VII Almost Complex Geometry

A. Donaldson's Results

The first result we would like to prove is

Th$^{\underline{m}}$ 1 (Donaldson):

> If (M,ω) is a closed symplectic manifold with
> $\frac{1}{2\pi}[\omega] \in H^2(M;\mathbb{Z})$
> then for large k, there is a codimension 2 symplectic submanifold Σ s.t. $[\Sigma]$ = Poincaré dual of $\frac{k}{2\pi}[\omega]$
> Moreover, $M-\Sigma$ is a Weinstein manifold

Remarks: This theorem gives a good way to try and see if a given manifold is symplectic or not.

More specifically, th$^{\underline{m}}$ says any symplectic $2n$-manifold breaks into 2 pieces:

① a symplectic D^2-bundle over a $(2n-2)$-manifold

② a Weinstein manifold

non-existence:

e.g. Does $\mathbb{C}P^2$ have an exotic smooth structure that supports a symplectic structure ("Symplectic Poincaré conjecture")

If X homeomorphic to $\mathbb{C}P^2$ and X admits a symplectic structure ω, then $X = A \cup B$ where

$A = D^2$-bundle over a surface and
B = Weinstein manifold

note: 1) ∂A concave $= \partial B$ convex

2) topology of B (e.g. homology) is somewhat determined by genus of A and $\mathbb{C}P^2$

does S^1-bundle ∂A with given contact structure have an appropriate Weinstein filling?

eg if A is D^2-bundle over a surface Σ s.t. $[\Sigma]$ generator of $H_2(\mathbb{C}P^2)$

then one can use fact that $c_1(\omega) = \pm 3$ ← Sect III F

to see $\Sigma \cong S^2$ (use "adjunction formula")

in this case $\partial A = S^3$ and unique choice for B ← Gromov

so $X = \mathbb{C}P^2$

similarly if $[\Sigma] = 2 \in H_2(\mathbb{C}P^2)$ then $\partial A = L(4,1)$ ← Lens space

and only 2 choices for B and only one has right homology so $X \cong \mathbb{C}P^2$

McDuff

what about for other Σ?

existence: Can you inductively prove $2n$-manifold with almost symplectic structure has symplectic structure?

Thm (Pancholi):

If X^n a manifold of dimension ≥ 4

$\alpha \in H^2(X;\mathbb{Z})$

then $\exists$ an oriented submanifold M^{n-2}

s.t. 1) $[M]$ = Poincaré Dual of α

2) $X - M$ admits a Morse function with no critical points $> \frac{n+1}{2}$

in $\pi_1 = 1$ case previously proven by Freedman and Kato-Matumoto

so if X^{2n} is a manifold with a non-degenerate 2 form ω_0 and a class $h \in H^2(X)$ s.t. $h^n \neq 0$

then thm above gives M Poincaré dual to h

s.t. $X - M$ has handles of index $\leq n$

so if $n>2$, then ω_0 can be deformed into a symplectic str on $X-M$ coming from a Stein str.

if $\omega_0|_M$ non-degenerate we could hope to inductively show M has a symplectic structure (note: h pulled back to M satisfies $h^{n-1} \neq 0$)

now a nbhd A of M has a symplectic structure and $B = X - A$ has one (it is Stein)

so existence comes down to

① Enhance Panchdi's result to get $\omega_0|_M$ non-degen

② show one can arrange contact structures on ∂A and ∂B to match up

③ Find a base case!

(e.g. say simply connected 6-manifolds then try for simply connected 8-manifolds by enhansing Pancholi's th^{m} so M is simply connected if X is)

Th^{m} 2 (Donaldson):

(M, ω) a symplectic manifold

suppose $[\omega] \in H^2_{DR}(M)$ is an integral class

For sufficiently large integers K there is a topological Lefschets pencil on M whose fibers are symplectic and homologous to the Poincaré dual of $k[\omega]$

Remark: In Sections II E-F we already discussed using this theorem to study symplectic manifolds

Th$^{\underline{m}}$ 3 (Auroux):

> any compact symplectic 4-manifold (M,ω) is a symplectic branched cover of $(\mathbb{C}P^2, \omega_{FS})$

All these theorems are proven by finding sections of line bundles, so we begin by studying complex line bundles.

