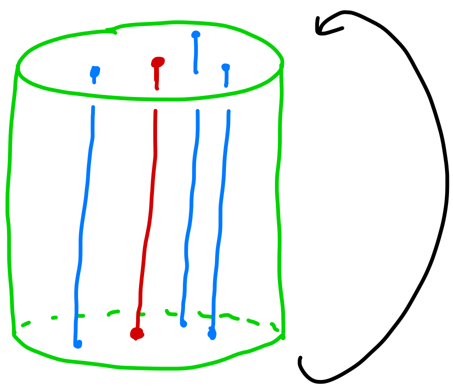
IV. Torus decompositions & Special 3-Manifolds

A. Torus Decomposition

a compact 3-manifold M is a Seifert Fibered Space (SFS) if M is a union of circles (called fibers) in such a way that every fiber has a nbhd which is a union of fibers and is fiber preserving homeo. to a fibered solid torus

$D^2 \times I$



h

h = rotation by $\frac{2\pi q}{p}$

$(p \geq 1, (p,q) = 1)$

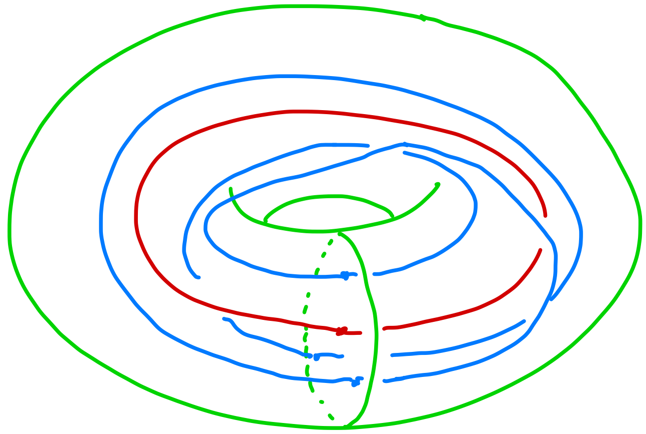
$V_{(p,q)} = D^2 \times I /_{\sim h}$ is a (p,q)-fibered solid torus

fibers are: $[\{x\} \times I] \cup [\{h(x)\} \times I] \cup \dots \cup [\{h^{p-1}(x)\} \times I]$

$x \neq 0$ are ordinary fibers

and: $[\{0\} \times I]$ for $x = 0$ is the exceptional fiber (if $p > 1$)

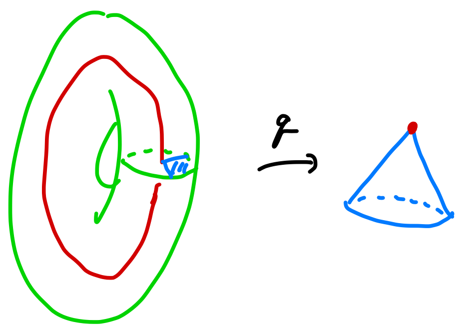
(a.k.a. singular fibers)



exercise: M has finitely many exceptional fibers

$V/\text{collapse each fiber} \cong D^2$

quotient map $p: V \to D^2$



note $p|_{\text{merid disk}}$ is a p-fold branched cover with one branch point

(i.e. merid. disk = unit disk in $\mathbb{C}$

$p|_D(z) = z^p$)

so if M is a SFS can get a quotient map

$$p: M \to B$$

B is called the base surface

a surface Σ is a 3-manifold M is boundary parallel if it is isotopic rel ∂ to a subsurface of ∂M

M is atoroidal if every incompressible torus in M is boundary parallel

a torus decomposition of M is a finite disjoint union $\mathcal{T}$ of incompressible tori $\subset \operatorname{int} M$ s.t.

(1) each component of $M \setminus \mathcal{T}$ is atoroidal or a Seifert fibered space, and

(2) $\mathcal{T}$ is minimal with respect to (1)

Th^{m} 1 (Torus decomposition theorem or JSJ decomposition Jaco-Shalen 1977 and Johannson 1979):

$$\boxed{\text{Every compact, irreducible 3-manifold has a torus decomposition unique up to isotopy}}$$

Remark: Existence is somewhat similar to prime decoposition theorem (Th^{m} 11.4) and uses "normal surfaces"

Uniqueness uses lots of properties of SFSs

for a proof see Hatcher's notes on 3-mfds

Part of the Geometrization Th^{m} (Perelman ~2003) says

$$\boxed{\text{if } M \text{ is an irreducible, atoroidal 3-manifold that is not a Seifert fibered space and is either closed or has torus boundary, then the interior of } M \text{ admits a hyperbolic structure}}$$

we say a 3-manifold is hyperbolic if it admits a complete Riemannian metric with all sectional curvatures $= -1$ and of finite volume

