III. Disk & Sphere Theorem

A. Recollections from Algebraic Topology

- $p: \tilde{X} \to X$ a covering space

  1) then $p_*: \pi_i(\tilde{X}) \to \pi_i(X)$ an isomorphism $\forall i \geq 2$

  2) if $f: Y \to X$ is a map s.t. $f_*(\pi_1(Y)) \subset p_*(\pi_1(\tilde{X}))$
     then f lifts to $\tilde{X}$ i.e.

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ Y \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

- X connected space

  $\exists$ Hurewicz map $h_n: \pi_n(X) \to H_n(X)$

  h_1 on π_1 is abelianization

  (i.e. h_1 onto and $\ker h_1 = [\pi_1(X), \pi_1(X)]$)

Hurewicz Th^m:

$\pi_1(X) = 1$ and $n \geq 2$

Then $\pi_i(X) = 0 \quad \forall\, 2 \leq i < n$

$$\Longleftrightarrow$$

$H_i(X) = 0 \quad \forall\, 2 \leq i < n$

and if this holds then $h_n: \pi_n(X) \to H_n(X)$ is an isomorphism

- **Whiteheads Th^{m}:** X, Y connected CW complexes
  $f: X \to Y$ a map

  1) $f_*: \pi_i(X) \to \pi_i(Y)$ an isomorphism for all i
     then f a homotopy equivalence
  2) $\pi_1(X) \cong \pi_1(Y) = 1$ and $f_*: H_i(X) \to H_i(Y)$
     an isomorphism for all i, then
     f is a homotopy equivalence

- X is a $K(\pi, 1)$ (or aspherical) if X is connected,
  $\pi_1(X) = \pi$, and $\pi_i(X) = 0 \quad \forall i \geq 2$

  if X, Y are $K(\pi, 1)$ CW complexes then $X \simeq Y$

  homotopy equivalence

- **Poincaré (Lefschetz) Duality:**
  M compact, oriented, n-manifold, then

$$H_q(M) \cong H^{n-q}(M, \partial M)$$
$$H_q(M, \partial M) \cong H^{n-q}(M)$$

- **Universal Coefficients Th^{ms}:**

$$H^n(X, A; \mathbb{Z}) \cong \mathrm{Free}(H_n(X, A; \mathbb{Z})) \oplus \mathrm{Tor}(H_{n-1}(X, A; \mathbb{Z}))$$

B. Algebraic Topology and 3-manifolds

we can use simple algebraic topology to understand certain 3-mfds upto homotopy

Lemma 1:

> M a closed connected 3-mfd
> $$\pi_1(M) = 1 \Leftrightarrow M \simeq S^3$$
> homotopy equiv.

later we will see much more is true

Proof: $(\Leftarrow)$ ✓

$(\Rightarrow)$ $\pi_1(M) = 1 \Rightarrow H_1(M) = 0$ (so M orientable)

$$H_2(M) \cong H^1(M) \cong \text{Free } H_1(M) \oplus \text{Tor } H_0(0) = 0$$

Poincaré Duality   Univ. Coeff. Th^{m}

$H_3(M) \cong H_0(M) \cong \mathbb{Z}$ (since closed, conn. 3-mfd)

thus Hurewicz th$^{\underline{m}}$ $\Rightarrow \pi_3(M) \cong H_3(M) \cong \mathbb{Z}$

$\therefore \exists f : S^3 \to M$ s.t. $[f]$ generates $\pi_3 = H_3$

so we see $f_* : H_3(S^3) \to H_3(M)$ an isom.

$\therefore f_*$ an isomorphism on $H_i \ \forall i$

since S^3, M simply connected, Whitehead's th$^{\underline{m}}$ implies f is a homotopy equiv. ▦

lemma 2:

M non-compact, connected 3-manifold with $\partial M = \emptyset$

$$\pi_1(M) \cong \pi_2(M) \cong 1 \iff M \cong \mathbb{R}^3$$

Proof: $(\Leftarrow)$ ✓

$(\Rightarrow)$ M non-compact $\Rightarrow H_i(M) = 0 \;\; \forall i \geq 3$

$\pi_1(M) = \pi_2(M) = 0 \Rightarrow H_1(M) = H_2(M) = 0$

$\therefore H_i(M) = 0 \;\; \forall i \geq 1$

let $f: M \to *$ be constant map

f induces isom. on all H_i $\therefore f$ is a homotopy equiv ▦

Earlier we looked at embedded 2-spheres

What about non-embedded ones (i.e. coming from $\pi_2(M)$)?

Th$^{\underline{m}}$ 3 (Sphere Th$^{\underline{m}}$; Papakyriakopoulos 1957, Whitehead 1958)

let M be an orientable 3-manifold

$f: S^2 \to M$ be a map s.t. $[f] \neq 0$ in $\pi_2(M)$

Then $\exists$ an embedding $e: S^2 \to M$ s.t.

$[e] \neq 0$ in $\pi_2(M)$

Th$^{\underline{m}}$ 4 (Disk Th$^{\underline{m}}$, Dehn's lemma, Papa 1957):

let M be an orientable 3-manifold, $\Sigma \subset \partial M$ a surface,

and $f: (D^2, S^1) \to (M, \Sigma)$ s.t. $[f|_{S^1}] \neq 1$ in $\pi_1(\Sigma)$

Then $\exists$ an embedding $e: (D^2, S^1) \to (M, \Sigma)$ s.t.

$e|_{S^1}$ is essential (i.e. doesn't bound a disk) in Σ

We prove the disk thm later (sphere thm similar)

but first let's see some consequences

Basically both theorems turn algebraic info into geometric info. This is rare and very helpful!

lemma 5:

M an orientable 3-manifold. Then

M irreducible $\iff \pi_2(M) = 0$

Proof: ($\Rightarrow$) $\pi_2(M) \neq 0 \underset{\text{Thm } 3}{\Longrightarrow} \exists$ embedded 2-sphere S

with $[S] \neq 0$ in $\pi_2(M)$

$\therefore$ $S \neq \partial(3\text{-ball})$

$\therefore$ M is reducible

($\Leftarrow$) for this we need

Poincaré Conj (proven by Perelman ~2003)

if M a 3-manifold $\simeq S^3$ then $M \cong S^3$

let $S \subset M$ be an embedded sphere

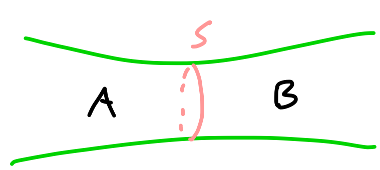
$\pi_2(M) = 0 \Rightarrow [S] = 0$ in $\pi_2(M)$

$\Rightarrow [S] = 0$ in $H_2(M)$

$\Rightarrow$ S separates M

exercise: prove this!

so $M = A \cup_S B$



let $\tilde{M} \xrightarrow{p} M$ be the universal cover

$$\left.\begin{array}{l} p^{-1}(A) = \text{copies of } \tilde{A} \ (\tilde{A} \text{ univ cover of } A) \\ p^{-1}(B) = \quad " \quad " \ \tilde{B} \ (\tilde{B} \quad " \quad\quad " \quad B) \end{array}\right\} \begin{array}{l}\text{check this}\\ \text{uses that } S\\ \text{a 2-sphere}\end{array}$$

