I. Introduction

While we don't have a complete classification of 3-manifolds, we do know a lot about them. we can break them into simple pieces

i.e. Seifert fibered spaces and
Hyperbolic manifolds

and we have lot's of tools to study each of these pieces

In this course we will

1) gives several constructions of 3-manifolds:
   identifying polyhedra
   Heegaard splittings
   Dehn surgery
2) Discuss the decompositions of 3-manifolds mentioned above:
   Prime and Torus Decompositions
3) Show how to convert algebra into topology!
   Disk and Sphere Theorems
4) Extensively study Dehn surgery
   eg. Kirby's Theorem, 3-mfd that are not surgery on a knot, knots with same surgeries, knots characterized by surgery, loosing & gaining properties via surgery ...

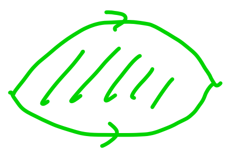
A. Examples and Constructions

simple examples

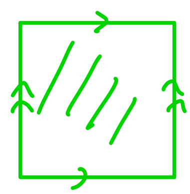
$$\mathbb{R}^3,\ S^3,\ D^3,\ T^3 = S^1 \times S^1 \times S^1,\ \Sigma \times S^1,\ \Sigma \times [0,1]$$

Construction 1: Identify faces of 3-dim'l polyhedra

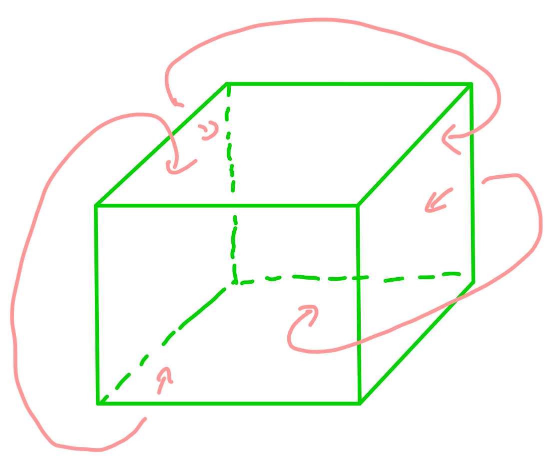
2D example: every surface is obtained this way

 $\cong S^2$  $\cong P^2$

projective plane

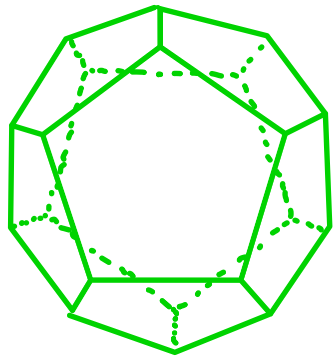
 $\cong T^2$ etc.

In 3-dimensions:

example:  $\cong T^3$

identify opposite faces by translation

example:



identify opposite faces of a regular dodecahedron after a rot^{n} of $\frac{2\pi}{10}$

this gives a manifold D called the Poincaré dodecahedral space

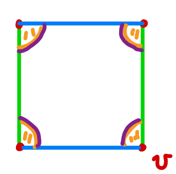
exercise: check D is a 3-manifold (easy for points on interior of dodecahedron and on faces and edges. need to be careful at vertices)

exercise (harder):

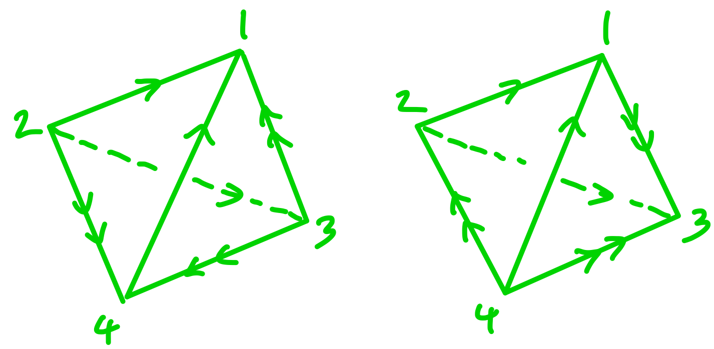
suppose X is obtained by identifying faces of a finite # of polyhedra

1) show X is a manifold iff the link of every vertex is a sphere (the link is formed by taking a nbhd of vertex in each simplex and gluing them & taking ∂

eg in 2D



2) let X be the result of the following identification

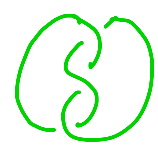


in the quotient space is a unique vertex and its nbhd is a cone on T^2 i.e. its link is T^2

so this is not a 3-mfd

but X − vertex is!

3) show X − vertex $\cong S^3 - K$

where $K =$ 

(this is quite hard! maybe google!)

4) show X is a 3-manifold iff

$\chi(X) = 0$

Euler characteristic

note: the dodecahedron has has 12 faces
30 edges.
20 vertices

edges are identified in 3's
vertices " " 4's
faces " " 2's

$\therefore$ we have

$$\left.\begin{array}{ll} 1 & \text{3-cell} \\ 6 & \text{2-cells} \\ 10 & \text{1-cells} \end{array}\right\} \Rightarrow \chi(D) = 0$$
$$\begin{array}{ll} 5 & \text{0-cells} \end{array}$$

so D a manifold follows from last exercise

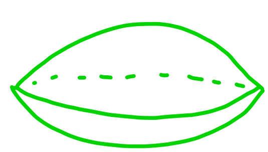
exercise:

$H_*(D) \cong H_*(S^3)$

but $\pi_1(D) \neq 1$

(extra credit $|\pi_1(D)| = 120$)

example: take a "lens"



identify top and bottom faces after a rotation through $\frac{2\pi q}{p}$

where $(p,q) = 1$

the quotient space is called a lens space and denoted $L(p,q)$

exercise:

1) Show $L(1,q) \cong S^3$, $L(0,q) \cong S^1 \times S^2$

2) Show $\pi_1(L(p,q)) = \mathbb{Z}/p$

3) let $S^3 \subset \mathbb{C}^2$ be the unit sphere

let $\mathbb{Z}/p$ act on S^3 by the action generated by

$$(z_1, z_2) \mapsto (e^{i2\pi/p} z_1, e^{i2\pi q/p} z_2)$$

show $L(p,q) \cong S^3/\sim$

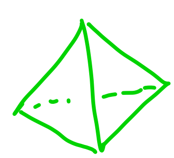
$\sim$ = quotient space gen by action

a topological space X is triangulable if $X \cong |K|$

where K is a finite simplicial complex

(i.e. obtained by identifying faces of a finite number of simplicies)

in dimension 3, we are gluing a finite number of tetrahedra

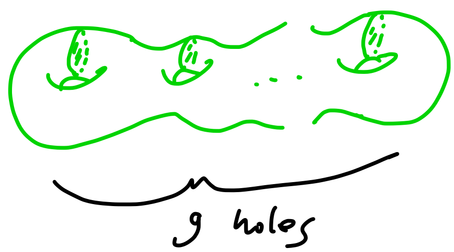


Fact (Moise 1952):

> any 3-manifold is triangulable and any 2 triangulations have "subdivisions" that are simplicially isomorphic

Construction 2: Heegaard splittings

a <u>handlebody of genus g</u> is a 3-manifold V homeomorphic to



that is consider the embedding of a surface of genus g in $\mathbb{R}^3$ shown above and V is the compact region bounded by it

lemma 1:

> M^3 is a handlebody of genus g
>
> $\iff$
>
> $\exists$ g embedded circles $c_1, \dots, c_g$ in ∂M s.t. $\exists$ disjoint properly embedded disks $D_1 \dots D_g$ in M st. $\partial D_i = c_i$ and $M \setminus (\text{tubular nbhd } (\cup D_i))$ is diffeomorphic to B^3

exercise: generalize this by replacing second statement with $D_1, \dots, D_{g+k}$ disks that cut M into $(k+1)$ 3-balls

we will prove this later, but for now

a <u>Heegaard splitting</u> of a closed 3-manifold M is

a decomposition of M

$$M = V_1 \cup_\Sigma V_2$$

where V_1, V_2 are genus g handlebodies

$$\Sigma = V_1 \cap V_2 = \partial V_1 = \partial V_2$$

Σ is called a <u>Heegaard surface</u> of the splitting, the <u>genus</u> of the splitting is the genus of Σ

another point of view

if V_1, V_2 are two handlebodies of genus g and $h: \partial V_1 \to \partial V_2$ is a(n orientation reversing) diffeomorphism,

then consider

$$V_1 \cup_h V_2 = V_1 \amalg V_2 \Big/ p \in \partial V_1 \sim h(p) \in \partial V_2$$

<u>exercise</u>: this is an oriented 3-mfd

we say (V_1, V_2, h) is a <u>Heegaard splitting</u> of M if

$$M \cong V_1 \cup_h V_2$$

exercise: show definitions are "equivalent" (and why "–"?)

lemma 2:

> given two n-manifolds M_1, M_2 and diffeomorphisms
> $$h_0, h_1 : \partial M_1 \to \partial M_2$$
> that are isotopic, then
> $$M_1 \cup_{h_0} M_2 \cong M_1 \cup_{h_1} M_2$$

Remark:

1) This is very important and even the proof is useful as we will see later!
2) this shows that the second definition of Heegaard splitting only depends on h upto isotopy

Proof:

let $h : [0,1] \times \partial M_1 \to \partial M_2$ be the isotopy from h_0 to h_1

and write $H : [0,1] \times \partial M_1 \to [0,1] \times \partial M_1$
$$(t,p) \longmapsto (t, h_0^{-1} \circ h(p,t))$$

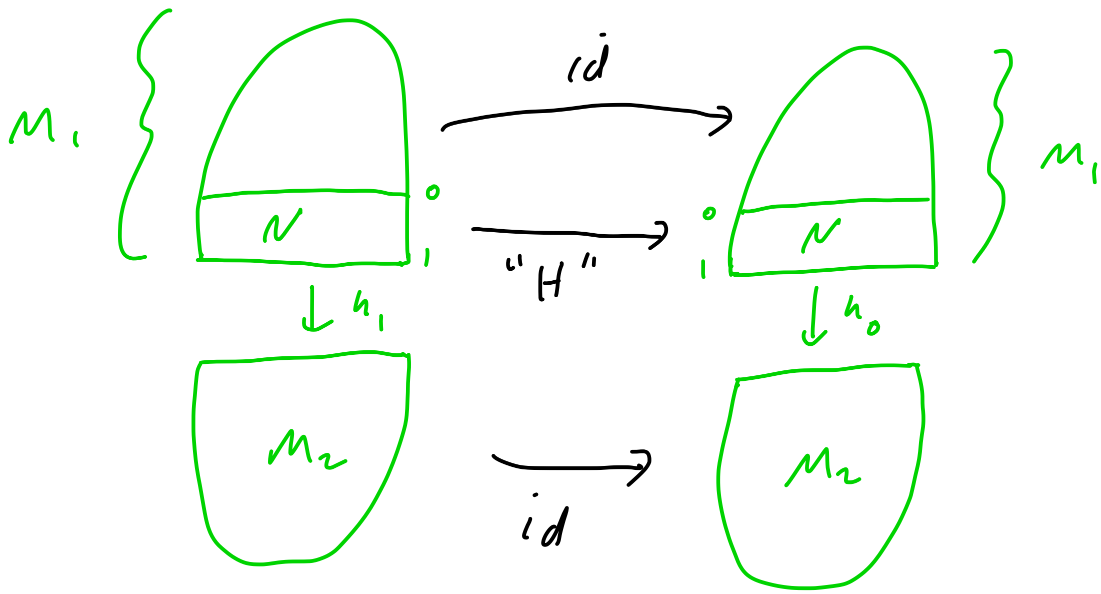
clearly H is a diffeomorphism (or exercise!)