Remark: so the Torus th$^{\underline{m}}$ and geometrization imply you can decompose any 3-mfd into its prime pieces and then along incomp. tori, so that each piece is

1) a Seifert fibered space or
2) hyperbolic

so we more or less "know" 3-manifolds if we know SFSs and hyperbolic manifolds!

we know a lot about these

B. Seifert fibered spaces

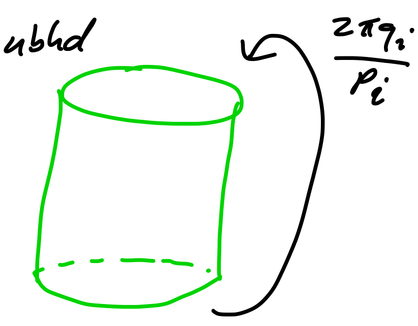
suppose M is a compact Seifert fibered space with projection

$$p : M \to B$$

to the base surface and singular fibers $C_1, \dots, C_n$ such that C_i has a nbhd

$N_i = p^{-1}(D_i)$ with

$N_i \cong (p_i, q_i)$ fibered torus



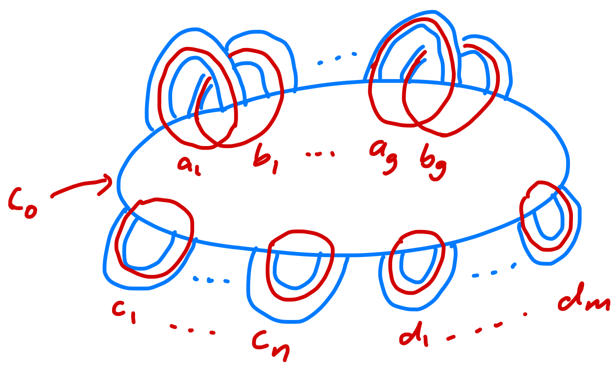
let N_0 = nbhd of a regular fiber = $p^{-1}(D_0)$

let $M_0 = M - \bigcup_{i=0}^{n} N_i$ and $B_0 = B - \bigcup_{i=0}^{n} D_i$

$p|_{M_0} : M_0 \to B_0$ an S^1- bundle

now we have 2-cases

(o) B_0 orientable

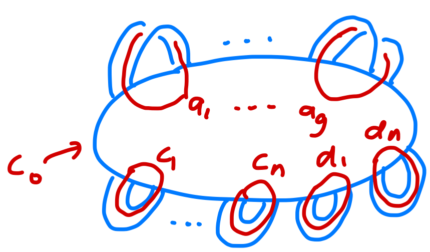


$c_0, \dots c_n$ correspond to ∂D_i

$d_1, \dots d_m$ correspond to ∂B

$\pi_1(B_0)$ = free group gen by $a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_n, d_1, \dots d_m$

(n) B_0 non-orientable



$\pi_1(B_0)$ = free group gen by $a_1, \dots, a_g, c_1, \dots c_n, d_1, \dots d_m$

let $\alpha_1, \dots, \alpha_k$ be arcs properly embedded in B_0

s.t. $B_0 \setminus \bigcup \alpha_i$ = disk D

$k = 2g+m+n$ (o)

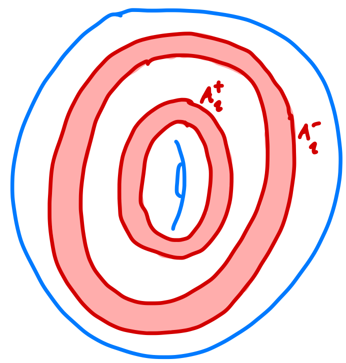
$\quad\; g+m+n$ (n)

$\pi^{-1}(\alpha_i) = A_i =$ annulus

$Q = M_0 \setminus \cup A_i = S^1 \times D^2$

$\pi_1(Q) \cong \mathbb{Z}$ gen by $h = [\text{fiber}]$

$\partial Q = S^1 \times \partial D \supset$ copies $A_i^{\pm}$ of A_i



M_0 obtained from Q by identifying A_i^+ with A_i^-

M_0 orientable $\Rightarrow$ identification is by

$id \times id : S^1 \times \alpha_i^+ \to S^1 \times \alpha_i^-$

except in case (n) for $a_1 \dots a_g$, then by

reflection $\times$ reflection

can assume $\alpha_i^+ \leftrightarrow \alpha_i^-$ in ∂D so $\exists$ a section

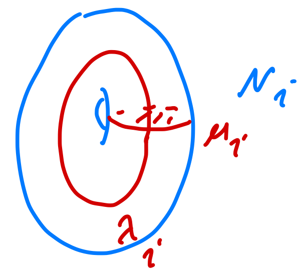
$\sigma : B_0 \to M_0$

$\therefore$ can think of $B_0 \subset M_0$

$$\pi_1(M_0) \cong \begin{cases} \pi_1(B_0) \times \underset{h}{\mathbb{Z}} & (o) \\ \langle a_1, \dots a_g, d_1 \dots d_m, c_1, \dots c_n, h \mid [h, d_i] = 1 = [h, c_i] & (n) \\ \qquad a_i^{-1} h a_i = h^{-1} \rangle & \end{cases}$$

