x_{n-k}, y_{n-k}$

set $\omega = \sum_{i=1}^{k} dq_i \wedge dp_i + \sum_{i=1}^{n-k} dx_i \wedge dy_i$

$v = \sum_{i=1}^{k} \left(2p_i \frac{\partial}{\partial p_i} - q_i \frac{\partial}{\partial q_i}\right) + \frac{1}{2} \sum_{i=1}^{n-k} \left(x_i \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_i}\right)$

<u>note</u>: $\mathcal{L}_v \omega = d\iota_v \omega + \cancel{\iota_v d\omega}^{\,0} = d\left(\sum_{i=1}^{k} -2p_i dq_i - q_i dp_i + \frac{1}{2}\sum_{i=1}^{n-k} x_i dy_i - y_i dx_i\right)$

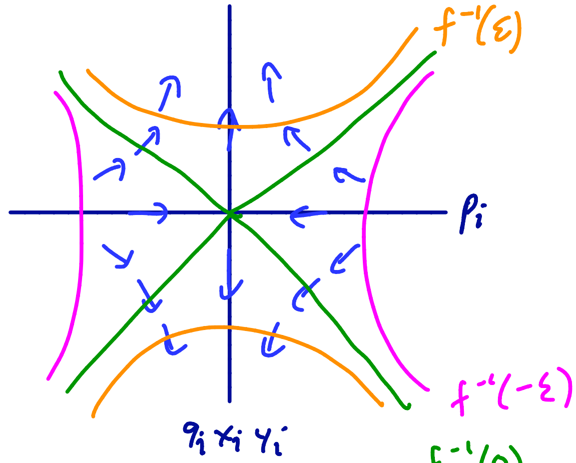
$= \sum dq_i \wedge dp_i + \sum_{i=1}^{n-k} dx_i \wedge dy_i = \omega$

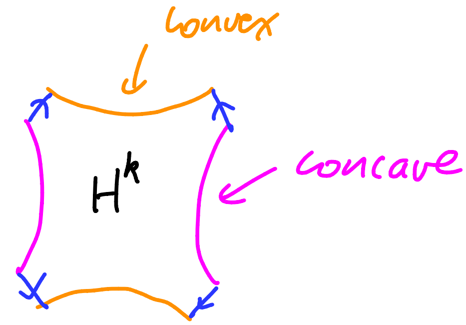
$f = \sum_{i=1}^{k} \left(q_i^2 - \frac{1}{2} p_i^2\right) + \frac{1}{4} \sum_{i=1}^{n-k} (x_i^2 + y_i^2)$

so $df(v) = \left[\sum -2q_i dq_i + p_i dp_i + \frac{1}{2} \sum x_i dx_i + y_i dy_i\right](v)$

$= \sum +2p_i^2 + 2q_i^2 + \frac{1}{4} \sum x_i^2 + y_i^2 > 0$ if $v \neq 0$

so v is gradient like for f



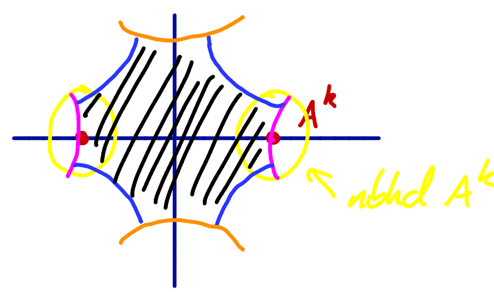


(H, ω, v, f) Weinstein handle

note:

1) $\partial_- H^k$ has an induced contact structure and $\partial_- H^k \cap \{p_1\text{-space}\}$ is a Legendrian $A^k = S^{k-1}$

2) We can choose a model for H^k so that in $\partial_- H^k$ is contained in any preassigned nbhd of A^k



now let S be an isotropic submanifold of (M, ξ)

let α be any 1-form s.t. $\xi = \ker \alpha$

$d\alpha|_\xi$ symplectic

recall $TS \subset TS^{\perp d\alpha}$

set $CSN(S) = \left(TS^{\perp d\alpha}\right) / TS$

$d\alpha$ induces a symplectic structure on $CSN(S)$ and a different contact form for ξ would induce a conformal symplectic structure on $CSN(S)$ so $CSN(S)$ is called the conformal symplectic normal bundle to S

note: $TM|_S = \xi \oplus TM/_\xi = (CSN(S) \oplus TS \oplus \xi/_{TS^\perp}) \oplus \mathbb{R}$