$\partial \tilde{A} = |\pi_1(A)|$ copies of S

$\partial \tilde{B} = |\pi_1(B)|$ " "

let $\tilde{S}_0$ be a lift of S

$$\left.\begin{array}{l} \pi_2(M) = 0 \Rightarrow \pi_2(\tilde{M}) = 0 \\ \qquad\qquad\qquad\ \ \pi_1(\tilde{M}) = 0 \end{array}\right\} \Rightarrow H_2(\tilde{M}) = 0$$

Hurewicz Th^{m}

$\therefore [\tilde{S}_0] = 0$ in $H_2(\tilde{M})$

$\therefore$ $\tilde{S}_0$ separates $\tilde{M} = X \cup_{S_0} Y$

Mayer-Vietoris gives

$$\underset{\mathbb{Z}}{\underset{\shortparallel}{H_2(S_0)}} \longrightarrow H_2(X) \oplus H_2(Y) \longrightarrow \underset{0}{\underset{\shortparallel}{H_2(\tilde{M})}}$$

exercise: show so $[S_0] = 0$ in $H_2(X)$ or $H_2(Y)$

say in $H_2(X)$

lemma 6:

> let M be a 3-manifold, Σ be a component of ∂M that is compact
> $[\Sigma] = 0$ in $H_2(M) \Leftrightarrow M$ is compact and $\partial M = \Sigma$

$\therefore$ $S_0 = \partial X$ and X compact

X must be $\tilde{A}$ or $\tilde{B}$, assume $\tilde{A}$

$\therefore \partial \tilde{A} = S_0$ and so $\pi_1(A) = 1$ (since $|\partial \tilde{A}| = 1$)

$\therefore A = \tilde{A}$

$A \cup B^3$ is a closed 3-mfd with $\pi_1 = 1$

$\therefore$ Poincaré $\Rightarrow$ $A \cup B^3 \cong S^3$ and

so $A \cong B^3$

$\therefore$ $S = \partial(A = B^3)$ so M irreducible ▦

Proof of lemma 6:

($\Leftarrow$) clear

($\Rightarrow$) $[\Sigma] = 0$ in $H_2(M) \Rightarrow \exists$ a compact submfd

$M_0 \subset M$ s.t. $[\Sigma] = 0$ in $H_2(M_0)$

so we can assume M is compact

need to show $\partial M = \Sigma$

Suppose not, long exact sequence of $(M, \partial M)$ gives

$$\begin{array}{ccccc} H_3(M) & \longrightarrow & H_3(M,\partial M) & \longrightarrow & H_2(\partial M) \\ \| & & \cong & & \cong \\ 0 & & H^0(M) & \xrightarrow{i^*} & H^0(\partial M) \\ & & \cong & & \cong \\ & & \mathbb{Z} & & \mathbb{Z}\oplus\cdots\oplus\mathbb{Z} \end{array}$$

\# of components of ∂M

the inclusion

$$\begin{array}{ccc} H_0(\partial M) & \xrightarrow{i_*} & H_0(M) \\ \cong & & \cong \\ \mathbb{Z}\oplus\cdots\oplus\mathbb{Z} & & \mathbb{Z} \end{array}$$

sends each generator of $H_0(\partial M)$ to ± 1 in $H_0(M)$ (say $+1$)

since i^* and i_* are dual we see $i^*(1) = (1, \dots, 1)$

$\therefore$ $[\Sigma]$ not in the image of i^* unless $\partial M = \Sigma$

$\therefore$ $[\Sigma] \neq 0$ in $H_2(M)$ unless $\partial M = \Sigma$ ▦

Thm 7:

let M be a closed 3-manifold with univ. cover $\tilde{M}$

1) if $\pi_1(M)$ is finite, then $\tilde{M} \cong S^3$

if $\pi_1(M)$ is infinite and M is prime then

2) $\tilde{M} \cong \mathbb{R}^3$ or

3) $M \cong S^1 \times S^2$ (so $\tilde{M} \cong \mathbb{R} \times S^2$)

Proof:

1) $\pi_1(M)$ finite $\Rightarrow \tilde{M}$ compact, $\pi_1(\tilde{M}) = 1$

$\therefore$ lemma 1 $\Rightarrow \tilde{M} \simeq S^3$

now Poincaré $\Rightarrow \tilde{M} \cong S^3$

if $\pi_1(M)$ infinite and M prime, then Thm II.1 $\Rightarrow M$ is $S^1 \times S^2$ or irreducible

if not $S^1 \times S^2$ then lemma 5 says $\pi_2(M) = 0$

$\therefore \pi_1(\tilde{M}) = \pi_2(\tilde{M}) = 0$

$\tilde{M}$ non-compact then $\Rightarrow \tilde{M} \simeq \mathbb{R}^3$ by lemma 2

the geometrization conjecture (discussed later) then $\Rightarrow \tilde{M} \cong \mathbb{R}^3$ $\boxplus$

Corollary 8:

1) if M is a closed prime 3-manifold with $\pi_1(M) \cong \mathbb{Z}$ then $M \cong S^1 \times S^2$

2) if M, N closed prime 3-manifolds with $\pi_1(M) \cong \pi_1(N)$ infinite, then $M \cong N$

Proof:

1) Claim: $\pi_2(M) \neq 0$

suppose not, then 2) of Th^{m} 7 must hold

$\therefore \pi_i(\tilde{M}) \cong \pi_i(M) = 0 \quad \forall i \geq 2$

let $f: S^1 \to M$ be a map s.t. $[f]$ generates $\mathbb{Z} \cong \pi_1(M)$

$\pi_i(S^1) = 0 \ \forall i \geq 2$

$\therefore f: \pi_i(S^1) \to \pi_i(M)$ an isomorphism $\forall i$

so f is a homotopy equivalence

$\therefore H_2(M) \cong H_2(S^1) = 0$

but $H_2(M) \cong H^1(M) \cong \text{free } H_1(M) \cong \mathbb{Z}$ $\otimes$

$\therefore \pi_2(M) \neq 0$

since $\pi_2(M) \neq 0$, case 3) of Th^{m} 7 holds and so $M \cong S^1 \times S^2$

2) M, N prime, $\pi_1(M) = \pi_1(N)$

if $\pi_1(M) \cong \mathbb{Z}$ then $M \cong S^1 \times S^2 \cong N$

if $\pi_1(M) \ncong \mathbb{Z}$ then Th^{m} 7 says $\tilde{M}$ & $\tilde{N} \cong \mathbb{R}^3$

$\therefore \pi_i(M) \cong \pi_i(N) \ \forall i$

so M and N are "$K(\pi(M), 1)$" spaces i.e. all higher homotopy groups vanish and π_1's are isom.

this $\Rightarrow M \simeq N$ (if you have not seen this before prove this!)

