# II. Prime Decompositions

$M_1$ and $M_2$ oriented, connected 3-manifolds

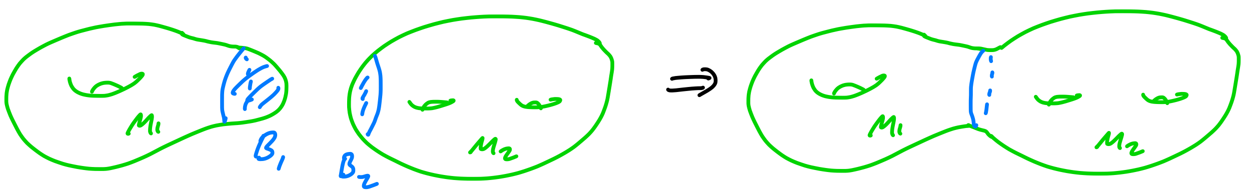
let $B_i \subset \operatorname{int} M_i$ be a 3-ball $i=1,2$

$\mathring{M}_i = \overline{M_i - B_i}$ $i=1,2$

$h : \partial B_1 \to \partial B_2$ be an orientation reversing diffeo.

the connected sum of $M_1$ and $M_2$ is

$$M_1 \# M_2 = \mathring{M}_1 \cup_h \mathring{M}_2$$




exercise:

1) $M_1 \# M_2$ is well-defined

Hint: need result of Palis & Cerf that any 2 orientation preserving embeddings of $B^n \to M^n$ are isotopic

(try to prove this!)

also need any to orientation reversing diffeos of $S^2$ are isotopic (Smale), could get around this by fixing a specific $\phi : \partial B^3 \to \partial B^3$

2) $\#$ is commutative, associative, and $S^3$ is the identity $M \# S^3 \cong M$

3) $\pi_1(M_1 \# M_2) \cong \pi_1(M_1) * \pi_1(M_2)$

$M$ is <u>prime</u> if $M \cong M_1 \# M_2 \Rightarrow M_1$ or $M_2 \cong S^3$

$M$ is <u>irreducible</u> if every embedded 2-sphere in $M$ bounds a 3-ball

Remark: clearly irreducible $\Rightarrow$ prime

if a 2-sphere $S \subset M$ does not bound a 3-ball call it <u>essential</u>

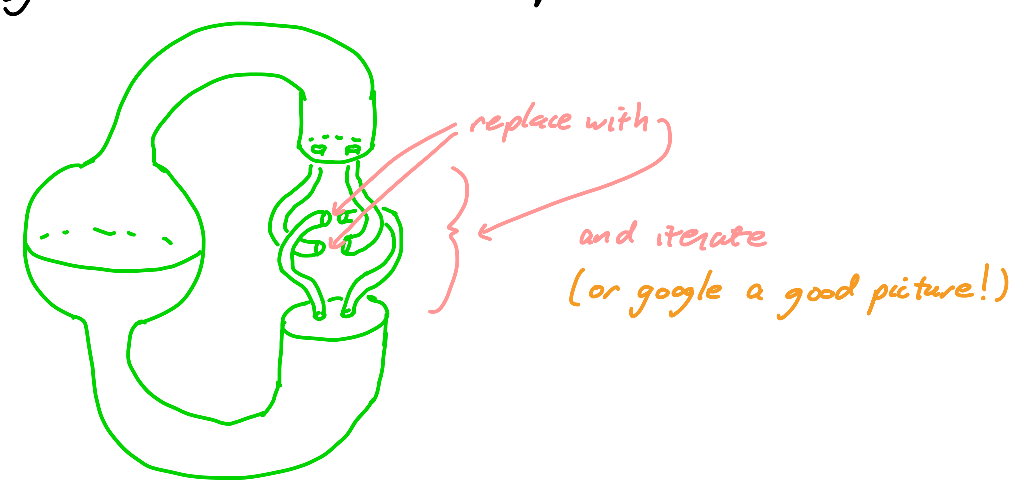
**Th$^{\underline{m}}$ (3D Schönflies Th$^{\underline{m}}$, Alexander 1930):** if $S \subset S^3$ is a smooth (or PL or "bicollared") then the closure of each component of $S^3 \setminus S$ is a 3-ball

Remark:

1) for a proof see Hatcher's notes on 3-mfds

2) false without hypothesis

eg. Alexander Horned sphere




3) analog for $S^1 \subset S^2$ true without extra hypoth.