$\pi_1(\partial N_i) \cong \mathbb{Z} \times \mathbb{Z}$ gen by h and $c_i \subset \partial B_0$

also by λ_i, μ_i in ∂N_i



$$h = \text{regular fiber} = \lambda_i^{p_i} \mu_i^{q_i}$$

so $c_i = \lambda_i^{r_i} \mu_i^{s_i}$ some r_i, s_i st.

$$p_i s_i - q_i r_i = 1$$

note: $c_i^{p_i} h^{-r_i} = \lambda_i^{r_i p_i} \mu_i^{s_i p_i} \lambda_i^{-p_i r_i} \mu_i^{-s_i r_i}$

$= \mu_i$

and $\mu_i = 1$ in $\pi_1(N_i)$

recall $p_0 = 1$ so set $r_0 = b \in \mathbb{Z}$

now: $c_0 = \begin{cases} \prod [a_i, b_i] \prod d_i \prod c_i & (o) \\ \prod a_i^2 \prod d_i \prod c_i & (n) \end{cases}$

now can use Van Kampen to prove

Thm 2:

(o) $\pi_1(M) \cong \langle a_1, b_1, \dots, a_g, b_g, d_1, \dots, d_m, c_1 \dots c_n, h \mid$

$[h, a_i] = [h, b_i] = [h, d_i] = [h, c_i] = 1$

$c_i^{p_i} = h^{r_i}, \ \prod [a_i, b_i] \prod d_i \prod c_i = h^b \rangle$

$g \geq 0, m \geq 0, n \geq 0 \quad p_i \geq 2, b \in \mathbb{Z}$

$$\boxed{\begin{aligned}(n)\ \pi_1(M) \cong \langle a_1 \ldots a_g, d_1 \ldots d_m, c_1, \ldots, c_n, h \mid\ & a_i^{-1} h a_i = h^{-1}, [h, d_i] = [h, c_i] = 1, c_i^{p_i} = h^{r_i} \\ & \textstyle\prod a_i^2 \prod d_i \prod c_i = h^b \rangle \\ g \geq 1, m \geq 0, n \geq 0, p_i \geq 2, b \in \mathbb{Z}\end{aligned}}$$

note: 1) Cyclic group $\langle h \rangle$ is normal in $\pi_1(M)$ and central in (0) case

2) if $\partial M \neq \emptyset$ (ie. $m \geq 1$) then can discard d_m and last relation

3) $p_*: \pi_1(M) \to \pi_1(B)$ is onto

$$\pi_1(M) / \langle h = 1, c_i = 1 \rangle = \pi_1(B)$$

4) recall $(r_i, p_i) = 1$ can arrange $0 < r_i < p_i$ and then b is uniquely determined

With this theorem can show lots of things

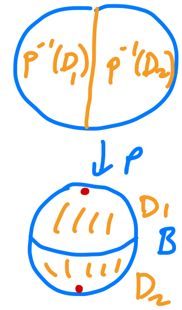
for example

Thm 3:

$$\boxed{\begin{aligned}&M \text{ a closed SFS} \\ &\quad \pi_1 M = 1 \iff M \cong S^3\end{aligned}}$$

(of course, now, we know with out SFS assumption, but proof much harder)

Proof: $\pi_1 M = 1 \Rightarrow \pi_1 B = 1 \Rightarrow B \cong S^2$



let $n = \#$ singular fibers

if $n \le 2$, then $M =$ union of 2 solid tori

exercise: given this, $\pi_1 M = 1 \Rightarrow M \cong S^3$

if $n \ge 3$, then

$$\pi_1 M / \langle h \rangle = \langle c_1, \dots, c_n \mid c_i^{p_i} = 1, \ \prod_{i=1}^{n} c_i = 1 \rangle$$

$$\downarrow \text{ quotient by } \langle c_4, \dots, c_n \rangle$$

$$\langle c_1, c_2, c_3 \mid c_1^{p_1} = c_2^{p_2} = c_3^{p_3} = 1, \ c_1 c_2 = c_3^{-1} \rangle$$