$\mathbb{R}$ spanned by a Reeb vector field

exercise: $T(T^*S) \cong TS \oplus \xi/_{TS^\perp}$

So the only part of $TM|_S$ not determined by S is $CSN(S)$

lemma 8 (Weinstein):

let (M^{2n}_i, ω_i) be a symplectic manifold $i = 1,2$

v_i an expanding vector field for v_i

Σ_i a hypersurface in M_i transverse to v_i (oriented so v_i positively transverse)

let $\xi_i = \ker(\iota_{v_i}\omega_i)|_{\Sigma_i}$ be the induced contact str

and Λ_i an isotropic S^k

Given a diffeomorphism $\phi: \Lambda_1 \to \Lambda_2$ that is covered by a bundle isomorphism $CSN(\Lambda_1) \to CSN(\Lambda_2)$

$\exists$ neighborhoods N_i of Λ_i in M_i and an extention of ϕ to a diffeomorphism

$$\Phi: N_1 \to N_2$$

such that

1) $\Phi(N_1 \cap \Sigma_1) = N_2 \cap \Sigma_2$

2) $\Phi^*(\iota_{v_2}\omega_2|_{\Sigma_2}) = \iota_{v_1}\omega_1|_{\Sigma_1}$

3) $\Phi^*\omega_2 = \omega_1$

4) $\Phi_*(v_1) = v_2$

Proof: let $\alpha_i = \iota_{v_i}\omega_i|_{\Sigma_i}$ be the contact forms on Σ_i

Claim: $\exists$ neighborhoods $\tilde{N}_1 \subset \Sigma_1$ of Λ_1 and a diffeomorphism $\Psi: \tilde{N}_1 \to \tilde{N}_2$ st. $\Psi^* \alpha_2 = \alpha_1$

given the claim, recall Th$^{\underline{m}}$ 5 gives an identification (using flow of v_1) of a nbhd N_1 of Λ_1 in M_1 with

$$((-\varepsilon,\varepsilon) \times \tilde{N}_1, d(e^t \alpha_1))$$

since Ψ preserves α_1 we can extend Ψ to $\Phi: N_1 \to N_2$ st. $\Phi^* \omega_2 = \omega_1$ and by construction 1)–4) in theorem are true!

Proof of Claim:

Step I: $\exists$ a hypersurface $R_i = \Lambda_1 \times D^{2n-k-2}$ such that

a) ξ_1 tangent to R_i along $\Lambda_1 = \Lambda_1 \times \{0\}$

b) $d\alpha_i$ symplectic on Σ_i

c) $\Sigma_1 \pitchfork$ to Reeb field X_{α_i}

to see this note the normal bundle to Λ_1 in ξ_1 is

$$\nu_1 = CSN(\Lambda_1) \oplus \xi_1 / T\Lambda_i$$

let R_1 be the image of a nbhd of the zero section in ν_i under the normal exponential map (use any metric)

note: R_1 satisfies all properties!

Step II: $\exists$ a symplectomorphism $\psi: (R_1, d\alpha_1) \to (R_2, d\alpha_2)$

Using the identifications of the CSN bundles in the hypothesis and the fact that Λ_1 is isotropic in $(R_1, d\alpha_1)$ we can build a diffeomorphism $R_1 \to R_2$ that takes $d\alpha_1$ to $d\alpha_2$ along Λ_1 (this is small extension of the argument in Cor III.5)

$Th^{\underline{m}}$ III.1 now says we can isotop diffeo to be a symplectomorphism in a nbhd of Λ_i

Step III: Build desired nbhds $\tilde{N}_1$ and $\Psi: \tilde{N}_1 \to \tilde{N}_2$

Using the Reeb vector field X_{α_i} build a nbhd $\tilde{N}_i$ of R_i in Σ_i of the form $(-\varepsilon, \varepsilon) \times R_i$.