more specifically, given a line bundle

$$\begin{tikzcd} \mathbb{C} \arrow[r] & L \arrow[d, "\pi"] \\ & M \end{tikzcd}$$

if $\sigma : M \to L$ a section that is transverse to the zero section Z, then $\sigma^{-1}(Z)$ is a codimension 2 submanifold of M!

so to prove Th$^{\underline{m}}$1 we just need to find the right line bundle and the right section

similarly we will see for Th$^{\underline{m}}$2 we just need to find 2 sections and for Th$^{\underline{m}}$3, 3 sections

B Complex line bundles

We consider complex line bundles over a manifold

$$\begin{tikzcd} \mathbb{C} \arrow[r] & L \arrow[d, "\pi"] \\ & M \end{tikzcd}$$

from Section IV.D we know that if $\{U_\alpha\}$ an open cover of M such that $\exists$ bundle isomorphisms

$$\begin{tikzcd} \pi^{-1}(U_\alpha) \arrow[rr, "\phi_\alpha"] \arrow[dr, "\pi"'] & & U_\alpha \times \mathbb{C} \arrow[dl, "pr"] \\ & U_\alpha & \end{tikzcd}$$

then we have

$$\phi_\alpha \circ \phi_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{C} \to (U_\alpha \cap U_\beta) \times \mathbb{C}$$
$$(x, v) \longmapsto (x, g_{\alpha\beta}(x) v)$$

where $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^* = \mathbb{C} - \{0\}$ are the transition functions and they satisfy

$$g_{\alpha\alpha}(x) = 1$$
$$g_{\alpha\beta} = g_{\beta\alpha}^{-1}$$
$$g_{\alpha\beta} \circ g_{\beta\gamma} = g_{\alpha\gamma}$$

moreover, given a cover $\{U_\alpha\}$ and $\{g_{\alpha\beta}\}$ satisfying the equations $\exists$ a line bundle realizing this data

When are 2 bundles L, L' the same?

we say L is isomorphic to L' if $\exists$ a bundle map

$$\begin{tikzcd} L \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & \circ & L' \arrow[dl, "\pi'"] \\ & M & \end{tikzcd}$$

that is an isomorphism on each fiber

if L and L' given by transition functions $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$

(note we can always assume cover $\{U_\alpha\}$ same for both L and L')

then ϕ gives us

$$\begin{tikzcd} U_\alpha \times \mathbb{C} \arrow[rrr, bend right, "{(x, v) \longmapsto (x, \lambda_\alpha(x)(v))}"'] & \pi^{-1}(U_\alpha) \arrow[l, "\phi_\alpha"'] \arrow[r, "\phi"] & (\pi')^{-1}(U_\alpha) \arrow[r, "\phi'_\alpha"] & U_\alpha \times \mathbb{C} \end{tikzcd}$$

where $\lambda_\alpha : U_\alpha \to \mathbb{C}^*$

exercise: 1) Show $g'_{\alpha\beta} = \lambda_\alpha g_{\alpha\beta} \lambda_\beta^{-1}$ $\circledast$

2) Show 2 line bundles L, L' are isomorphic

iff their trasistion functions are related by $\circledast$

now given 2 line bundles L and L' given by transition functions $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ (note we can always assume cover $\{U_\alpha\}$ same for both L and L')

we define their tensor product $L\otimes L'$ to be the bundle with transition functions $\{g_{\alpha\beta}\cdot g'_{\alpha\beta}\}$

we also define the inverse of L, denoted L', to be the bundle associated to $\{g^{-1}_{\alpha\beta}\}$

exercise: the set $Line(M)$ of isomorphism classes of complex line bundles is an abelian group with respect to the operation $\otimes$.

example: Consider $\mathbb{C}P^1$

$\mathbb{C}P^1 = \mathbb{C}\cup\{\infty\}$ set $U_\alpha = \mathbb{C}$ and $U_\beta = \mathbb{C}P^1 - \{0\}$

the map $g^n_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathbb{C}^* : z \mapsto z^n$

defines a line bundle L^n

let $L = L^1$ note $L^n = \underbrace{L\otimes L \ldots \otimes L}_{n \text{ times}} = L^{\otimes n}$

exercise: $Line(\mathbb{C}P^1) \cong \mathbb{Z}$

note: If $L \to M$ has local trivializations $\{\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}\}$ with transition maps $\{g_{\alpha\beta}\}$ then a section $s: M \to L$ gives functions