$$\cong$$

$$\langle c_1, c_2 \mid c_1^{p_1} = c_2^{p_2} = (c_1 c_2)^{-p_3} = 1 \rangle$$

this is called the triangle group $T(p_1, p_2, p_3)$ $p_i \ge 2$

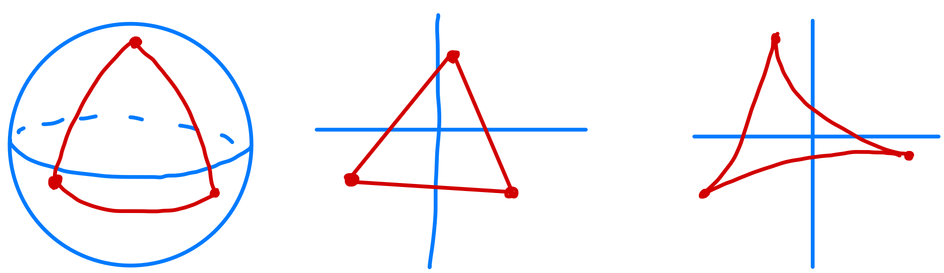
Claim: $T(p_1, p_2, p_3) \ne 1$ $\quad (\cancel{\ast} \Rightarrow n \le 2)$

idea: $\exists$ triangles with verticies $A_1, A_2, A_3 \subset \begin{cases} S^2 & \text{round sphere} \\ E^2 & \text{flat } \mathbb{R}^2 \\ \mathbb{H}^2 & \text{hyperbolic space} \end{cases}$

with angles $\frac{\pi}{p_1}, \frac{\pi}{p_2}, \frac{\pi}{p_3}$

according as $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} \begin{cases} > 1 \\ = 1 \\ < 1 \end{cases}$

eg



let ρ_{ij} = reflection in $A_i A_j$

$\gamma_1 = \rho_{13}\rho_{12}$ = rotation about A_1 through $\frac{2\pi}{p_1}$

$\gamma_2 = \rho_{21}\rho_{23}$ = " " A_2 " " $\frac{2\pi}{p_2}$

$\gamma_3 = \gamma_1 \circ \gamma_2 = \rho_{13}\rho_{23}$ " " A_3 " " $\frac{2\pi}{p_3}$

$\Gamma(p_1, p_2, p_3)$ = subgroup of isometries of $\left\{ \begin{matrix} S^2 \\ E^2 \\ \mathbb{H}^2 \end{matrix} \right\}$ generated by γ_1, γ_2

so $\Gamma(p_1, p_2, p_3) \neq 1$ 

other results one can show are

1) M closed SFS, $\tilde{M}$ its universal cover

then $\tilde{M} \cong S^3, S^2 \times \mathbb{R}, \mathbb{R}^3$

(so M irreducible or $\tilde{M} \cong S^2 \times \mathbb{R}$)

2) M closed SFS then $\pi_1(M)$ is finite <u>or</u>

h has infinite order in $\pi_1(M)$

3) M SFS:

∂M incompressible unless $B = D^2$ and $n \leq 1$

4) M SFS if (o) $g > 0$ or $g = 0$ and $n \geq 4$

or (n) $g > 1$ or $g = 1$ and $n \geq 2$

then M contains an incompressible torus exercise: this follows from

lemma 4:

$$\boxed{\begin{array}{l} M \text{ a SFS, then} \\ \quad M \cong S^1 \times D^2 \iff B \cong D^2 \text{ and } n \leq 1 \end{array}}$$

Proof:

($\Leftarrow$) clear

($\Rightarrow$)
$$\left.\begin{array}{l} |\partial M| = 1 \Rightarrow |\partial B| = 1 \\ \pi_1(M) \cong \mathbb{Z} \Rightarrow \pi_1(B) \text{ cyclic} \end{array}\right\} \Rightarrow B \cong \begin{cases} D^2 & \text{or} \\ \text{Möbius band} \end{cases}$$