extend $\psi: R_1 \to R_2$ to $\phi: \tilde{N}_1 \to \tilde{N}_2$

since the Reeb flow of α_i preserves $d\alpha_i$ we see

$$\phi^*(d\alpha_2) = d\alpha_1$$

also $\phi^* \alpha_2 = \alpha_1$ on $T\Sigma_1|_{\Lambda_1}$ (have same kernel and eval same on X_{α_1})

we now build an embedding $f_1: N_1 \to N_1$ such that

might need to shrink

i) $f_1 = id_{\Lambda_1}$

ii) $f_1^*(\phi^* \alpha_2) = \alpha_1$

so $\Psi = \phi \circ f_1$ is desired map

to build f consider

$$\beta_t = \alpha_1 + t(\phi^* \alpha_2 - \alpha_1)$$

note: β_t is contact near Λ_1

(so on N_1 if N_1 small enough)

this is because at Λ_1, $\ker \beta_t = \ker \alpha_1$

and $d\beta_t = d\alpha_1$ on $\ker \beta_t$ is sympl.

$\therefore$ $d\beta_t$ sympl. on $\ker \beta_t$ near Λ_1

now $d(\phi^*\alpha_2 - \alpha_1) = 0$ and

$\phi^*\alpha_2 - \alpha_1 = 0$ on Λ_1

$\therefore \exists$ some function $h : N_1 \to \mathbb{R}$ s.t.

$$dh = -(\phi^* \alpha_2 - \alpha_1) \quad \text{and}$$

$$h = 0 \quad \text{on} \quad \Lambda_1$$

if X_t is the Reeb vector field of β_t

then set $v_t = h X_t$ and denote the flow of

v_t by f_t

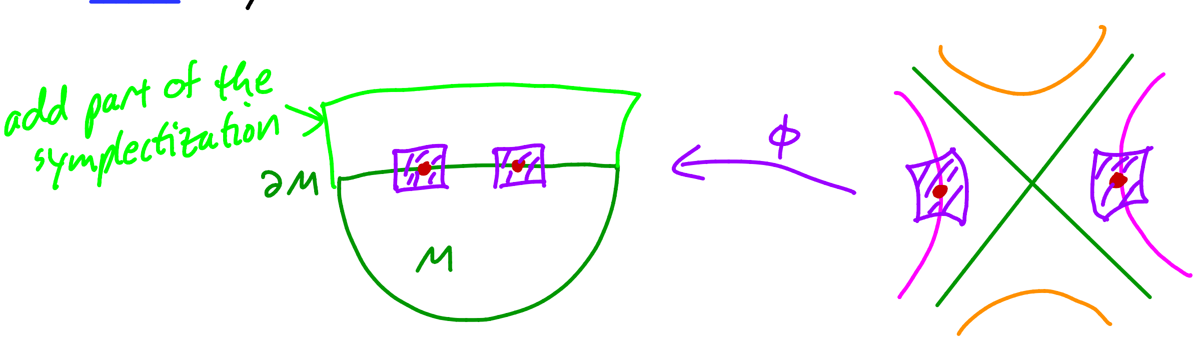
now
$$\frac{d}{dt} f_t^* \beta_t = f_t^* \left(\mathcal{L}_{v_t} \beta_t + \frac{d}{dt} \beta_t \right)$$
$$= f_t^* \left(d\iota_{v_t} \beta_t + \cancel{\iota_{v_t} d\beta_t}^{\,0} - dh \right)$$
$$= f_t^* (dh - dh) = 0$$