$$\begin{tikzcd} U_\alpha \arrow[r, "s"] \arrow[rrr, bend right, "s_\alpha"'] & \pi^{-1}(U_\alpha) \arrow[r, "\phi_\alpha"] & U_\alpha \times \mathbb{C} \arrow[r, "pr"] & \mathbb{C} \end{tikzcd}$$

and the $\{s_\alpha\}$ satisfy $g_{\alpha\beta}\, s_\beta = s_\alpha$ on $U_\alpha \cap U_\beta$

moreover any collection of functions $s_\alpha : U_\alpha \to \mathbb{C}$ satisfying this relation gives a section

i.e. "sections are twisted functions on M"

in particular, if $s,t : M \to L$ are 2 sections then

if $s \neq 0$, $t/s : M \to \mathbb{C}$ is a function.

We now want to talk about differentiating sections of a line bundle, for this we introduce connections

a connection on a line bundle L over M is a linear map

$$\nabla : \Gamma(L) \to \Gamma(T^*M \otimes L)$$

satisfying $\nabla(f \otimes s) = df \otimes s + f \nabla s \quad \forall f \in C^\infty(M),\ s \in \Gamma(L)$

so given $v \in \mathfrak{X}(M)$ we write $\nabla_v : \Gamma(L) \to \Gamma(L) : s \longmapsto (\nabla s)(v)$

and think of this as a directional derivative

example: if $L = M \times \mathbb{C}$ so $\Gamma(L) = C^\infty(M)$ then for any $\beta \in \Omega^1(M)$

$$\Gamma(L) \longrightarrow \Gamma(T^*M \otimes L) = \Gamma(T^*M)$$
$$f \longmapsto df + f\beta$$

is a connection (later we will see this is all connections on $M \times \mathbb{C}$)

lemma 4:

If s,t are sections of L that agree near $x \in M$ then $\nabla s(x) = \nabla t(x)$

"connections are local operators"

Proof: if $s = 0$ near x, then let f be a bump function s.t. $f = 1$ outside support of s an $= 0$ near x.

then $\nabla s(x) = (\nabla f s)(x) = (df \otimes s + f \nabla s)(x) = 0$

∴ if $s = t$ in a nbhd of x, then $\nabla s(x) = \nabla t(x)$ ▨

note: lemma $\Rightarrow$ given any connection ∇ on L and open set $U \subset M$ $\exists!$ connection ∇^U on $L|_U$ s.t. $(\nabla s)|_U = \nabla_U(s|_U)$ and ∇ is determined by its restrictions to open sets

lemma 5:

every line bundle $L \to M$ has a connection and the set of connections on L is

$$\mathcal{A}(L) = \{\nabla^0 + \beta : \beta \in \Omega^1(M)\}$$

where ∇^0 is any one connection on L.

Proof: to construct a connection let $\{\phi_\alpha : U_\alpha \to M\}$ be a collection of local trivializations

let $s_\alpha : U_\alpha \to L$ be local frames (ie. non zero sections)

now given any section $s : M \to L$ note that

$f_\alpha = s/s_\alpha : U_\alpha \to \mathbb{C}$ are functions and

$s = f_\alpha s_\alpha$ on U_α

let $\{\psi_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$

set $\nabla s = \sum_\alpha \psi_\alpha (df_\alpha \otimes s_\alpha)$

exercise: Check this is a connection.

now if ∇^0 and ∇ are two connections on L

then note $(\nabla - \nabla^0)(fs) = f(\nabla - \nabla^0)s$

for any $f \in C^\infty(M)$ and $s \in \Gamma(L)$

exercise: Show this $\Rightarrow (\nabla - \nabla^0)(s)(x)$ only depends on $s(x)$ and this $\Rightarrow \exists$ a 1-form β s.t. $(\nabla - \nabla^0)s = \beta \otimes s$

$\blacksquare$

now if $s: U_\alpha \to L$ is a local non zero section

then $\nabla s \in \Gamma(T^*M \otimes L)$

exercise: $\exists$ 1-form $A_\alpha \in \Omega^1(U_\alpha)$ s.t. $\nabla s = A_\alpha \otimes s$