Case 1: $B \cong D^2$

$$\pi_1(M) \cong \langle h, c_1, \ldots, c_n \mid c_i^{p_i} = h^{r_i}, [h, c_i] = 1 \rangle$$

$$\pi_1(M)/\langle h \rangle \cong \mathbb{Z}/p_1 * \ldots * \mathbb{Z}/p_n$$

$$\pi_1(M) \cong \mathbb{Z} \Rightarrow n \leq 1$$

Case 2: $B \cong$ Möbius band

$$\pi_1(M) \cong \langle h, a, c_1, \ldots c_n \mid a^{-1} h a = h, c_i^{p_i} = h^{r_i}, [h, c_i] = 1 \rangle$$

$$\pi_1(M)/\langle h \rangle \cong \mathbb{Z} * \mathbb{Z}_{p_1} * \ldots * \mathbb{Z}_{p_n}$$

$$\pi_1(M) \cong \mathbb{Z} \Rightarrow n = 0$$

so $\pi_1(M) \cong \langle h, a \mid a^{-1} h a = h^{-1} \rangle$

exercise: show this not isomorphic to $\mathbb{Z}$

hint: mod out by $\langle h^2 \rangle$

this $\otimes$ says $B \ncong$ Möbius band

5) M closed SFS then either

1) $\pi_1(M)$ finite $\quad (\Leftrightarrow \tilde{M} \cong S^3)$

2) $\pi_1(M) \supset \mathbb{Z} \times \mathbb{Z} \quad (\Leftrightarrow \tilde{M} \cong \mathbb{R}^3)$

3) $M \cong S^1 \times S^2$ or $\mathbb{RP}^3 \# \mathbb{RP}^3$ $(\Leftrightarrow \tilde{M} \cong S^2 \times \mathbb{R})$

for last result we need

Th$^{\underline{m}}$ 5:

M a closed SFS

$\tilde{M} \cong S^2 \times \mathbb{R} \Leftrightarrow M \cong S^2 \times S^1$ or $\mathbb{RP}^3 \# \mathbb{RP}^3$

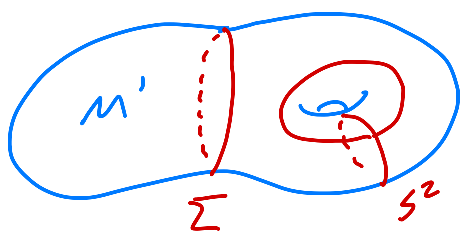
Proof: $(\Rightarrow)$ $\tilde{M} \cong S^2 \times \mathbb{R}$

so $\pi_2(M) \cong \pi_2(\tilde{M}) \neq 0$

Sphere Th$^{\underline{m}}$ $\Rightarrow \exists$ essential 2-sphere $S \subset M$

Case 1: S is non-separating

then $M \cong S^2 \times S^1 \# M'$ (Th$^{\underline{m}}$ II.1 + Remark)



let Σ be connect sum sphere

lift Σ to $\tilde{\Sigma} \subset \tilde{M} \cong S^2 \times \mathbb{R} = \mathbb{R}^3 - \{(0,0,0)\}$

$\tilde{\Sigma} = \partial \tilde{B}$, $\tilde{B}$ a 3-ball in $\mathbb{R}^3$

Claim: $\tilde{B} \subset \mathbb{R}^3 - \{(0,0,0)\}$

suppose $(0,0,0) \in \tilde{B}$

let $\tilde{S}$ be a lift of S to $\tilde{M}$

note $\tilde{S}$ and $\tilde{\Sigma} = \partial \tilde{B}$ both embedded

non null-homotopic spheres

$\therefore$ $\tilde{S} \simeq \tilde{\Sigma}$ in $\tilde{M}$ and so $S \simeq \Sigma$

$\otimes$ S non-separating, Σ separating

<u>exercise</u>: $p|_{\tilde{B}} : \tilde{B} \to p(\tilde{B})$ is a homeo.

thus $p(\tilde{B}) = B$ is an embedded ball with $\partial B = \Sigma$

$\therefore$ $M' = S^3$ and $M \cong S^1 \times S^2$

<u>Case 2</u>: S separating

$M \cong M_1 \# M_2$

$\pi_1(M) = A * B$ $\qquad$ S not $\simeq * \Rightarrow$ $A = \pi_1(M_1) \neq 1$, $B = \pi_1(M_2) \neq 1$

recall $\langle h \rangle \triangleleft \pi_1(M)$

<u>lemma 6</u>:

> a free product $A * B$ $(A \neq 1 \neq B)$ has a non-trivial cyclic normal subgroup $\Leftrightarrow A \cong B \cong \mathbb{Z}/2$

we prove this later

$\therefore$ $\pi_1(M) \cong \mathbb{Z}/2 * \mathbb{Z}/2$

center $(\mathbb{Z}/2 * \mathbb{Z}/2) = 1 \Rightarrow M$ is a SFS of case (n)

the "Kurosh subgroup th$^{\underline{m}}$" $\Rightarrow$ any abelian subgroup of $\mathbb{Z}_2 * \mathbb{Z}_2$ is either $\mathbb{Z}$ or conjugate into a factor

(see my undergrad alg. top. notes)