**Corollary:**

$$\boxed{S^3, \mathbb{R}^3 \text{ are irreducible (and hence prime)}}$$

**exercise:**

$M$ a 3-mfd $\subset S^3$ with $\partial M$ connected then $M$ is irreducible

(e.g. Knot complements, handlebodies)

**Th$^{\underline{m}}$ 1:**

$$\boxed{M \text{ prime} \iff M \text{ irreducible or } M \cong S^1 \times S^2}$$

**Proof:**

($\Leftarrow$) irred $\Rightarrow$ prime was noted above

<u>$S^1 \times S^2$ is prime</u>

suppose $S^1 \times S^2 \cong M_1 \# M_2 = \mathring{M}_1 \cup_S \mathring{M}_2$ ($S \leftarrow$ 2-sphere)

by van Kampen's Th$^{\underline{m}}$

$$\mathbb{Z} \cong \pi_1(S^1 \times S^2) \cong \pi_1(\mathring{M}_1) * \pi_1(\mathring{M}_2)$$

$\therefore \pi_1(\mathring{M}_1) = 1$ (or $\pi_1(\mathring{M}_2)$)

(since free products are never abelian unless one of the groups is trivial)

consider the universal cover

$$\mathbb{R}^1 \times S^2 \longrightarrow S^1 \times S^2$$

$\pi_1(\mathring{M}_1) = 1 \Rightarrow \mathring{M}_1$ lifts to $\mathbb{R}^1 \times S^2 = \mathbb{R}^3 - \{(0,0,0)\} \subseteq \mathbb{R}^3$

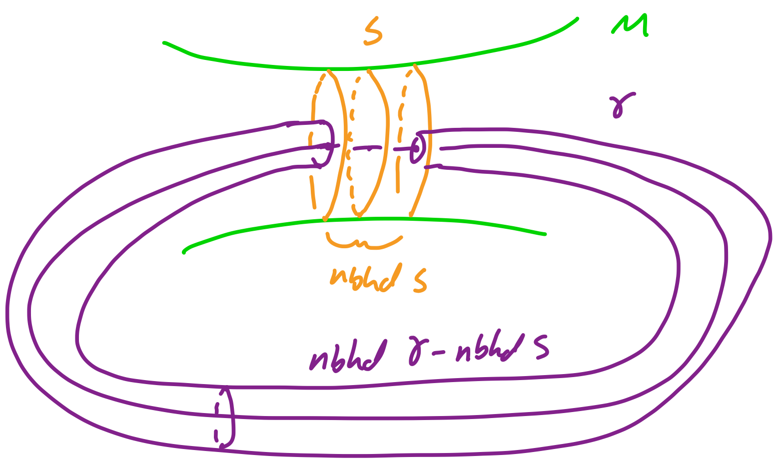
since $\mathbb{R}^3$ is irreducible and $\mathring{M}_1$ is compact we must have $\mathring{M}_1 = B^3$ ($\mathbb{R}^3 \setminus \partial \mathring{M}_1 = B^3 \cup$ non cpt by irreduc.)

for ($\Rightarrow$) we show if $M$ prime, but not irreducible then $M \cong S^1 \times S^2$

$M$ contains an essential 2-sphere $S$

since $M$ is prime $S$ is non-separating

$\therefore \exists$ an embedded loop $\gamma \subset M$ s.t. $\gamma$ meets $S$ in one point and they are transverse




let $N$ = regular nbhd of $S \cup \gamma$ in $M$

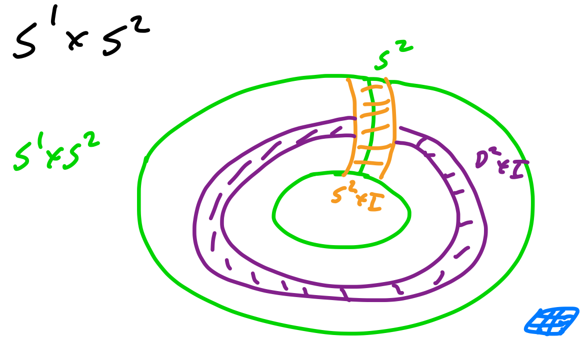
$\cong S \times [0,1] \cup D^2 \times [0,1]$