(i.e. $A_\alpha = \nabla s / s$)

so for any $t: U_\alpha \to L$, there is some function $f: U_\alpha \to \mathbb{C}$ st $t = fs$ so

$$\nabla t = \nabla fs = df \otimes s + f A_\alpha \otimes s$$

$\therefore$ in the frame s, the connection is $(d + A_\alpha)$

if $s': U_\alpha \to L$ is another non zero section then $\exists$ non-zero $f: U_\alpha \to \mathbb{C}$ st. $s' = fs$

if in the frame s we have $\nabla = d + A_\alpha$ and

" " s' " " $\nabla = d + A_\alpha'$

then $A_\alpha' \otimes s = \nabla s' = \nabla fs = df \otimes s + f \nabla s$

$= df \otimes \frac{1}{f} s' + f A_\alpha \otimes \frac{1}{f} s' = (\frac{1}{f} df + A_\alpha) \otimes s'$

$$\therefore \quad A_\alpha' = A_\alpha + \underbrace{\frac{1}{f} df}$$

note this is $d(\ln f)$ so closed

so given a cover of M by local trivializations

$$\{\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}\}$$

let s_α be the section $U_\alpha \to L$ given by $\phi_\alpha^{-1}(1)$

constant 1 section in $U_\alpha \times \mathbb{C}$

then a connection ∇ gives $A_\alpha \in \Omega^1(U_\alpha)$ st. $\nabla s_\alpha = A_\alpha \otimes s_\alpha$

and they are related by

$$\circledast \quad A_\beta = A_\alpha + \frac{1}{g_{\alpha\beta}} dg_{\alpha\beta} \qquad \text{on } U_\alpha \cap U_\beta$$

also the $\{A_\alpha\}$ determine ∇

moreover any collection of 1-forms $\{A_\alpha \in \Omega^1(U_\alpha)\}$ satisfying $\circledast$ gives a connection on L.

now given ∇ on L, let $\{A_\alpha\}$ be the associated 1-forms

note: $dA_\alpha = dA_\beta$ on $U_\alpha \cap U_\beta$

so dA_α glue a global closed 2-form $F_\nabla \in \Omega^2(M)$!

F_∇ is called the curvature of ∇

given $\nabla : \Gamma(L) \to \Gamma(T^*M \otimes L)$ we can extend ∇ to

$$\nabla : \Gamma(\Lambda^k T^*M \otimes L) \to \Gamma(\Lambda^{k+1} T^*M \otimes L)$$

by defining $\nabla(\beta \otimes s) = d\beta \otimes s + (-1)^k \beta \wedge \nabla s$

lemma 6:

1) $\nabla^2 : \Gamma(L) \to \Gamma(\Lambda^2 T^*M \otimes L)$ is tensor with F_∇

2) $F_{\nabla+\alpha} = F_\nabla + d\alpha$ (recall any other connection on L is $\nabla + \alpha$ some $\alpha \in \Omega^1(M)$)

Proof:

1) in a local chart with non-zero section s

$$\nabla s = A \otimes s$$

and for any section $t = fs$ we have

$$\begin{aligned} \nabla^2 t &= \nabla(df \otimes s + fA \otimes s) \\ &= \cancelto{0}{d(df)} \otimes s - df \wedge \nabla s + d(fA) \otimes s - fA \wedge \nabla s \\ &= -df \wedge A \otimes s + (df \wedge A + f dA) \otimes s + f\cancelto{0}{A \wedge A} \otimes s \\ &= dA \otimes fs = dA \otimes \underline{t} \end{aligned}$$

2) locally $F_\nabla = dA$ and since for $\nabla + \alpha$ the associated 1-form is $A + \alpha$ we see $F_{A+\alpha} = d(A+\alpha) = F_A + d\alpha$ ∎

suppose $\begin{matrix} L \\ \downarrow \\ M \end{matrix}$ is a Hermitian line bundle, recall this means

there is a fiberwise Hermitian inner product $\langle\cdot,\cdot\rangle : L_x \times L_x \to \mathbb{C}$

and is equivalent to transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to S^1$