$\therefore \mathbb{Z}/2 * \mathbb{Z}/2 \not\supset \mathbb{Z}\times\mathbb{Z}$ and so <u>no</u> incomp. tori

so fact 4) above $\Rightarrow$ we are in case (n) $g=1$ and $n \leq 1$

if $n=1$: singular fiber of multiplicity p

so $\pi_1(M)/_{\langle h\rangle} = \langle a, c : c^p = 1, a^2 c = 1\rangle$

$\cong \mathbb{Z}/_{2p}$

$\therefore p = 1$ since $\mathbb{Z}/_2 * \mathbb{Z}/_2 \twoheadrightarrow \mathbb{Z}/_{2p} \Rightarrow p = 1$

so fiber not singular! $\otimes$

if $n=0$: $\pi_1(M) \cong \langle a, h : a^{-1}ha = h^{-1}, a^2 = h^b\rangle$

<u>$b \neq 0$:</u> a^2 is central so $a^2 = 1$ since $\mathbb{Z}/_2 * \mathbb{Z}/_2$ has trivial center

thus $\pi_1(M) \cong \langle a, h : a^{-1}ha = h^{-1}, a^2 = h^b = 1\rangle$

$\cong D_b$ a dihedral group!

$\otimes$ $\pi_1(M)$ infinite, D_b finite so $b=0$: so M is an S^1-bundle over $\mathbb{R}P^2$

and b was the obstruction to a section

$\therefore \exists\, \sigma: \mathbb{R}P^2 \to M$ and we think of

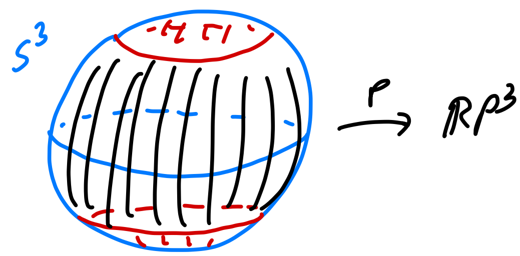
$\mathbb{R}P^2 \subset M$ via $\sigma(\mathbb{R}P^2)$

now $N(\mathbb{R}P^2) \subset M$ is a twisted I-bundle over $\mathbb{R}P^2$ call it P

exercise: $\exists$ unique orientable I-bundle over $\mathbb{R}P^2$

Claim: $P \cong \mathbb{R}P^3 \setminus B^3$

indeed $\mathbb{R}P^3 = S^3/_{\text{antipodes}}$



$p^{-1}(B^3) = S^2 \times I$

$\mathbb{R}P^3 \setminus B^3 \cong p^{-1}(B^3)/_\sim \cong S^2 \times I/_\sim$

I bundle over $\mathbb{R}P^2$

since orientable must be P

now $M \setminus N(\mathbb{R}P^2)$ also I-bundle over $\mathbb{R}P^2$

$\therefore$ also P

so $M = P \cup_\partial P \cong \mathbb{R}P^3 \# \mathbb{R}P^3$ ✓

($\Leftarrow$) clear ▦

Proof of lemma 6:

($\Leftarrow$) $\mathbb{Z}/2 * \mathbb{Z}/2 \cong \langle a, b \mid a^2 = b^2 = 1 \rangle$

$$\underset{\substack{\uparrow \\ \text{exercise}}}{\cong} \langle a, c : a^2 = 1,\ aca = c^{-1} \rangle = D_\infty$$

infinite dihedral group

$$1 \to \underset{\substack{\uparrow \\ \text{gen by } c}}{\mathbb{Z}} \to D_\infty \to \mathbb{Z}/2 \to 1$$

$\langle c \rangle$ normal subgroup $\cong \mathbb{Z}$

($\Rightarrow$) suppose $\langle h \rangle \triangleleft A * B$ $\quad A \neq 1 \neq B,\ h \neq 1$

i) $h = a$ or b $\quad a \neq 0 \in A,\ b \neq 0 \in B$

ii) $h = a_1 b_1 \cdots a_m b_m$ $\quad a_i \neq 1,\ b_i \neq 1$
or $a_1 b_1 \cdots a_m$, or $b_1 a_1 \cdots b_m a_m$ or $b_1 a_1 \cdots b_m$

$\langle h \rangle$ normal $\Rightarrow ghg^{-1} = h^{n(g)}$ some $n(g) \in \mathbb{Z}$ $\forall g \in A * B$