recall, $\exists$ a neighborhood N of ∂M_1 in M_1

and a diffeomorphism $\psi : [0,1] \times \partial M_1 \to N$

s.t. $\psi(1,p) = p \in \partial M_1$

we now build our diffeomorphism



$$\Phi : M_1 \cup_{h_1} M_2 \to M_1 \cup_{h_0} M_2 : p \mapsto \begin{cases} p & p \in M_2 \cup (\overline{M_1 - N}) \\ \psi \circ H \circ \psi^{-1}(p) & p \in N \end{cases}$$

note: Φ well-defined on $M_1 \amalg M_2$ since

on $N \cap \overline{M_1 - N} = \{0\} \times \partial M_1$

$$H(0,p) = h_0^{-1} \circ h(0,p) = h_0^{-1} \circ h_0(p) = p$$

so $\psi \circ H \circ \psi^{-1}(p) = p$

(if you make h constant near $t=0$, this is clearly smooth too)

Φ decends to give a well-defined map on the quotient space because

$$p \in \partial M_1 \sim h_1(p) \text{ on left}$$

and $h_1(p) \in \partial M_2 \xrightarrow{\Phi} h_1(p)$

while $p \in \partial M_1 \xrightarrow{\Phi} \psi \circ H \circ \psi^{-1}(p) = \psi \circ H(1,p)$

$$= \psi \circ (1, h_0^{-1} \circ h_1(p))$$

$$= h_0^{-1} \circ h_1(p)$$

and on the right $h_0^{-1} \circ h_1(p)$ is identified to $h_1(p)$

exercise:

1) show Φ is a homeomorphism

2) show, with care, can assume Φ is a diffeomorphism

Hint: first recall or look up why $M_1 \cup_{h_i} M_2$ is a smooth mfd!

exercise:

Extend this lemma to show

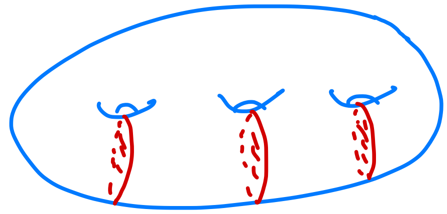
if $A \subset \partial M_1$ is a compact domain and $h_0, h_1 : A \to \partial M_2$ are isotopic embeddings

then $M_1 \cup_{h_1} M_2 \cong M_1 \cup_{h_2} M_2$

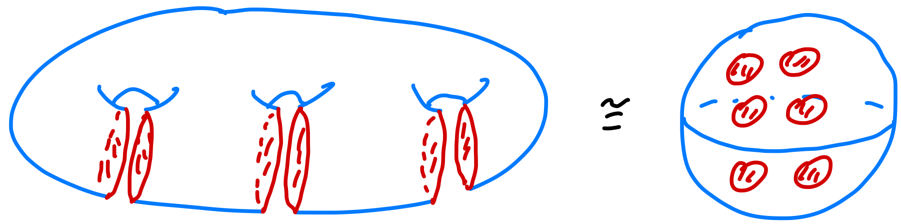
maybe check homeomorphi (unless you know how to put smooth structure on it)

Proof of lemma 1:

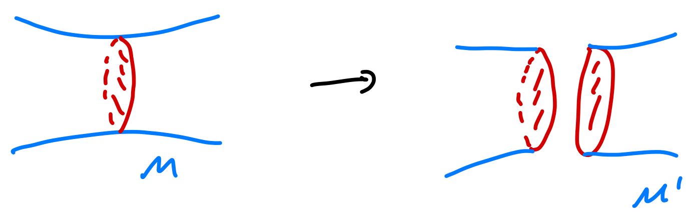
($\Rightarrow$) clear



$\downarrow$ remove nbhds to get



($\Leftarrow$) note if $M' = M \setminus \text{nbhd}(\text{properly embedded } D^2)$



to get back M just glue $D^2 \times [0,1]$ back to M' where $D^2 \times \{0,1\}$ is identified with the copies of $D \subset \partial M'$

so if $\exists$ g disks $D_1, \dots D_g$ in M s.t.

$$M - \bigcup \text{tubular nbhd}(D_i) \cong B^3$$

then $M = B^3$ with g copies of $D^2 \times [0,1]$ glued on along embeddings of $D^2 \times \{0,1\}$

but so is a handlebody of genus g!

now done by

Fact any two oriented embedding of a collection of disks into a connected surface are isotopic

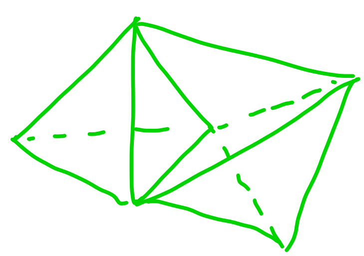
(maybe try to prove this!)

Theorem 3:

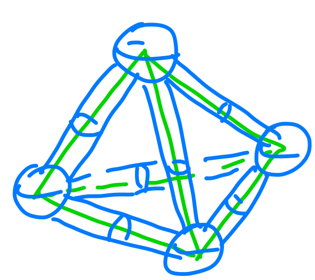
$$\boxed{\text{every close (oriented) 3-manifold has a Heegaard decomposition}}$$

Proof: let K be a triangulation of M

i.e. $M \cong |K|$ K is a bunch of tetrahedra glued together along their faces



let V_1 = nbhd of 1-skeleton



exercise: V_1 is a handlebody of genus

$\#$ edges $-$ $\#$ verticies $+1$

Hint: cut along one disk per edge use lemma 1 exercise: $V_2 = \overline{M - V_1}$ is also a handlebody