now again geometrization $\Rightarrow M \cong N$ (since prime) & π_1 infinite

Th$\underline{\text{m}}$ 9:

$$\boxed{\text{let } M \text{ be a compact, irreducible 3-manifold with } \pi_1(M) \text{ free, then } M \text{ is a handlebody (or } S^3)}$$

need 3 lemmas

lemma 10:

$$\boxed{\text{let } \Sigma \text{ be a closed surface } \not\cong S^2 \text{ then } \pi_1 \Sigma \text{ is not free}}$$

Proof: suppose $\pi_1 \Sigma$ is free for rank n

let $X = \bigvee_{i=1}^{n} S^1$  wedge of n circles

then $\exists\, f : X \to \Sigma$ s.t. $f_* : \pi_1 X \to \pi_1 \Sigma$ is an isom.

the universal cover of Σ is $\tilde{\Sigma} \cong \mathbb{R}^2$

$$\therefore \pi_i(\Sigma) = 0 \quad \forall i \geq 2$$

we also know $\pi_i(X) = 0 \ \forall i \geq 2$

$\therefore$ Hurewicz says f is a homotopy equivalence

so we must have $f_* : \underset{\underset{0}{\shortparallel}}{H_2(X)} \to \underset{\underset{\mathbb{Z}}{\shortparallel}}{H_2(\Sigma)}$ an isom $\not\emptyset$

Lemma 11:

$\boxed{\text{any subgroup of a free group is free}}$

Proof: G a free group then

$G \cong \pi_1(X)$ some $X = \hat{\bigvee_{i=1}} S^1$

(if G not finitely generated use $\cdots \bigcirc\bigcirc \cdots$)

let H be a subgroup of G

then $\exists$ a covering space $\tilde{X} \to X$ s.t. $\pi_1(\tilde{X}) \cong H$

but $\tilde{X}$ a 1-complex so $\pi_1(\tilde{X})$ is free

Lemma 12:

M a compact orientable 3-manifold with $H_1(M)$ finite then $\partial M \cong \amalg S^2$

Proof: $H_2(M, \partial M) \underset{\text{Poincaré duality}}{\cong} H^1(M) \underset{\text{univ. coeff. thm}}{\cong} \text{free } H_1(M) = 0$

now the exact sequence for $(M, \partial M)$ gives

$$\underset{\substack{\shortparallel \\ 0}}{H_2(M, \partial M)} \to H_1(\partial M) \to \underset{\substack{\shortparallel \\ \text{finite}}}{H_1(M)}$$

$\therefore$ $H_1(\partial M)$ finite and since the only finite group that is $H_1(\text{orientable sfc})$ is 0

we see $H_1(\partial M) = 0$

$\therefore$ $\partial M = \amalg S^2$

Proof of 9: suppose $\pi_1(M)$ free of rank n

we prove theorem by induction on n

$\underline{n=0}$: $\pi_1(M) = 1$

if $\partial M = \emptyset$ then from Th$^{\underline{m}}$ 7 $M \cong S^3$

if $\partial M \neq \emptyset$ then $\partial M = \amalg S^2$ (lemma 12)

M irreducible $\Rightarrow M \cong D^3$

(i.e. handle body of genus 0)

$\underline{n \geq 1}$: M irreducible $\Rightarrow \pi_2(M) = 0$ (lemma 5)

$\pi_1(M)$ infinite $\Rightarrow$ universal cover $\tilde{M}$ is non-compact

$\therefore H_i(\tilde{M}) = 0 \quad \forall i \geq 3$

we know $\pi_i(\tilde{M}) \cong \pi_i(M) \quad \forall i \geq 2$

$\therefore \pi_2(\tilde{M}) = 0$ and $\pi_i(\tilde{M}) = 0 \quad \forall i \geq 3$ by Hurewicz

let $X = \bigvee_{i=1}^{n} S^1$

$\exists f : X \to M$ s.t. $f_* : \pi_i(X) \to \pi_i(M)$ is isom $\forall i$

$\therefore f$ is a homotopy equivalence by Whitehead

$\therefore f_* : H_i(X) \to H_i(M)$ an isom. $\forall i$

so $H_3(M) \cong H_3(X) = 0 \quad \therefore \partial M \neq \emptyset$

if some component of ∂M is S^2 then M irred

$\Rightarrow M \cong D^3 \Rightarrow \pi_1(M) = 1$ $\otimes$

so let Σ be a component of ∂M with genus $\Sigma > 0$

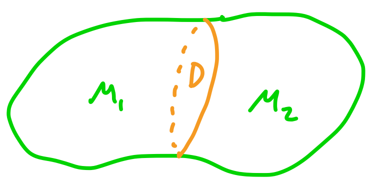
by lemma 10 & 11 $\pi_1(F) \to \pi_1(M)$ is <u>not</u>

one-to-one

$\therefore$ Disk Th$^{\underline{m}}$ (Th$^{\underline{m}}$ 4) $\exists$ embedded disk $D \subset M$
such that $\partial D = D \cap \partial M$ is
essential in Σ

<u>2 cases:</u>

1) D separates M



so $\overline{M \setminus N(D)} = M_1 \amalg M_2$

$\pi_1(M) \cong \pi_1(M_1) * \pi_1(M_2)$

$\therefore \pi_1(M_i)$ free of rank n_i by lemma 11
with $n_1 + n_2 = n$
and $\partial M_i \neq \emptyset$

<u>Claim:</u> $n_i > 0$
if not, say, $n_1 = 0$, then $M = D^3$
$\therefore \partial D$ bounds disk in Σ $\otimes$

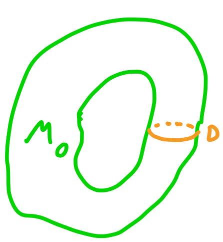
$\therefore n_i < n$

Clearly M_i is irreducible (check if not clear!)

$\therefore$ by induction the M_i are handlebodies

$\therefore M$ is a handlebody (lemma I.1) ✓

2) D does not separate M



so $\overline{M \setminus N(D)} = M_0$

$\pi_1(M) \cong \pi_1(M_0) * \mathbb{Z}$ (check)

so $\pi_1(M_0)$ free of rank $< n$

M_0 irreducibly and $\partial M_0 \neq \emptyset$

$\therefore M_0$ a handlebody

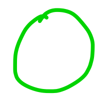
so M is too ▦

recall a knot K is the image of an embedding $f_K : S^1 \to S^3$

$K_1 \sim K_2$ (equivalent) if $\exists$ an isotopy from f_{K_1} to f_{K_2}

(recall isotopy extension says $\exists$ an isotopy $F_t : S^3 \to S^3$ s.t. $F_0 = id$ and $f_{K_2} = F_1 \circ f_{K_1}$

so $\exists$ a diffeo $F_1 : S^3 \to S^3$ s.t. $F_1(K_1) = K_2$)