$\uparrow$ unit circle in $\mathbb{C}$

a connection ∇ on such a bundle is called Hermitian

if for 2 sections s_1, s_2 we have

$$d\langle s_1, s_2\rangle = \langle \nabla s_1, s_2\rangle + \langle s_1, \nabla s_2\rangle$$

now given such a connection, let s be a local section with length 1, then we have the 1-form A s.t. $\nabla s = A \otimes s$ and we see

$$\langle s,s\rangle = 1 \quad \text{so} \quad 0 = d\langle s,s\rangle = \langle \nabla s, s\rangle + \overline{\langle s, \nabla s\rangle} = \langle A\otimes s, s\rangle + \langle s, A\otimes s\rangle$$

$$= \langle A\otimes s, s\rangle + \overline{\langle A\otimes s, s\rangle} = (A + \bar{A})\langle s,s\rangle$$

$$= A + \bar{A}$$

so the real part of A is 0 i.e. $A \in i\Omega^1(M)$

before we were thinking of A as a complex valued 1-form

now this means A is purely imaginary

<u>exercise</u>: given a Hermitian line bundle L and Hermitian connection ∇, let $U(L)$ be the unit norm bundle in L so $U(L)$ is an S^1-bundle

Show $\exists$ a 1-form A on $U(L)$ s.t. for any local trivialization $s_\alpha : U_\alpha \to L$ by a unit section of L

we have $ds_\alpha = A_\alpha$ and $A_\alpha = s_\alpha^* A$

Thm 7 (Chern-Weil):

let L be a Hermitian line bundle over M with a Hermitian connection ∇

let $s: M \to L$ be a section transverse to the zero section Z

set $S = s^{-1}(Z)$

Then

$$c_1(L) = \text{Poincaré dual } [S]$$

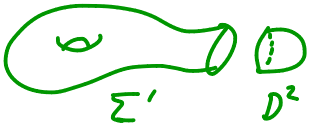
and for any 2-cycle C transverse to S

$$S \cdot C = \frac{i}{2\pi} \int_C F_\nabla = \langle c_1(L), C \rangle$$

Proof:

for the first part recall c_1 for complex line bundles over a surface Σ

given L over Σ, let $\Sigma = \Sigma' \cup D^2$



we can take the structure group of L to be $U(1) = S^1$ (All complex bundles have Hermitian structure)

$L|_{\Sigma'} = \Sigma' \times \mathbb{C}$

when we attach the 2-cell D^2 to Σ we must glue $\partial\Sigma' \times \mathbb{C} \to \partial D^2 \times \mathbb{C}$

by an element n of $\pi_1(U(1)) \cong \mathbb{Z}$

this element is precisely $c_1(L)$ evaluated on D^2, which can be taken to be the generator of the chain group $C_2(\Sigma) = \mathbb{Z}$

so $\langle c_1(L), [\Sigma] \rangle = n$

the above gluing map can be taken to be

$$\begin{aligned} S^1\times\mathbb{C} &\longrightarrow S^1\times\mathbb{C} \\ (e^{i\theta}, v) &\longmapsto (e^{i\theta}, e^{in\theta} v) \end{aligned}$$

so if we take the section $s:\Sigma' \to L|_{\Sigma'} = \Sigma'\times\mathbb{C}$
$x \longmapsto (x,1)$

then on ∂D^2 we have the section

$$\begin{aligned} \partial D^2 = S^1 &\to S^1\times\mathbb{C} \subseteq D^2\times\mathbb{C} \\ e^{i\theta} &\mapsto (e^{i\theta}, e^{in\theta}) \end{aligned}$$

we can extend s over D^2 by $z \longmapsto (z, z^n)$

exercise: Show s can be perturbed to be transverse to the zero section and intersect it n times counted with sign.

$\therefore \langle c_1(L), [\Sigma]\rangle = \# s^{-1}(0)$

now for a general $\begin{matrix} L \\ \downarrow \\ M \end{matrix}$ if $\Sigma \subset M$ a surface let $i:\Sigma\to M$ be inclusion, then

$$\begin{aligned} \langle c_1(L), [\Sigma]\rangle &= \langle i^*c_1(L), [\Sigma]\rangle \\ &= \langle c_1(i^*L), [\Sigma]\rangle \\ &= \text{intersection } \# \, n \text{ of a generic section w/ zero section} \end{aligned}$$

but if $s: M\to L$ a section transvers to z then this gives a generic section of i^*L and $n = \Sigma\cdot s^{-1}(z)$

so $\langle c_1(L), [\Sigma]\rangle = \Sigma\cdot s^{-1}(z)$

i.e. $[s^{-1}(Z)]$ is Poincaré dual to $c_1(L)$

for the second part let

$$M_\varepsilon = \{x \in M : |s(x)| \geq \varepsilon\}$$

and $C_\varepsilon = M_\varepsilon \cap C$

on M_ε we have $\frac{\nabla s}{s}$ is a well-defined complex valued 1-form

and $d\left(\frac{\nabla s}{s}\right) = F_\nabla$

so $\int_C F_\nabla = \lim\limits_{\varepsilon \to 0} \int_{C_\varepsilon} F_\nabla = \lim\limits_{\varepsilon \to 0} \int_{C_\varepsilon} d\left(\frac{\nabla s}{s}\right) = \lim\limits_{\varepsilon \to 0} \int_{\partial C_\varepsilon} \nabla s/s$

when $\varepsilon \to 0$ the last integral is supported in a small nbhd of $C \cap S$ and each such point is in some local trivialization of L, let s_α be the trivializing section near one of these points