Hint: cut along faces of K in V_2

M a closed 3-manifold, the Heegaard genus $g(M)$ is

$$g(M) = \min\{\text{genus}(\Sigma) : \Sigma \text{ a Heegaard sfc for } M\}$$

note: if $g(M) = 0$, then $M \cong S^3$

indeed, let D^3 = unit disk in $\mathbb{R}^3$

S^3 = unit sphere in $\mathbb{R}^4$

set $f : D^3 \to S^3 : (x,y,z) \mapsto (x,y,z,\sqrt{1-x^2-y^2-z^2})$

$g : D^3 \to S^3 : (x,y,z) \mapsto (x,y,z,-\sqrt{1-x^2-y^2-z^2})$

$S^3 = \text{im } f \cup \text{im } g$ and $\text{im } f \cap \text{im } g = S^2 \subset \mathbb{R}^3 \subset \mathbb{R}^4$

so $g(S^3) = 0$

now if $h = g|_{\partial D^3} \circ (f|_{\partial D^3})^{-1}$ then

$$S^3 = D^3 \cup_h D^3$$

if h' is any homomorphism $S^2 \to S^2$

then we claim $D^3 \cup_{h'} D^3 \cong D^3 \cup_h D^3 \cong S^3$

to see this we note

Fact: any homeomorphism $\phi : S^2 \to S^2$

extends over B^3

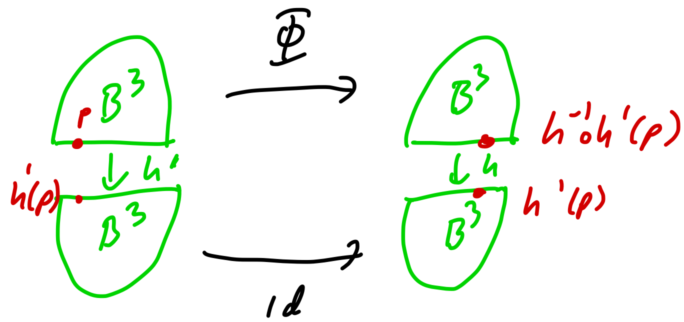
Proof: just cone

$$B^3 = S^2 \times [0,1] / S^2 \times \{0\}$$

$$\Phi : S^2 \times [0,1] / \sim \longrightarrow S^2 \times [0,1] / \sim$$
$$(p,t) \longmapsto (\phi(p),t)$$

is a homeomorphism ✓

now let $\phi = h^{-1} \circ h'$ then ϕ extends to $\Phi : B^3 \to B^3$ and



gives a homeomorphism
(prove this if not obvious!) ▦

now consider $g(M) = 1$

then $M = S^1 \times D^2 \cup_h S^1 \times D^2$

for $h : S^1 \times \partial D^2 \longrightarrow S^1 \times \partial D^2$
(orientation reversing)

Fact (see Rolfsen):
h is isotopic to

$$h(\theta,\phi) = \begin{bmatrix} a & p \\ b & q \end{bmatrix} \begin{bmatrix} \theta \\ \phi \end{bmatrix} = (a\theta + p\phi, b\theta + q\phi)$$

where $aq - bp = -1$

if $p = 0$, then $aq = -1$ so we can assume $a = 1$, $q = -1$ and

$h(\theta, \phi) = (\theta, b\theta - \phi)$

if $b = 0$ then we have

this circle ($\phi = \text{const}$) glues to this $\phi = \text{const}$ circle

this gives an S^2 in $M = S^1 \times D^2 \cup_h S^1 \times D^2$

and we get one for each $\phi \in S^1$

intuitively we get $S^1 \times S^2$

rigorously let $f : D^2 \to S^2$
$(x, y) \mapsto (x, y, \sqrt{1 - x^2 - y^2})$

$g : D^2 \to S^2$
$(x, y) \mapsto (x, y, -\sqrt{1 - x^2 - y^2})$

we get $\tilde{\Phi} : S^1 \times D^2 \cup S^1 \times D^2 \to S^1 \times S^2$

$(\phi, p) \longmapsto (\phi, f(p))$

$(\phi, p) \longmapsto (\phi, g(p))$

easy to check $\tilde{\Phi}$ induces a homeomorphism

$$S^1\times D^2 \cup_h S^1\times D^2 \longrightarrow S^1\times S^2$$

exercise:

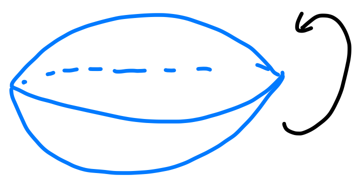
1) check this

2) check you get $S^1\times S^2$ for any b (with $p=0$)

now if $p\neq 0$ then

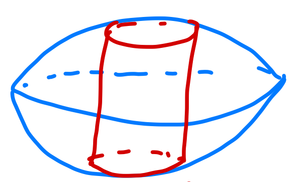
Claim: $M = L(p,q)$

to see this recall $L(p,q)$ is



glue top to bottom with $\frac{2\pi q}{p}$ twist

now



this is a torus Σ after gluing

exercise:

1) show components of $L(p,q)-\Sigma$ are tori and compute the gluing map h

2) show for any h w/ $p \neq 0$ you get $L(p,q)$

3) $\pi_1(L(p,q)) \cong \mathbb{Z}_p$

4) $L(p,q) \cong L(p,q')$ if $q \equiv \pm q' \bmod p$ (this is iff but only if harder)

5) Show $L(m,q)$ is a p fold cover of $L(mp,q)$

In particular, all $L(p,q)$ are covered by $L(1,q) \cong L(1,0) \cong S^3$

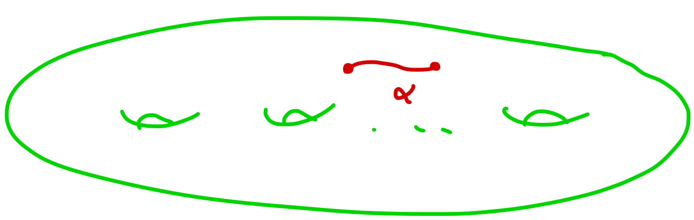
show this!