(i) $\Rightarrow ba^{-1}b = a^n$ $\not\!\!O$ for $b \neq 1 \in B$

(ii) case $h = a_1 b_1 \cdots a_m b_m$
pick $a \in A$, $a \neq 1$

$$a^{-1} h a = \underbrace{a^{-1}(a_1 b_1 \cdots a_m b_m) a}_{\substack{\text{length } 2m \text{ if } a = a_1, \\ 2m+1 \text{ otherwise}}}$$

$$= \underset{\substack{\uparrow \\ \text{length } 2m|n|}}{h^n}$$

$\therefore\ n = \pm 1$ and $a = a_1$

if $a = a_1$, then

$$b_1 \cdots a_m b_m a = (a_1 b_1 \cdots a_m b_m)^{\pm 1}$$

so $n = -1$ $\quad a = a_1^{-1} = a_1 \Rightarrow a_1^2 = 1$

since a was any $a \neq 1$ in A and a_1 fixed, $A = \mathbb{Z}/2$

similarly $B = \mathbb{Z}/2$

exercise: check other cases $h = a_1 b_1 \ldots a_m \ldots$ $\blacksquare$

Some much harder results about SFS are

6) Torus Thm (Scott 1980):

$\boxed{\begin{array}{l} M \text{ irreducible closed oriented 3-mfd} \\ \text{s.t. } \mathbb{Z}\times\mathbb{Z} < \pi_1(M) \\ \text{Then either} \\ \text{1) } M \text{ contains an embedded incompressible torus, or} \\ \text{2) } \pi_1(M) \text{ has an infinite cyclic normal subgroup} \end{array}}$

7) Seifert fibered th$^{\underline{m}}$ (Mess, Tukai, Gabai, Casson-Jungreis):

> If M irreducible and $\pi_1(M)$ has an infinite cyclic normal subgroup, then M is a SFS

We finish our discussion of SFS with a very useful result

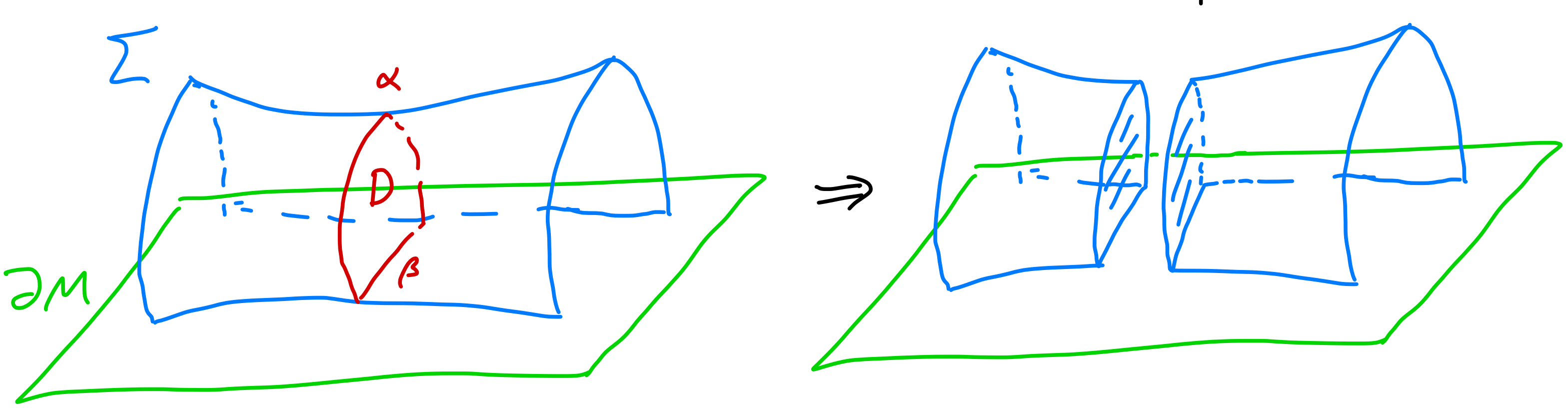
Th$^{\underline{m}}$ 7:

> M an irreducible SFS
> $\Sigma \subset M$ an incompressible, ∂-incompressible, orientable surface
> Then Σ can be isotoped to be either
>
> <u>vertical</u> (= a union of regular fibers)
> (so a torus or annulus)
>
> or <u>horizontal</u> (= transverse to all fibers)

let Σ be a surface with $\partial\Sigma \neq \emptyset$, properly embedded in a 3-manifold M

given a disk $D \subset M$ such that

1) $\partial D = \alpha \cup \beta$, α, β arcs $\alpha \cap \beta = \partial\alpha = \partial\beta$
2) $D \cap \Sigma = \alpha$ and $D \cap \partial M = \beta$



on can do surgery of Σ along D to get

$$\Sigma^* = (\Sigma - (\alpha \times I)) \cup (D^2 \times \partial I)$$

if α doesn't separate Σ into 2 components one of which is a disk, then D is a boundary compressing disk for Σ and Σ is boundary compressible

if Σ is not boundary compressible and no component is a disk, then it is boundary incompressible