K is trivial if $\sim$ the unknot $U =$ 

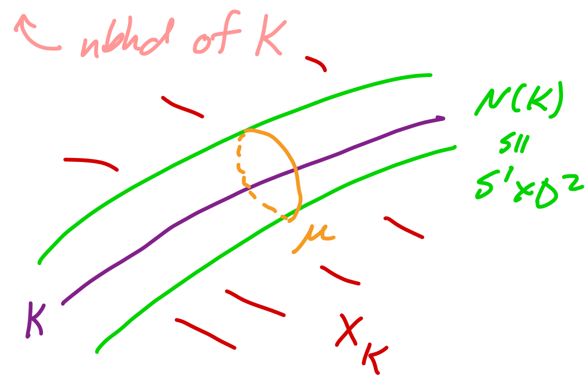
the group of K is $\pi_1(S^3 \setminus K)$

$$K_1 \sim K_2 \Rightarrow \pi_1(S^3 - K_1) \cong \pi_1(S^3 - K_2)$$

the exterior of K is $X_K = \overline{S^3 - N(K)}$

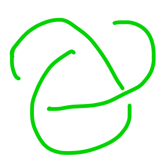
$$\partial X_K = T^2$$

$$\pi_1(X_K) \cong \pi_1(S^3 - K)$$



note: $X_U \cong S^1 \times D^2$

$$\pi_1(X_U) \cong \mathbb{Z}$$

example:  $T =$ trefoil

$\pi_1(X_T) \cong \langle x, y \mid x^2 = y^3 \rangle$ (check)

note $\pi_1(X_T)$ maps onto $\langle x, y \mid x^2 = 1 = y^3 \rangle$

$\cong \mathbb{Z}_2 * \mathbb{Z}_3$

non-abelian

$\therefore T \not\sim$ unknot

to what extent does $\pi_1(X_K)$ determine K?

Th$^{\underline{m}}$ 13 (Dehn 1910 modulo his "lemma"):

$$\boxed{\pi_1(X_K) \cong \mathbb{Z} \iff K \sim U}$$

first a lemma

lemma 14:

K a knot in S^3 then

$$H_i(X_K) \cong \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z} & i=1 \text{ gen by } \mu \\ 0 & i \geq 2 \end{cases}$$

$\mu = [\partial \text{ meridional disk in } N(K)] \in H_1(\partial N) \subset H_1(\partial X_K)$

let λ be a simple closed curve on ∂X_K that intersects μ transversely in one point

Proof: apply Mayer-Vietoris to X_K and $N(K)$

$$0 \to \underset{\mathbb{Z}}{\underset{\cong}{H_3(S^3)}} \xrightarrow{\Delta} \underset{\mathbb{Z}}{\underset{\cong}{H_2(\partial N)}} \to H_2(X_K) \oplus \underset{0}{\underset{\|}{H_2(N(K))}} \to \underset{0}{\underset{\|}{H_2(S^3)}}$$

$$\to \underset{\underset{\mu \quad \lambda}{\mathbb{Z} \oplus \mathbb{Z}}}{\underset{\cong}{H_1(\partial N)}} \xrightarrow{\phi} H_1(X_K) \oplus \underset{\underset{\text{gen by } \lambda}{\mathbb{Z}}}{\underset{\cong}{H_1(N(K))}} \to \underset{0}{\underset{\|}{H_1(S^3)}}$$

exercise: Δ an isomorphism (recall def$^{\underline{n}}$ of Δ)

$\therefore\ H_2(X(K)) = 0$

note: $H_1(X(K)) \cong \mathbb{Z}$

now we know

$$\phi(\mu) = (a, 0) \quad \text{some } a \text{ and } b$$
$$\phi(\lambda) = (b, 1)$$

for these to generate $\mathbb{Z} \oplus \mathbb{Z}$ need $a = 1$

so $\mu \subset X_K$ generates $H_1(X_K)$ $\blacksquare$

Proof of Thm 13:

($\Leftarrow$) clear!

($\Rightarrow$) $\pi_1(X_K) \cong \mathbb{Z}$

X_K irreducible (corollary of Schönflies)

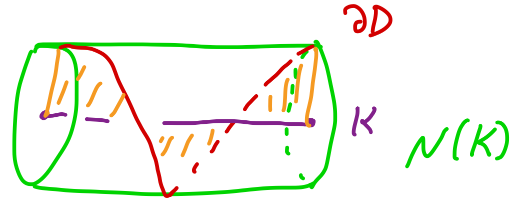
$\therefore X_K \cong S^1 \times D^2$ by Thm 9

let $D = \{pt\} \times D^2 \subset X_K$

$\partial D = c\mu + d\lambda$ in $H_1(X_K)$ $\quad$ c,d rel prime since ∂D embedded

$[\partial D] = 0$ in $H_1(X_K)$

$\therefore c = 0$ and $|d| = 1$



so $\exists$ an annulus $A \subset N(K)$ s.t.

$$\partial A = K \cup \partial D$$

$\therefore K = \partial(D \cup A)$ and K the unknot $\blacksquare$

Remark: 1) $\pi_1(X_K)$ does not determin K in general

these are not isotopic
but have isom π_1

2) If K_1 is prime and $\pi_1(X_{K_1}) \cong \pi_1(X_{K_2})$
then $\exists$ homeomorphism $\phi : S^3 \to S^3$ s.t. $\phi(K_1) = K_2$

A surface Σ embedded in M^3 is

- <u>compressible</u> if $\exists$ a disk $D \subset M$ s.t.
  - $D \cap \Sigma = \partial D$
  - ∂D is essential in Σ

  (D is a compressing disk)
- <u>incompressible</u> if $\Sigma \neq S^2$ and not compressible

<u>Thm 15:</u>

Σ connected surface properly embedded in a 3-manifold M

Σ is incompressible
$\Leftrightarrow$
the inclusion $i : \Sigma \to M$ induces an injection $i_* : \pi_1(\Sigma) \to \pi_1(M)$

<u>Proof:</u> ($\Leftarrow$) Σ compressible $\Rightarrow \exists$ disk $D \subset M$ s.t.
$[\partial D] \subset \Sigma$ is essential but
$i_*(\partial D) = 0$ in M so $\ker i_* \neq 0$ ✓

($\Rightarrow$) let $M \setminus \Sigma = \overline{M - N(\Sigma)}$

$N(\Sigma) = \Sigma \times [-1,1]$

get 2 copies $\Sigma_\pm = \Sigma \times \{\pm 1\}$ in $\partial(M \setminus \Sigma)$

Claim: $\pi_1(\Sigma) \longrightarrow \pi_1(M)$ one-to-one

$\Longleftrightarrow$

$\pi_1(\Sigma_\pm) \longrightarrow \pi_1(M \setminus \Sigma)$ one-to-one for + or −

indeed: ($\Rightarrow$) suppose $\pi_1(\Sigma_+) \to \pi_1(M \setminus \Sigma)$ is not one-to-one

then
$$\begin{tikzcd} \pi_1(\Sigma_+) \arrow[r] \arrow[rr, bend right, "\text{not one-to-one}"'] & \pi_1(M \setminus \Sigma) \arrow[r] & \pi_1(M) \end{tikzcd}$$