exercise: Show Poincaré homology sphere has a Heegaard 2 splitting

if Σ is a Heegaard surface for M separating M into $V_1 \cup_\Sigma V_2$, then

let α be an embedded arc in Σ

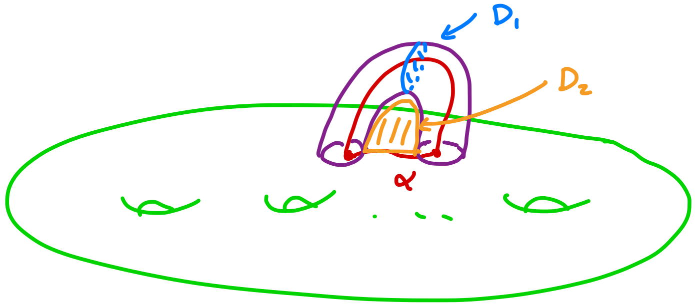
push the interior of α into V_1 or V_2, say V_2 for now, call it $\tilde{\alpha}$



note: $\alpha \cup \tilde{\alpha} = \partial$ disk D

let $V_1' = V_1 \cup N(\tilde{\alpha})$ a tubular nbhd of $\tilde{\alpha}$

$V_2' = \overline{V_2 - N(\tilde{\alpha})}$



note: $N(\alpha) = D^2 \times I$ let $D_1 = D^2 \times \{pt\}$

and $\exists$ disk $D_2 \subset V_2'$ given by $D \cap V_2'$

now $\overline{V_1' - \text{nbhd } D_1} \cong V_1$ and

$\overline{V_2' - \text{nbhd } D_2} \cong V_2$

so V_1' and V_2' are handlebodies (check this if not obvious)

we say $\Sigma' = \partial V_1'$ is the Heegaard surface obtained from Σ by stabilizing

exercise: show stabilization, upto isotopy, is independent of α or whether interior α pushed into V_1 or V_2

Fact: any 2 Heegaard splittings of M are isotopic after some finite number of stabilizations

(think about how to prove this)

Construction 3: Dehn surgery on links

let M be a closed 3-manifold with

$T \subset \partial M$ a torus

a Dehn filling of M is any manifold obtained by gluing M and $S^1 \times D^2$ along T and $\partial(S^1 \times D^2)$

i.e. $M \cup_\phi S^1 \times D^2 = M \amalg S^1 \times D^2 \Big/ {}_{p \in \partial(S^1 \times D^2) \sim \phi(p)}$

where $\phi : \partial(S^1 \times D^2) \to T$ is a homeomorphism

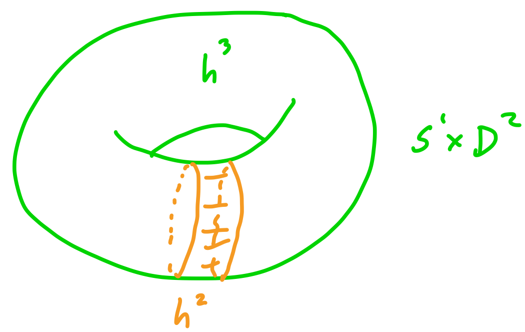
denote this by $M(\phi)$

lemma 4:

$$\boxed{\text{the homeomorphism type of } M(\phi) \text{ is determined by } \alpha = \phi(\mu) \text{ up to isotopy, where } \mu = \{pt\} \times \partial D^2}$$

meridian of $S^1 \times D^2$

Proof: $S^1 \times D^2 = \underbrace{I \times D^2}_{\substack{= h^2 \\ = \text{nbhd of } \{pt\} \times D^2}} \cup \underbrace{I \times D^2}_{\substack{= h^3 \\ = \text{rest of } S^1 \times D^2}}$



suppose ϕ_1 and $\phi_2 : \partial(S^1 \times D^2) \to T$ st.

$\alpha_1 = \phi_1(\mu)$ isotopic to $\alpha_2 = \phi_2(\mu)$

so isotop ϕ_1 so that $\phi_1 = \phi_2$ on $h_2 \cap \partial(S^1 \times D^2)$

define a homeomorphism

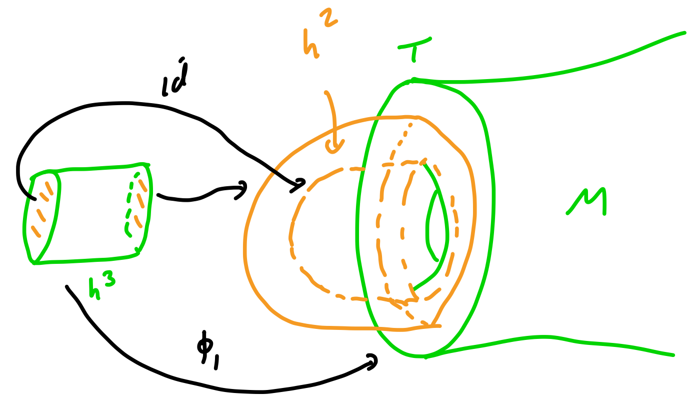
$$M \cup_{\phi_1} h^2 \longrightarrow M \cup_{\phi_2} h^2$$

by the identity map

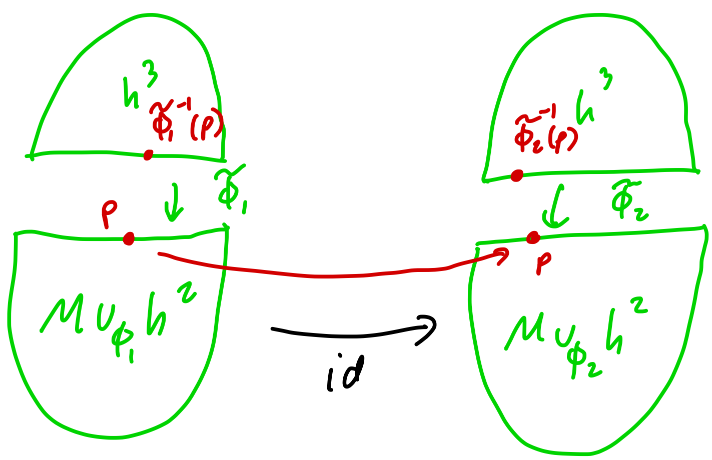
to get $M(\phi_1)$ need to glue h^3 to $M \cup_{\phi_1} h^2$

by ϕ_1 on $h^3 \cap (\partial S^1 \times D^2)$ and the identity on rest of ∂h^3

denote this map by $\tilde{\phi}_1 : \partial h^3 \to M \cup_{\phi_1} h^2$



and similarly for $M(\phi_2)$, $\tilde{\phi}_2 : \partial h^3 \to M \cup_{\phi_2} h^2$