<u>exercise</u>: we can choose coordinates on nbhd U so that

$$s = z s_\alpha \quad \text{or} \quad s = \bar{z} s_\alpha$$

now $\nabla s = dz \otimes s_\alpha + z A_\alpha \otimes s_\alpha$ or $\nabla s = d\bar{z} \otimes s_\alpha + \bar{z} A_\alpha \otimes s_\alpha$

and $\underbrace{\nabla s/s = \frac{1}{z} dz + A_\alpha}_{\text{we focus on this one}}$ or $\nabla s/s = \frac{1}{\bar{z}} d\bar{z} + A_\alpha$

$z = re^{i\theta}$ so $dz = e^{i\theta} dr + rie^{i\theta} d\theta$

and $\frac{1}{z} dz = \frac{1}{r} dr + i d\theta$

$= d(\ln r) + i d\theta$

since A_α is a well-defined 1-form on U, we see

$$\lim_{\varepsilon \to 0} \int_{\partial C_\varepsilon \cap U} A_\alpha = 0$$

$\therefore \lim\limits_{\varepsilon \to 0} \int_{\partial C_\varepsilon \cap U} \nabla s/s = \lim\limits_{\varepsilon \to 0} \int_{\partial C_\varepsilon \cap U} d(\ln r) + i d\theta = \lim\limits_{\varepsilon \to 0} \int_{\partial C_\varepsilon \cap U} i d\theta$

$= \lim\limits_{\varepsilon \to 0} \overset{\uparrow}{-i} 2\pi = -i 2\pi$

note as ∂C_ε or$^{\underline{n}}$ is <u>opposite</u> as ∂ disk so for each positive intersection point of $C \cap S$ we get a contribution of $-i2\pi$ to $\int_C F_\nabla$

similarly for each negative intersection we get $i2\pi$

$$\therefore\ C \cap S = \frac{i}{2\pi} \int_C F_\nabla$$

▦

Th^m 8:

Given any 2-form ω such that $[\omega] \in H^2(M;\mathbb{Z})$ there is a complex line bundle $L \to M$ and a Hermitian connection ∇ on L such that

$$\frac{i}{2\pi} F_\nabla = \omega$$

ie. $c_1(L) = [\omega]$

Proof: We give 2 proofs

1st Proof: *exercise:* Given $h \in H^2(M;\mathbb{Z})$, $\exists$ codim 2-submanifold Σ such that $[\Sigma] =$ Poincaré Dual to $h \in H_{n-2}(M)$

Hint: $H^2(M;\mathbb{Z}) \cong [M : \mathbb{C}P^\infty]$ ← homotopy classes of maps

↖ Brown representation th^m

take $f : M \to \mathbb{C}P^\infty$ representing h

can homotop f so $\operatorname{im} f \subset \mathbb{C}P^{\lceil n/2 \rceil}$ & $f \pitchfork \mathbb{C}P^{\lceil n/2 \rceil - 1}$

let $\Sigma = f^{-1}(\mathbb{C}P^{\lceil n/2 \rceil - 1})$

now a nbhd N of Σ in M is a D^2-bundle over Σ and we can assume structure group $U(1)$, ie. $\exists$ a $\mathbb{C}$-bundle

$$\begin{tikzcd} E \arrow[d, "\pi"] \\ \Sigma \end{tikzcd}$$

such that $N = \{ v \in E : \|v\| \leq 1 \}$

let $\tilde{\pi} = \pi|_N$ and set $F = \tilde{\pi}^* E$ this is a complex line bundle over N

let $s : N \to F : v \mapsto (v, v)$

note: s is a section of F and if we restrict s to any fiber $\tilde{\pi}^{-1}(x) = D^2$ of N we get the section $z \mapsto (z, z)$