$\therefore$ $\pi_1(\Sigma) \to \pi_1(M)$ not one-to-one

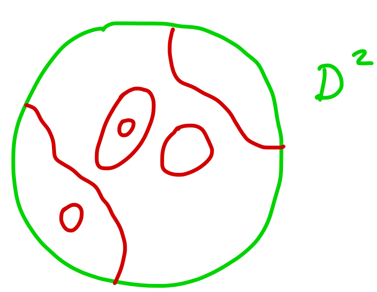
since Σ is isotopic to Σ_+ ✓

($\Leftarrow$) suppose $\pi_1(\Sigma) \longrightarrow \pi_1(M)$ is not one-to-one

so $\exists$ $f: (D^2, S^1) \to (M, \Sigma)$ s.t. $[f|_{S^1}] \neq 0$ in Σ

make f transverse to Σ

then $f^{-1}(\Sigma) = \coprod$ simple closed curves $\coprod$ arcs



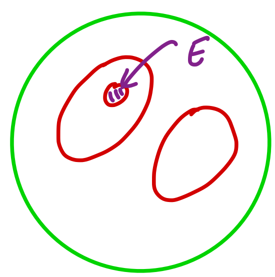
Can assume no arcs: since arcs

break D^2 into subdisks $D_1 \dots D_n$
with ∂ on Σ

one must have $\partial \neq 0$ in Σ or
else $[\partial D^2] = 0$ in Σ

just restrict f to this disk

let γ be an innermost s.c.c.
and E the disk it bounds

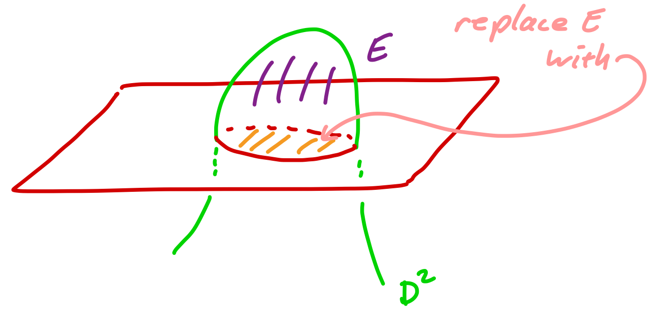


<u>Case 1</u>: $f|_\gamma$ inessential in Σ

define $f_1 : D^2 \to M$ by

$$f_1|_{\overline{\Sigma - E}} = f|_{\overline{\Sigma - E}}$$

$$f_1|_E \subset \Sigma$$



small homotopy of f_1 gives
new disk with fewer s.c.c. of $\cap$ w/ Σ

<u>Case 2</u>: $f|_\gamma$ is essential in Σ

by shrinking $N(\Sigma)$ can assume

$N(\Sigma) \cap E =$ annulus A

let $E_0 = \overline{E - A}$

$f(\partial E_0) \subset \Sigma_\pm$, say Σ_+

then $\pi_1(\Sigma_+) \to \pi_1(M \setminus \Sigma)$

is not one-to-one

now we know $\pi_1(\Sigma_+) \to \pi_1(M \setminus \Sigma)$

is note one-to-one

by the Disk Theorem $\exists$ disk $D \subset M \setminus \Sigma$

such that ∂D essential in Σ_+

$\therefore \exists$ a disk $D^+ \subset M$ s.t. $D^+ \cap \Sigma = \partial D^+$

is essential in Σ (add $\partial D \times [0,1]$ in $N(\Sigma)$)

$\therefore \Sigma$ is compressible

a 3-manifold M is called Haken if it is compact, irreducible, and contains an incompressible surface

Facts: 1) M Haken $\Rightarrow \pi_1(M)$ is infinite

(lemma 5 $\Rightarrow \pi_2(M) = 0$

$\therefore$ by lemma 2 universal cover $\simeq \mathbb{R}^3$
and $\pi_i(M) = 0 \quad \forall\, i \geq 2$)

2) If M irreducible and $H_1(M)$ is infinite
then M Haken

One can iteratively cut a Haken manifold along incompressible surfaces until all thats left are 3-balls

Using this one can easily prove lots of things for example

Th$^{\underline{m}}$:

> M, N closed irreducible 3-mfds with N Haken
> if $f: M \to N$ s.t. $f_*: \pi_1(M) \to \pi_1(N)$ an isom
> then $f \simeq$ homeomorphism

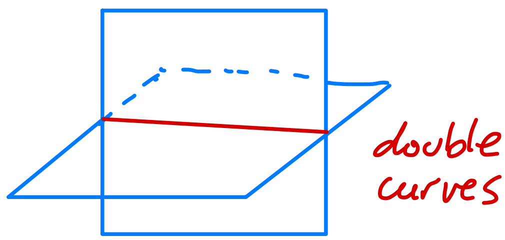
now let's prove

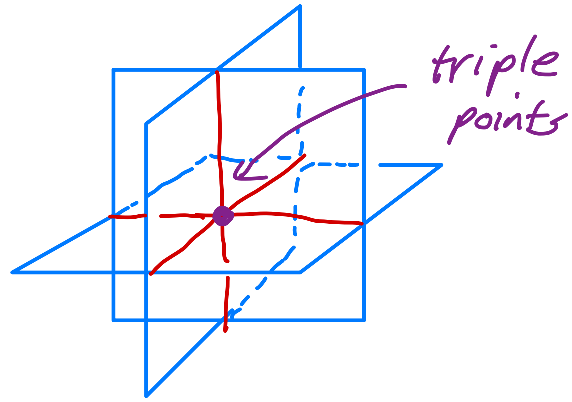
Th$^{\underline{m}}$ 4 (Disk Th$^{\underline{m}}$, Dehn's lemma, Papa 1957):

> let M be an orientable 3-manifold, $\Sigma \subset \partial M$ a surface, and $f: (D^2, S^1) \to (M, \Sigma)$ s.t. $[f|_{S^1}] \neq 1$ in $\pi_1(\Sigma)$
> Then $\exists$ an embedding $e: (D^2, S^1) \to (M, \Sigma)$ s.t. $e|_{S^1}$ is essential (i.e. doesn't bound a disk) in Σ

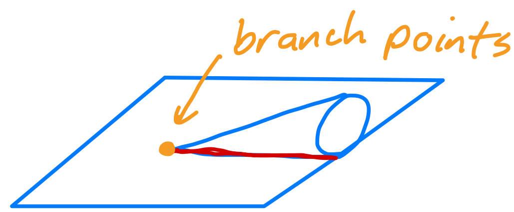
for this we need

Fact: $f:\Sigma^2 \to M^3$ a generic smooth map, then the singularities (non-embedded points) will consist of