so we need $\tilde{\phi}_2^{-1} \circ \tilde{\phi}_1 : \partial h^3 \to \partial h^3$ to extend over h^3

recall, above we saw any homeo. of $S^2 = \partial h^3$ extends to $B^3 = h^3$

so $\exists\, \Phi : h^3 \to h^3$ that extends $\tilde{\phi}_2^{-1} \circ \tilde{\phi}_1$

now

$$M(\phi_1) = (M \cup_{\phi_1} h^2) \cup_{\tilde{\phi}_1} h^3 \longrightarrow M(\phi_2) = (M \cup_{\phi_2} h^2) \cup_{\tilde{\phi}_2} h^3$$
$$p \longmapsto p$$
$$p \longmapsto \Phi(p)$$

is a homeomorphism

exercise: Show that for any simple closed curve $\alpha \subset T$ that doesn't bound a disk $\exists$ a homeomorphism $\phi : \partial(S^1 \times D^2) \to T$ such that $\phi(\mu) = \alpha$

so Dehn fillings determined by a s.c.c. $\alpha \subset T \subset \partial M$

denote this by $M(\alpha)$

given a basis λ, μ for $H_1(T^2)$ any simple closed curve $\alpha \subset T^2$ that doesn't bound a disk is represented by

$$[\alpha] = \pm(a\lambda + b\mu)$$

for a pair of co-prime integers a, b

exercise: Check this

thus non-trivial s.c.c. on T are in one-to-one correspondence with

$$\mathbb{Q} \cup \{\infty\}$$

$$\alpha \longmapsto b/a$$

so if we have a basis for $H_1(T)$ then Dehn fillings can be denoted by $M(b/a)$

Common situation: K a knot in M^3 (K = image of embedding S^1 in M^3)

let $N(K)$ = small tubular neighborhood of K
$\cong S^1 \times D^2$

$M_K = \overline{M \setminus N(K)}$

$T = \partial N(K) \subset \partial M_K$

then Dehn filling $T \subset \partial M_K$ is called Dehn surgery on K and is denoted

$M_K(\alpha)$ or $M_K(b/a)$

note: in T we have the curve $\mu = \{pt\} \times \partial D^2$
this is called the meridian of K

any curve $\lambda \subset T$, with μ, forms a basis for $H_1(T)$ is called a longitude for K it is also called a framing

note: infinitely many longitudes $\lambda + n\mu$ any $n \in \mathbb{Z}$

given a longitude we can express Dehn surgery using $\mathbb{Q} \cup \{\infty\}$

note: $M_K(\infty) = M_K(\mu) \cong M$!

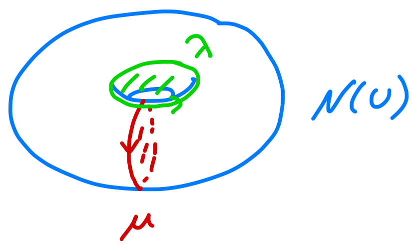
exercise: if K is null-homologous in M then
$\exists !$ simple closed curve $\lambda \subset T \subset \partial M_K$
that is trivial in M_K
moreover, λ, μ forms a basis for $H_1(T)$

in particular since $H_1(S^3) = 0$ we see we can use rational numbers to describe Dehn surgery on knots in S^3!

example:

1) if U is the unknot in S^3

then $S^3_U \cong D^2 \times S^1$ (recall $S^3 = S^1 \times D^2 \cup D^2 \times S^1$)



exercise:

1) $S^3_U(0) \cong S^1 \times S^2$

2) $S^3_U(-p/q) \cong L(p,q)$

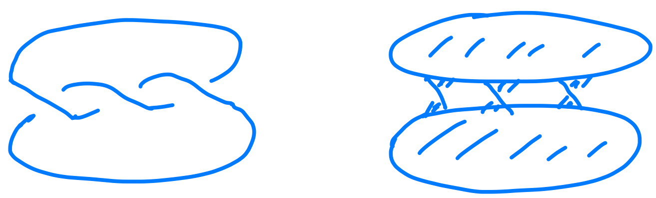
(? might have gotten orientation wrong in def^n above ... so maybe

$S^3_U(p/q) \cong L(p,q)$

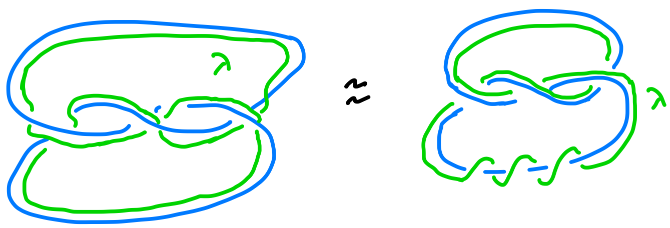
but we want to orient so we the claimed statement)

2) any knot $K \subset S^3$ bounds a Seifert surface

that is a surface $\Sigma \subset S^3$ with $\partial \Sigma = K$

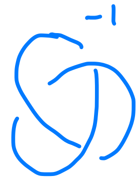


note: $\lambda = \partial N(K) \cap \Sigma$



we use this λ to describe Dehn surgeries using $\mathbb{Q} \cup \{\infty\}$

exercise: show



$\cong$ Poincare homology sphere

(this is hard!)