with $D^2 \times \{i\}$ attached to $S \times \{i\}$ $i = 0,1$

note $\partial N = S^2$ a separating 2-sphere in $M$

$\therefore \partial M$ bounds a ball $B^3$ disjoint from $N$

exercise: show $N \cup_{S^2} B^3 \cong S^1 \times S^2$



**note:** Proof shows: $S \subset M^3$ a non-separating 2-sphere then $M \cong M' \# S^1 \times S^2$

**Th$^{\underline{m}}$ 2:**

$$\boxed{\begin{array}{l}\text{let } \tilde{M} \xrightarrow{p} M \text{ be a regular covering space} \\ \text{Then } \tilde{M} \text{ irreducible} \Leftrightarrow M \text{ irreducible}\end{array}}$$

**Proof:** ($\Rightarrow$) $S$ a 2-sphere in $M$

$\pi_1(S) = 1 \Rightarrow S$ lifts to $\tilde{M}$

let $\tilde{S}$ be a lift of $S$ to $\tilde{M}$

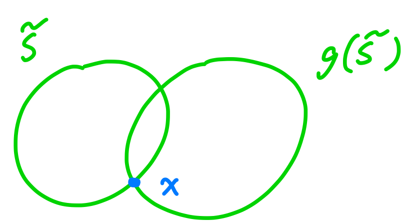
set $G$ = group of covering transforms of $\tilde{M} \to M$

($\tilde{M}$ regular $\Rightarrow \tilde{M}/G \cong M$)

so $p^{-1}(S) = \coprod_{g \in G} g(\tilde{S})$

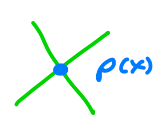
**note:** $g(\tilde{S})$ & $g'(\tilde{S})$ disjoint $\forall g, g'$ since $S$ embedded in $M$

indeed




$p(x)$ has an evenly covered nbhd $U$ let $V$ be nbhd of $x$ s.t. $p|_V : V \to U$ homeo

so a nbhd of $p(x)$ in $U$ looks like




i.e. a double pt!

$\tilde{M}$ irreducible $\Rightarrow \exists\, \tilde{B}$ a 3-ball in $\tilde{M}$ s.t. $\partial \tilde{B} = \tilde{S}$

now if $g \neq 1$ then $g\tilde{B} \cap \tilde{B} = \emptyset$

if not then since $g\tilde{S} \cap \tilde{S} = \emptyset$

we must have $g\tilde{B} \subset \tilde{B}$

(or $\tilde{B} \subset g\tilde{B}$)

then Brower fixed pt th$^{\underline{m}}$ $\Rightarrow$ $g$ has

a fixed point $\times$ $g \neq 1$

$\therefore$ $p|_{\tilde{B}}$ is a homeomorphism onto $p(\tilde{B}) = B$

so $S = \underline{\partial B}$ ✓

($\Leftarrow$) Proof is much harder, uses minimal surfaces

(Meeks-Yau 1980 "equivariant sphere th$^{\underline{m}}$")

**Cor 3:** $\boxed{\text{lens spaces are irreducible}}$

**Proof:** $S^3$ covers lens spaces

**Th$^{\underline{m}}$ 4:**

(Kneser 1929)

every closed oriented 3-manifold is a finite connected sum of prime mfds

(Milnor 1962)

if $M_1 \# \ldots \# M_m$ and $N_1 \# \ldots \# N_n$ are two such decompositions of $M$ (and no $M_i, N_j \cong S^3$) then $m = n$ and (after reordering) $M_i \cong N_i \; \forall i$

The proof uses normal surface theory a very useful tool, but we will not prove this here See Hatcher's notes on 3-mfds

But for us, this means if we want to understand 3-manifolds it suffices to understand prime ones (we will see below how to recognize when a 3-mfd is a connected sum)