and

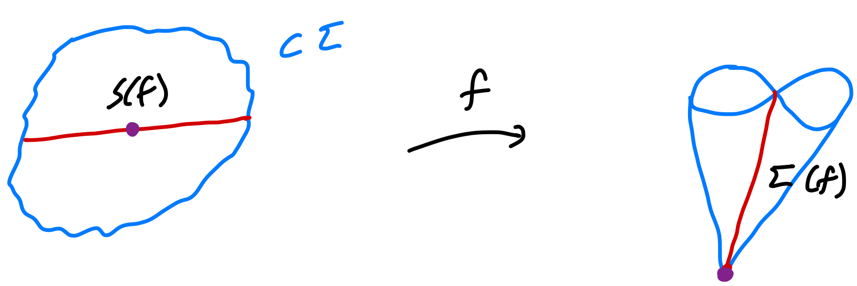


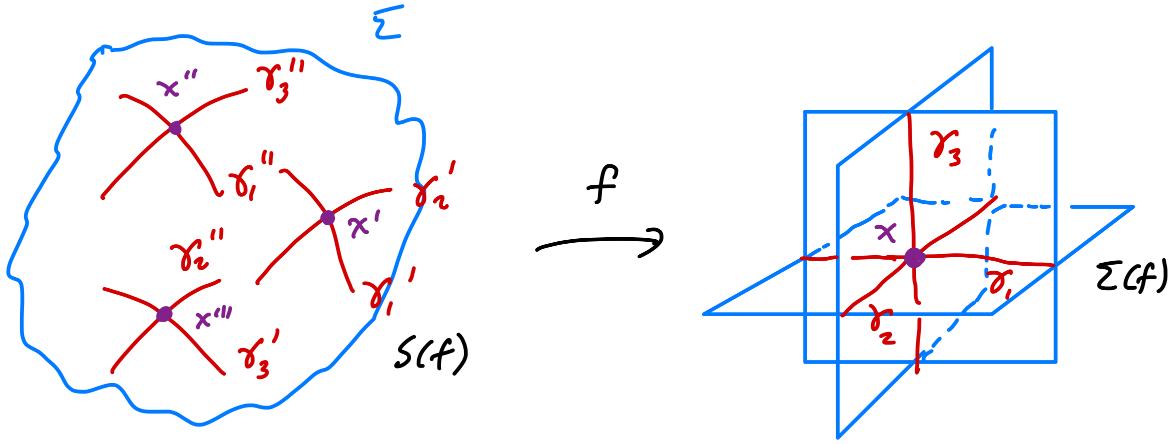
by a homotopy we can assume all our maps are generic and we do that from now on

let $S(f) = \overline{\{x \in \Sigma : f^{-1}(f(x)) \neq \{x\}\}}$

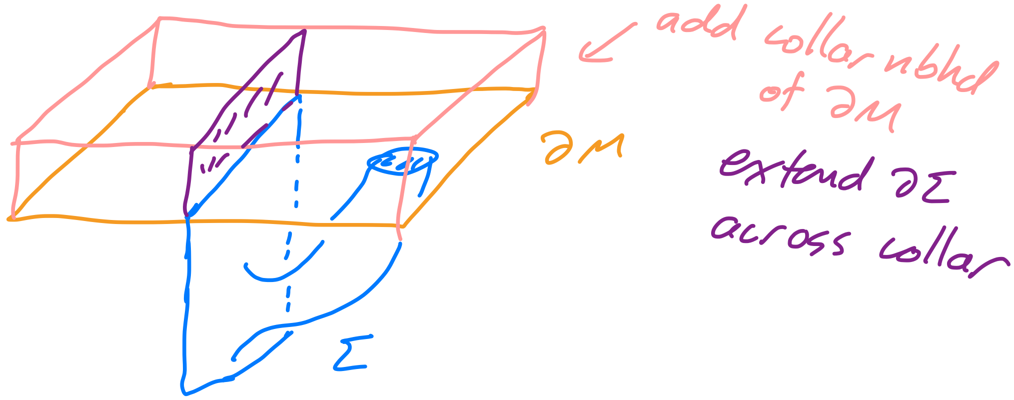
$= \bigcup$ immersed circles and properly embedded arcs both with transvers $\cap$'s and self $\cap$'s

$\Sigma(f) = f(S(f)) \subset M$ = union of double curves and arcs





note: we can assume $f(\text{int}\,\Sigma) \cap \partial M = \emptyset$



we will always assume this

exercise: given $f: \Sigma \to M$ generic

Show $\exists$ a homotopy (but not nec. smooth) to a smooth map $\bar{f}: \Sigma \to M$ with no branch points

hint: "merge" branch points or "push them" off the boundary

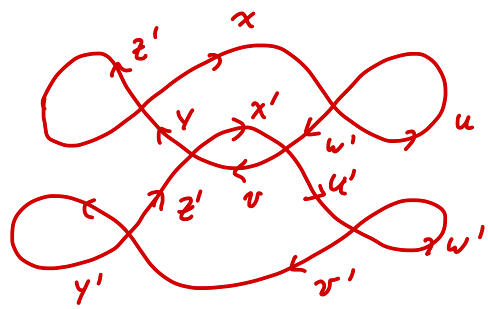
a double curve is simple if it is homeomorphic to S^1 (it may intersect other double curves) when trying to prove something, try simple cases first

lemma 16:

> Let M, Σ, f be as in the Disk Theorem
> and $f|_{\text{nbhd}\,\partial D}$ embedded
> if $\Sigma(f)$ contains only simple double curves
> then the conclusion of the Disk Theorem holds

not all double curves simple

e.g.

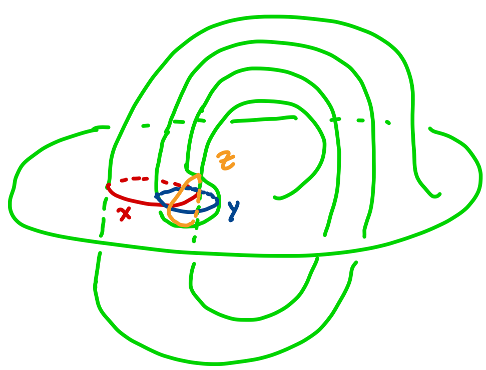


exercise: try to visualize this!

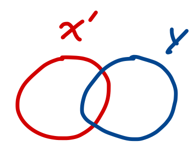
if $\Sigma(f)$ not simple, then use covering trick!

"intersections simplify in covers"

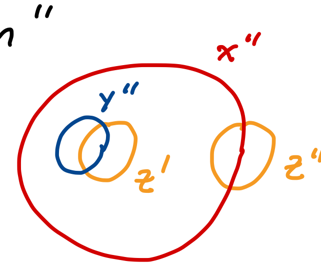
e.g.



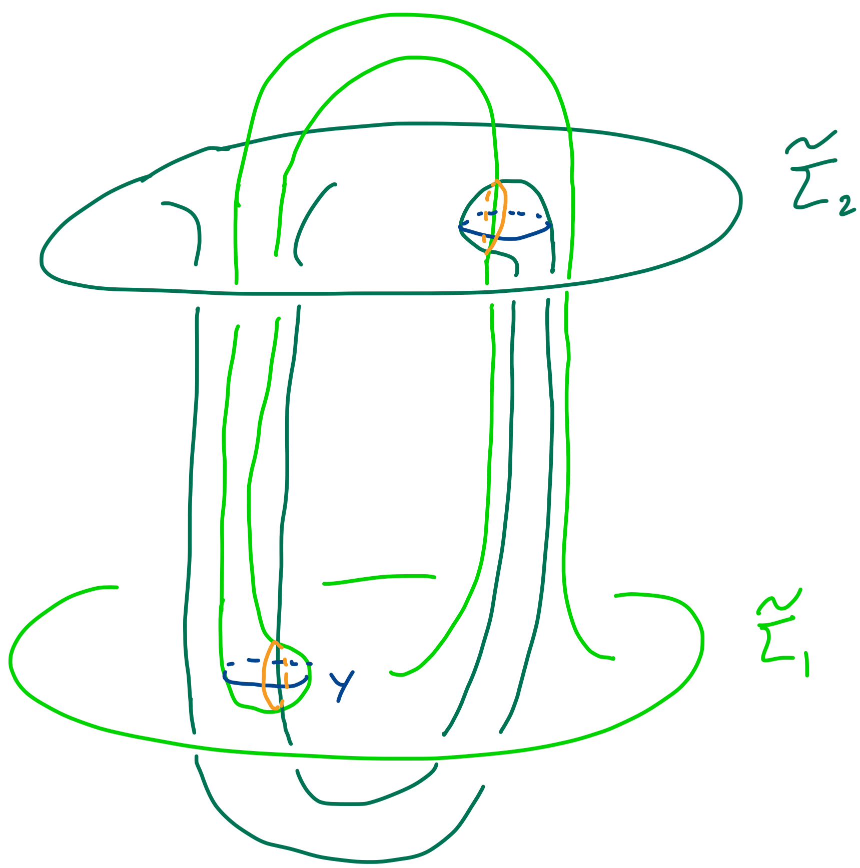
"on disk"