Theorem 5 (Lickorish, Wallace ~1960):

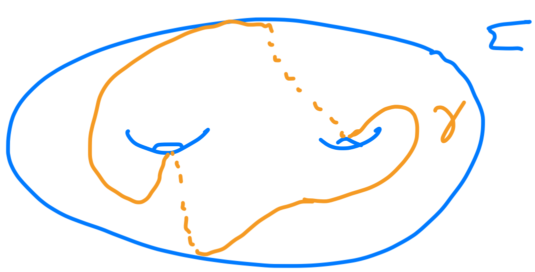
> every closed, oriented 3-manifold can be obtained by Dehn surgery on a link in S^3

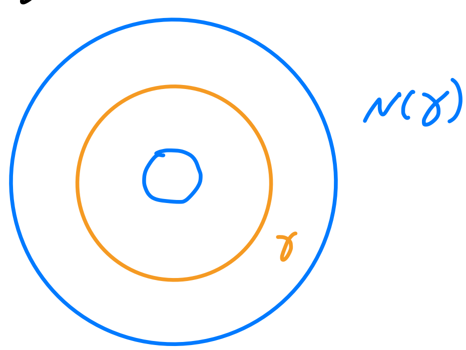
to prove this, we need two results, the first is about homeomorphisms of surfaces

let Σ be an orientable surface

γ a simple closed curve on Σ

$N(\gamma) = S^1 \times [0,1]$ a regular neighborhood of γ





a positive Dehn twist about γ is a homeomorphism

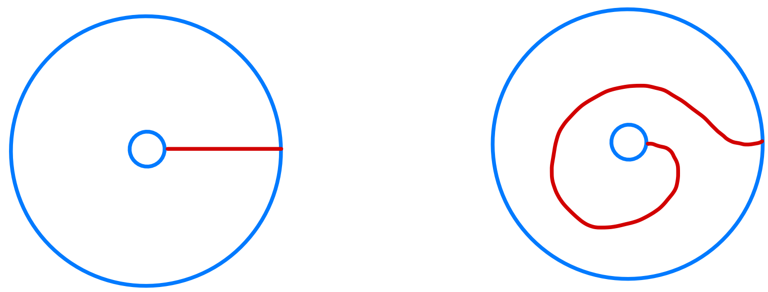
$$\tau_\gamma : \Sigma \to \Sigma$$

defined by

$$\tau_\gamma = \text{identity on } \overline{\Sigma \setminus N(\gamma)}$$

and

$$\begin{array}{ccccc}
\tau_\gamma : & N(\gamma) & \longrightarrow & N(\gamma) \\
& \| & & \| \\
& S^1 \times [0,1] & & S^1 \times [0,1] \\
& (\theta, t) & \longmapsto & (\theta - 2\pi t, t)
\end{array}$$



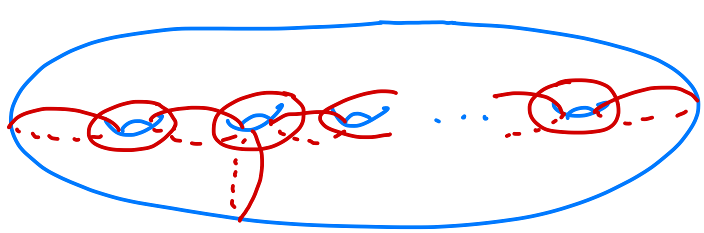
exercise: τ_γ is well-defined up to isotopy (i.e. if you isotop γ or choose a different neighborhood of γ then resulting homeo. is isotopic to τ_γ)

$\tau_\gamma^{\pm 1}$ is called a Dehn twist about γ

Fact (Dehn-Lickorish):

any orientation preserving homeomorphis of a surface is isotopic to a composition of Dehn twists

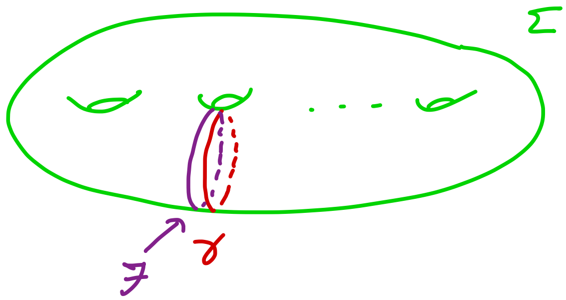
in fact, Humphries showed you only need Dehn twists about



The second result we need is

lemma 6:

let $\Sigma \subset M^3$ be a surface and M' be the result of cutting M^3 along Σ and regluing by a Dehn twist $\tau_\gamma^{\pm 1}$ then $M' \cong M_\gamma(\mathcal{F} \mp 1)$ where $\mathcal{F}$ is the framing on γ induced by Σ



very useful! we will use this again!

here is one more simple lemma

lemma 7:

Suppose the homeomorphism $\phi : \Sigma \to \Sigma$ is a composition $\phi = \phi_1 \circ \phi_2$ of two homeomorphisms

let $\Sigma \subset M$ and $N = \Sigma \times [-1,1]$ be a nbhd of Σ w/ $\Sigma = \Sigma \times \{0\}$

set $\Sigma' = \Sigma \times \{1/2\}$

let $M' = M$ cut along Σ and reglued by ϕ and $M'' = M$ cut along Σ and Σ' and reglued by ϕ_2 along Σ' and ϕ_1 along Σ

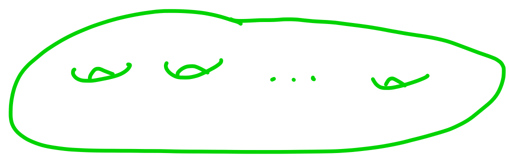
Then $M' \cong M''$

Proof of $\text{Th}^{\underline{m}}$ 5:

given M then $\exists$ a genus g Heegaard splitting

$$M = V_1 \cup_\Sigma V_2$$

by stabilizing we know S^3 has a genus g splitting



S^3

$$S^3 = V_1 \cup_\Sigma V_2$$

so $\exists$ some orien. pres. homeomorphism $\phi : \Sigma \to \Sigma$ s.t.