Proof of Th$^{\underline{m}}$ 7:

let $C = \bigcup_{i=0}^{n} C_i$ where $C_1 \dots C_n$ are the exceptional fibers and C_0 a regular fiber

isotop Σ so $C \pitchfork \Sigma$ and minimize $\Sigma \cap C$

$$\Sigma \cap N(C) = \coprod \text{meridional disks}$$

let $M_0 = M \setminus N(C)$
$\Sigma_0 = \Sigma \cap M_0$

exercise: Σ_0 is incompressible and ∂-incomp. in M_0

restrict $p: M \to B$ to $p: M_0 \to B_0$ to get an S^1-bundle over B_0 ($\partial B_0 \neq \emptyset$)

let α_i be arcs in B_0 s.t. $B_0 \setminus \bigcup \alpha_1 =$ disk D

$A_1 = p^{-1}(\alpha_1)$ annuli in M_0

Isotop Σ_0 so $\Sigma_0 \pitchfork A$ and minimize $\Sigma_0 \cap A$

Set $M_1 = M_0 \setminus \cup A_i$

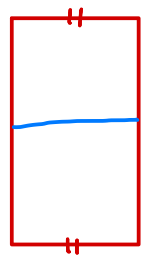
note $M_1 = D^2 \times S^1$ and each A_i gives $A_i^+, A_i^- \subset \partial M_1$

set $\Sigma_1 = \Sigma_0 \setminus \cup A_i$

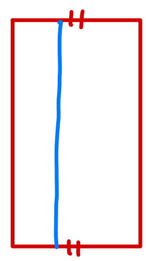
exercise:

1) each component of $\Sigma_1 \cap A_i^{\pm}$ is horizontal or vertical

ie.



or



2) Σ_1 is incompressible in M_1

3) a connected, orientable, incompressible surface in $S^1 \times D^2$ is a meridional disk or a ∂-parallel annulus

3 cases to consider:

i) some component of Σ_1 is an annulus with boundary horizontal

ii) some component of Σ_1 is a meridional disk

iii) all components of Σ_1 are annuli with vertical boundary

in Case i): let G be an innermost such annulus
(i.e. no other component of Σ_1 between G and ∂M_1)

$\exists$ a ∂-compressing disk D for G in M_1 s.t. $D \cap A_1^{\pm} = \emptyset$

so D is ∂-compressing for Σ_0 in M_0 $\otimes$

$\therefore$ Case i) doesn't happen

in Case ii): if some component a disk then all are and easy to see Σ horizontal.

in Case iii): if all components vertical annuli then clearly Σ vertical

C Hyperbolic Manifolds

recall a manifold M is hyperbolic if it admits a complete Riemannian metric with sectional curvature -1 and finite volume

(can relax complete and finite volume, but we want)

A lot is known about hyperbolic manifolds, we could do the whole course on them! But here we just mention a few facts

Mostow-Prasad rigidity theorem 1968, 1973:

> let M, N be (complete, finite volume) hyperbolic manifolds of dimension ≥ 3.
> If $f: M \to N$ is a homotopy equivalence then f is homotopic to an isometry!

Remark: This says a geometric invariant of a hyperbolic manifold is also a topological invariant!

e.g. if two hyperbolic manifolds (of dim ≥ 3) have different volumes then they are <u>not</u> homeomorphic

Suppose M is a 3-manifold with n torus boundary components (and no others)

we say M is hyperbolic if the interior of M has a complete, finite volume metric with sectional curvature -1

fix a basis for the homology of each boundary component of M and let $M(r_1, \dots, r_n)$ be the Dehn filling of M with slopes $r_1, \dots, r_n$

Thurston's hyperbolic Dehn surgery theorem, 1979:

There are a finite set of slopes $\mathcal{E}$ on ∂M such that $M(r_1, \dots, r_n)$ has a hyperbolic structure if $r_i \notin \mathcal{E}$ for all i

Moreover, for a larger set $\mathcal{E}' \supset \mathcal{E}$ if $r_i \notin \mathcal{E}'$ then the cores of the surgery tori form disjoint geodesics in $M(r_1, \dots, r_n)$ with small length and these are the shortest geodesics in $M(r_1, \dots, r_n)$

so, for example, if $K \subset S^3$ is a knot s.t. $S^3 - K$ is hyperbolic, then $S^3_K(r)$ is hyperbolic for all but finitely many r

call r exceptional if $S^3_K(r)$ not hyperbolic

let M hyperbolic 3-manifold with one torus boundary component

Lackenby-Meyerhoff 2015: then M has at most 10 exceptional slopes

Agol 2010: there are only finitely many such M with 9 or more exceptional slopes