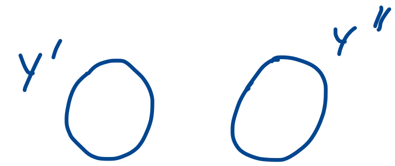
"on finger"



(note all are simple so could use lemma 16 but let's not) in a 2-fold cover of a nbhd of $f(D^2)$ you see



on $\tilde{\Sigma}_1$ only see



lemma 17:

Let M, Σ, f be as in the Disk Theorem
and $f|_{\text{nbhd}\, \partial D}$ embedded
let N be a regular nbhd of $f(D^2)$ and $\tilde{N}$ a 2-fold cover
if $\exists$ an embedding $f_1 : D^2 \to \tilde{N}$ s.t. $p \circ f_1(\partial D^2)$ is essential in Σ, then $\exists$ an embedding $e_1 : D^2 \to M$ s.t. $e(\partial D^2)$ is essential in Σ

but what if there is no 2-fold cover of N?

lemma 18:

> let M, Σ, f, and N be as in lemma 17
> Suppose N does not have a 2-fold cover
> then the conclusion of the Disk Theorem holds

Proof of Th$^{\underline{m}}$ 4:

assume $f|_{\text{nbhd}\,\partial D^2}$ an embedding

let N = regular nbhd of $f(D^2)$

- done if N has no 2-fold cover
- done if th$^{\underline{m}}$ true in a 2-fold cover

so use a tower

$$\begin{tikzcd}
 & N_n \subset M_n \arrow[d, "\text{2-fold cover}"] \\
 & N_{n-1} \\
 & \vdots \\
 & N_2 \subset M_2 \arrow[d, "\text{2-fold cover}"] \\
 & N_1 \subset M_1 \arrow[d, "\text{2-fold cover}"] \\
D^2 \arrow[uuuuur, bend left, "f_n"] \arrow[uuuur, bend left, "f_{n-1}"] \arrow[uur, "f_2"] \arrow[ur, "f_1"] \arrow[r, "f_0 = f"'] & N_0 \subset M_0
\end{tikzcd}$$

$N_0 = N$, $M_0 = M$

N_i a reg nbhd of $f_i(D^2)$ in M_i
M_i 2-fold cover of N_{i-1}
f_i a lift of f_{i-1} to M_i

this is called a <u>tower</u> for $f : D \to M$

lemma 19:

> for f as above, f has a finite tower
> such that N_n has no 2-fold cover

lemmas 16-19 complete the Disk theorem when
$f|_{\text{nbhd}\,\partial D^2}$ is an embedding so thm done
by

lemma 20:

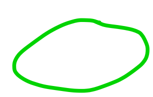
> Let M, Σ, f be as in the Disk Theorem
> there is another map $g:(D^2, \partial D^2) \longrightarrow (M, \Sigma)$
> s.t. g is an embedding near ∂D^2 and
> $g(\partial D^2)$ is essential in Σ

we must now go back and prove lemmas

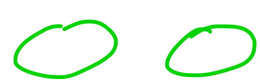
Proof of lemma 16:

let $\gamma \subset \Sigma(f)$

$f|_{f^{-1}(\gamma)}$ is a 2 to 1 covering map of S^1

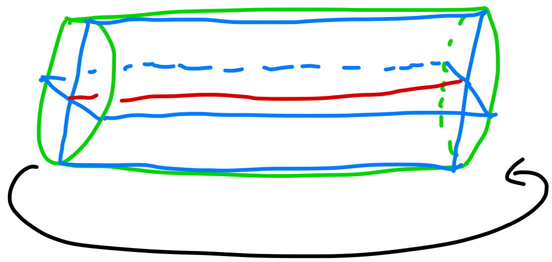
so $f^{-1}(\gamma) =$  one simple close curve

or

 two simple closed curves

let $N(\gamma)$ be a nbhd of γ in M

so $N(\gamma) = S^1 \times D^2$ (since M orientable)



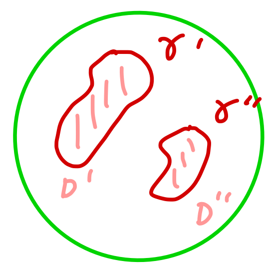
ends glued by a rotation of $\begin{cases} (1) & 0 \quad (+2n\pi) \\ (2) & \pi/2 \quad (+2n\pi) \\ (3) & \pi \quad (+2n\pi) \end{cases}$

In case (2) and (3) $D \cap N(\gamma)$ is a Möbius band (or 2 bands) $\otimes$ D orientable

$\therefore$ in case (1) and $D \cap N(\gamma) = 2$ annuli

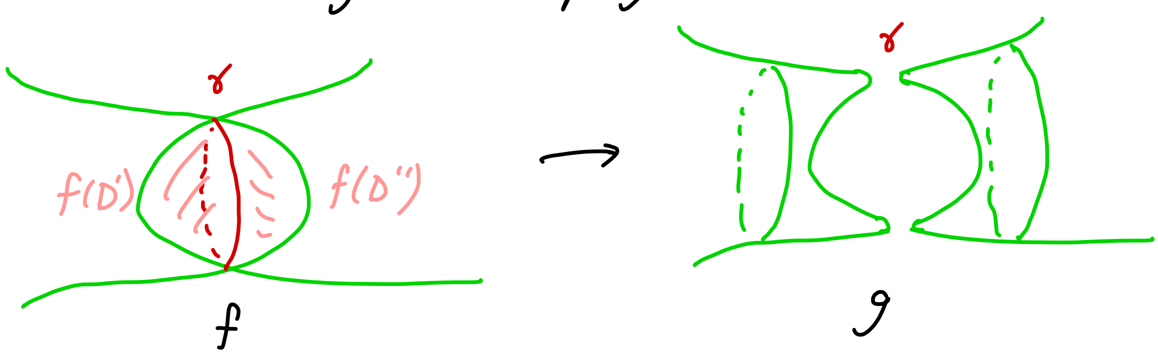
$\therefore f^{-1}(\gamma) = \gamma' \amalg \gamma''$