$M = S^3$ cut along Σ and reglued by ϕ

now Dehn-Lickorish $\Rightarrow$ $\phi = \tau_{\gamma_1}^{\varepsilon_1} \circ \dots \circ \tau_{\gamma_n}^{\varepsilon_n}$

for some curves γ_i and $\varepsilon_i = \pm 1$

let $N = \Sigma \times [-1,1]$ be a nbhd of $\Sigma \subset S^3$

now let $\Sigma_i = \Sigma \times \{\frac{1}{i}\}$ $i = 1, \dots n$

and think of γ_i as sitting on Σ_{n-i+1}

now $M = S^3$ cut along the Σ_i and reglued

along Σ_i by $\tau_{\gamma_{n-i+1}}$ by lemma 7

by lemma 6 each regluing by $\tau_{\gamma_{n-i+1}}$ is a

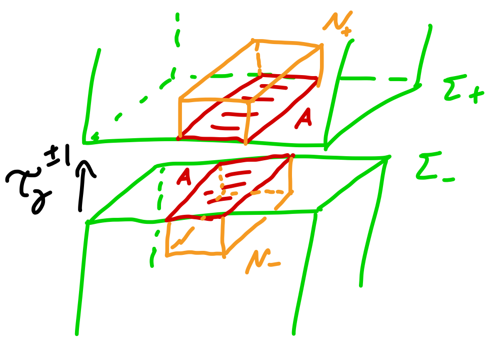
Dehn surgery on γ_{n-i+1}

$\therefore$ $M = S^3$ after Dehn surgery along $\gamma_1 \cup \dots \cup \gamma_n$ $\boxplus$

exercise: Using Humphries show M can be obtained from S^3 by Dehn surgery on a link of unknots with surgery coeff. ± 1

Proof of lemma 6:

when we cut open M along Σ we get 2 copies of Σ, $\Sigma_\pm$



let $A = \gamma \times [-1, 1]$ be a nbhd of γ on $\Sigma = \Sigma_\pm$

and $N_\pm = A \times [0,1]$ a nbhd of $\gamma \subset \Sigma_\pm$ in M

assume $\tau_\gamma^{\pm 1}$ is supported in A

note: we think of $\tau_\gamma^{\pm 1} : \Sigma_- \to \Sigma_+$ and

$\tau_\gamma^{\pm 1}\big|_{\Sigma_+ - A} : (\Sigma_- - A) \to (\Sigma_+ - A)$ is the identity map

let $N = N_- \cup N_+ \subset M$ and $N' = N_- \cup_{\tau_\gamma^\pm} N_+ \subset M'$

so in $(M - N)$ if we cut along Σ and reglue by $\tau_\gamma^\pm$ we get back M!

so $M - N$ and $M' - N'$ are homeomorphic!

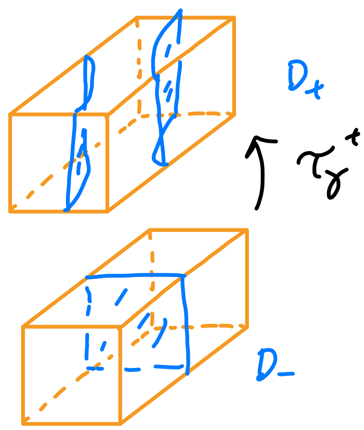
i.e. $M' = (M - \text{solid torus}) \cup (\text{solid torus})_{/\sim}$

i.e. M' is obtaind from M by Dehn surgery on γ

now to figure out which surgery

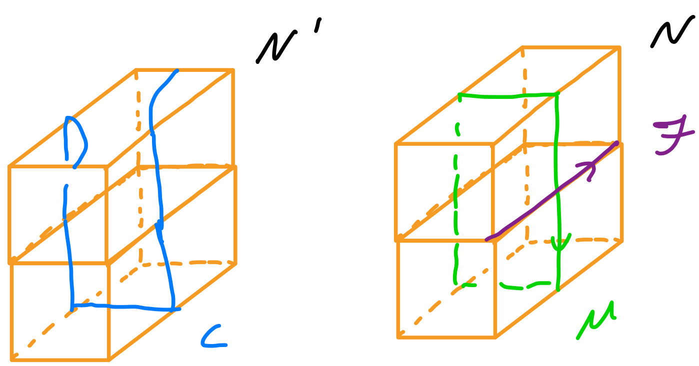
note: $N_+ \cup_{\tau_\gamma^{\pm}} N_-$ is a solid torus

let's find its meridian



so meridian is $D_+ \cup D_-$

i.e.



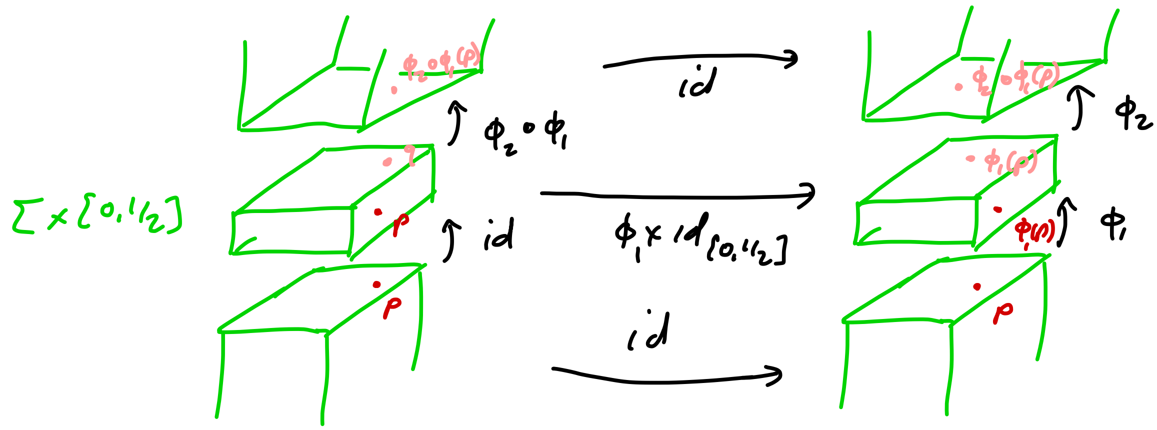
so $c = \mathcal{F} - \mu$ curve

i.e. $M' = M_K(\mathcal{F} - 1)$

you can check τ_γ^{-1} gives $M_K(\mathcal{F}+1)$

Proof of lemma 7:

we build the homeomorphism



exercise: show this is a homeomorphism ▦

exercise:

1) Show how to get a Heegaard splitting of M if M is given by surgery on a link in S^3

   Hint: put link of a Heegaard surface for S^3 so the framing from Heegaard surface is nice.

2) Given a Heegaard decomposition of M find a Dehn surgery description.

Remark: Two other useful descriptions of 3-mfds are open book decompositions and branched covers

exercise: look these up!