$\therefore f_t^* \beta_t = \beta_0$ ▦

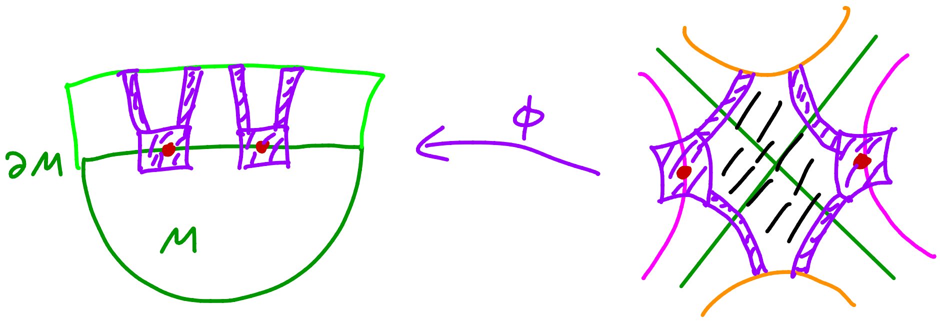
lemma 9:

> If (M, ω) a symplectic manifold with convex boundary and ξ induced contact structure on ∂M
> then for any isotropic sphere Λ^{k-1} in $(\partial M, \xi)$ and choice of trivialization of $SCN(\Lambda)$ (this might not exist)
> one may attach a Weinstein k-handle to M to get a new symplectic manifold with convex boundary
> Moreover, if (M, ω) was Weinstein, then the result of attaching the k-handle is also Weinstein

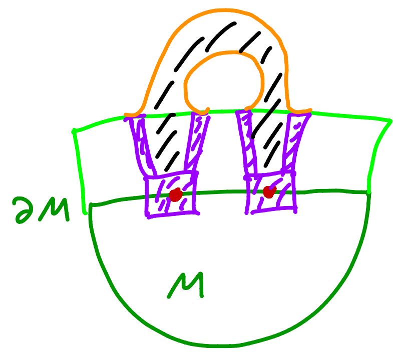
Proof: by lemma 8 we have an identification



now extend along part of the symplectization



glue in handle



exercise: Show everything fits together ∎

exercise: Show if (M, ω, v, ϕ) is Weinstein, then M is obtained from $\emptyset$ by a sequence of Weinstein handle attachments as in lemma 9.

Th$^{\underline{m}}$ (Cieliebak - Eliashberg):

> let $\phi: M^{2n} \to \mathbb{R}$ be a proper Morse function, bonded below with no critical points of index $> n$
>
> let η be a non-degenerate 2-form on M
>
> If $n > 2$, then $\exists$ a Weinstein structure (ω, v, ϕ) on M such that η and ω are homotopic through non-degenerate 2-forms

We call a complex manifold (M, J) Stein if it admits an exhausting J-convex function $\phi: M \to \mathbb{R}$

exhausting means: ϕ is bounded below and proper

J-convex (a.k.a. (strictly) plurisubharmonic) if

$$\omega_\phi = -d(d\phi \circ J)$$

is symplectic

exercise: 1) J is compatible with ω_ϕ

2) ω_ϕ, J define a metric g_ϕ
If X_ϕ is the gradient with respect to g_ϕ
then $\mathcal{L}_{X_\phi} \omega_\phi = \omega_\phi$

(so if ϕ is Morse, then $(M, \omega_\phi, X_\phi, \phi)$
is a Weinstein manifold!)

3) if $\Sigma^2 \subset M$ is an immersed holomorphic curve
then $\phi|_\Sigma$ can have no interior local maxima
and any maxima on $\partial\Sigma$ is not a critical point
($\phi|_\Sigma$ is subharmonic)

Thm (Grauert):

A complex manifold (M, J) is Stein
$\Longleftrightarrow$
(M, J) can be properly holomorphically embedded in $\mathbb{C}^N$ for some N

note: $\mathbb{C}^N \to \mathbb{R} : (z_1, \dots, z_N) \mapsto \sum |z_i|^2$
is plurisubharmonic so given $M \subset \mathbb{C}^N$ as in thm
clearly get plurisubharmonic

so the hard part of the theorem is $(\Rightarrow)$

note: if $X \subset \mathbb{C}P^n$ is any complex submanifold
and $H \subset \mathbb{C}P$ is a hyperplane
then $X - H$ is a Stein manifold

Thm (Cieliebak-Eliashberg):

$\boxed{\text{If } (M, \omega, v, \phi) \text{ a Weinstein manifold then there is a Stein structure on } M \text{ such that its induced Weinstein structure is homotopic to } (M, \omega, v, \phi)}$