γ', γ'' bound disks $D', D'' \subset D$

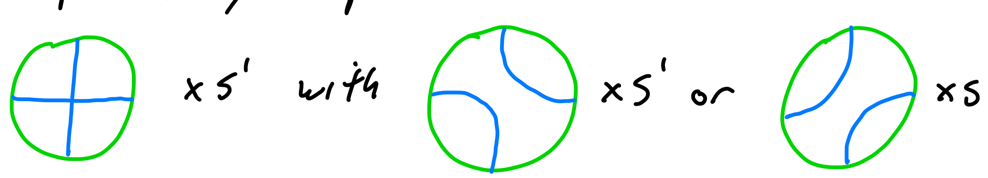


Case 1: D', D'' disjoint

In M "surgery" f along γ to get a map g



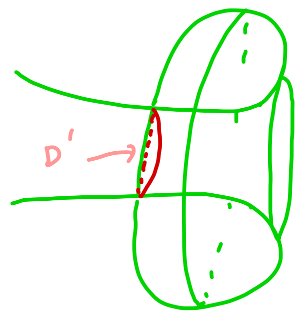
more precisely replace

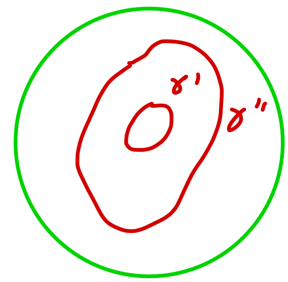


<u>note</u> $g:(D^2, \partial D^2) \to (M, \Sigma)$

and $g|_{\partial D^2} = f|_{\partial D^2}$

<u>Case 2</u>: $D' \subset D''$ (or $D'' \subset D'$)





form g by f on $D - D''$

and f on D'

note: domain of g is a disk

and $g|_{\partial D^2} = f|_{\partial D^2}$

we have elliminated γ from $\Sigma(f)$ continue with other curves in $\Sigma(f)$ untill you get embedding ▦

<u>Proof of lemma 17</u>: let $\bar{f} = p \circ f_1$

by hypothesis $\bar{f}(\partial D^2)$ is essential in Σ

now $\Sigma(\bar{f})$ contains only simple double curves

(check this if not clear)

$\therefore$ lemma 16 $\Rightarrow \exists$ embedded disk $e: D^2 \to M$

with $e(\partial D^2)$ essential in Σ ▦

<u>Proof of lemma 18</u>: having no 2-fold covers implies there is no nontrivial homomorphism $\pi_1(N) \to \mathbb{Z}_2$

thus no nontrivial homomorphism $H_1(N) \to \mathbb{Z}_2$

(since $\pi_1(N) \to H_1(N)$ is abelianization)

now $H^1(N;\mathbb{Z}_2) = \underbrace{\mathrm{Hom}(H_1(N),\mathbb{Z}_2)}_{=0} \oplus \underbrace{\mathrm{Ext}(H_0(N),\mathbb{Z}_2)}_{=0}$ universal coefficient thm

$= 0$

the exact sequence for $(N,\partial N)$ gives

$$
\begin{tikzcd}
H_2(N,\partial N;\mathbb{Z}_2) \arrow[r] \arrow[d, phantom, "\cong"{rotate=90}, "\text{Poincaré duality}"'] & H_1(\partial N;\mathbb{Z}_2) \arrow[r] & H_1(N;\mathbb{Z}_2) \arrow[d, phantom, "\cong"{rotate=90}, "\text{universal coefficients}"'] \\
H^1(N;\mathbb{Z}_2) \arrow[d, phantom, "="{rotate=90}] & & H^1(N;\mathbb{Z}_2) \arrow[d, phantom, "="{rotate=90}] \\
0 & & 0
\end{tikzcd}
$$

so $H_1(\partial N;\mathbb{Z}_2)=0$ and $\partial N = \coprod S^2$

$\exists\, S^2 \subset \partial N$ s.t. $\partial D \subset S^2$

let D' be a disk in S^2 that ∂D bounds

push the interior of D' into N and D' is an embedded disk with $\partial D' = \partial D$

Proof of lemma 19:

consider

$$
\begin{tikzcd}
 & N_i \arrow[r, phantom, "\subset"] & M_i \arrow[dl, "p_i"] \\
D^2 \arrow[ur, "f_i'"] \arrow[r, "f_{i-1}"'] & N_{i-1} &
\end{tikzcd}
$$

clearly $S(f_i) \le S(f_{i-1})$

Claim: $S(f_i) \neq S(f_{i-1})$

indeed, if $S(f_i) = S(f_{i-1})$ then

$$p_i|_{f_i(D)} : f_i(D) \to f_{i-1}(D) \text{ is an isom. on } \pi_1$$

these are the same quotient spaces

so

$$\begin{tikzcd}
\pi_1(f_i(D)) \arrow[r] \arrow[d, "\cong"] & \pi_1(M_i) \arrow[d, "(p_i)_*"] \\
\pi_1(f_{i-1}(D)) \arrow[r, "\cong"'] & \pi_1(N_{i-1})
\end{tikzcd}$$

since N_{i-1} reg nbhd of $f_{i-1}(D)$

$\therefore (p_i)_*$ is onto $\pi_1(N_{i-1})$

this contradicts $p_i : M_i \to N_{i-1}$ a 2-fold cover

thus f_i has strictly <u>fewer</u> singularities than f_{i-1}

$\therefore \exists$ finite tower since if $S(f_n) = \emptyset$ then, there is no 2-fold cover

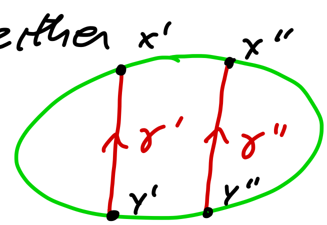
<u>Proof of lemma 20</u>: let x be a double point on $f(\partial D^2)$

so $\exists\, x', x'' \in \partial D^2$ s.t. $f(x') = f(x'') = x$

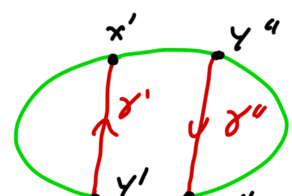
and $\exists$ double point arc γ in $f(\partial D)$ s.t. $x \in \partial\gamma$

since we arranged no branch points $\partial\gamma = \{x, y\}$ with y another double pt in $f(\partial D^2)$

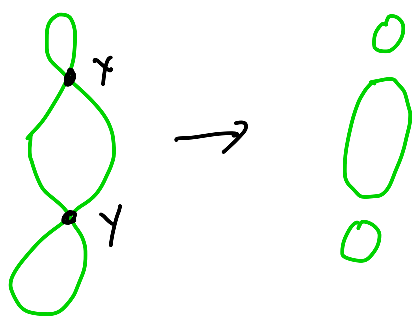
we now have either



or



in the first case we can surger along γ to get



if bounds a disk can use it to change f to remove x, y

the proof of the sphere theorem is similar

(or one can use the disk th$^{\underline{m}}$ to prove it)