i.e. $s \neq 0$ on ∂N, $s \pitchfork$ zero section, and $\Sigma = s^{-1}$(zero section)

so s trivializes $F|_{\partial N} \cong \partial N \times \mathbb{C}$

now glue $A \times \mathbb{C}$ to F along $\partial A \times \mathbb{C} \cong \partial N \times \mathbb{C}$

to get the complex line bundle
$$\begin{tikzcd} \mathbb{C} \arrow[r] & L \arrow[d] \\ & M \end{tikzcd}$$

and s extends to all of to be non-zero on A

$\therefore$ $s \pitchfork$ zero section and

$$[s^{-1}(\text{zero section})] = [\Sigma] = \text{Poincaré Dual}[\omega]$$

from Th$^{\underline{m}}$ 7 we know $c_1(L) = [\omega]$

2nd Proof:

let $\{U_\alpha\}$ be an open cover of M such that U_α and $U_\alpha \cap U_\beta$ contractible for all α, β (i.e. good cover from diff. topology)

we want to construct transition functions $\{g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^*\}$ defining a bundle L and $\{A_\alpha \in \Omega^1(U_\alpha)\}$ defining a connection ∇ on L s.t. $\frac{1}{2\pi} F_\nabla = \omega$ then done by Th$^{\underline{m}}$ 7

to this end note $d\omega|_{U_\alpha} = 0 \Rightarrow \exists \eta_\alpha \in \Omega^1(U_\alpha)$ s.t. $d\eta_\alpha = \omega|_{U_\alpha}$

now on $U_\alpha \cap U_\beta$ we see $d(\eta_\alpha - \eta_\beta) = \omega - \omega = 0$

so $\exists f_{\alpha\beta}$ s.t. $df_{\alpha\beta} = \eta_\alpha - \eta_\beta$

on $U_\alpha \cap U_\beta \cap U_\gamma$ we have

$$d(f_{\beta\gamma} - f_{\alpha\gamma} + f_{\alpha\beta}) = \eta_\beta - \eta_\gamma - \eta_\alpha + \eta_\gamma + \eta_\alpha - \eta_\beta = 0$$

so $\exists$ constants $a_{\alpha\beta\gamma}$ such that $f_{\beta\gamma} - f_{\alpha\gamma} + f_{\alpha\beta} = a_{\alpha\beta\gamma}$

exercise: Show, since $[\omega] \in H^2(M;\mathbb{Z})$, we can choose f st. all $a_{\alpha\beta\gamma} \in \mathbb{Z}$, $f_{\alpha\alpha} = 0$, and $f_{\alpha\beta} = -f_{\beta\alpha}$

Hint: Not so bad with sheaves, but maybe bit challenging without.

now set $g_{\alpha\beta} = e^{2\pi i f_{\alpha\beta}}$ and $A_\alpha = -2\pi i\, \eta_\alpha$

note: 1) $g_{\alpha\alpha}(x) = 1$

$$g_{\alpha\beta}^{-1} = e^{-2\pi i f_{\alpha\beta}} = e^{2\pi i f_{\beta\alpha}} = g_{\beta\alpha}$$

$$g_{\alpha\beta}\, g_{\beta\gamma} = e^{2\pi i (f_{\alpha\beta} + f_{\beta\gamma})} = e^{2\pi i f_{\alpha\gamma}} = g_{\alpha\gamma}$$

so $\{g_{\alpha\beta}\}$ satisfy conditions for trasition functions

$\therefore \exists$ a complex line bundle L over M realizing them

2) $A_\beta - A_\alpha = 2\pi i (\eta_\alpha - \eta_\beta) = 2\pi i\, d f_{\alpha\beta}$

now $\frac{1}{g_{\alpha\beta}} d g_{\alpha\beta} = e^{-2\pi i f_{\alpha\beta}}\, 2\pi i\, e^{2\pi i f_{\alpha\beta}}\, d f_{\alpha\beta} = 2\pi i\, d f_{\alpha\beta}$

so $A_\beta = A_\alpha + \frac{1}{g_{\alpha\beta}} d g_{\alpha\beta}$

$\therefore \{A_\alpha\}$ satisfy conditions for a connection ∇ on L

and $F_\nabla|_{U_\alpha} = dA_\alpha = -2\pi i\, d\eta_\alpha = -2\pi i\, \omega$

so $\frac{i}{2\pi} F_\nabla = \omega$ ▦